Subject Name	**:**	**Trigonometry**
Subject Code	**:**	**FMA12**
Unit	**:**	**III**
Class	**:**	**I B.Sc., Mathematics**

UNIT - III

Hyperbolic function

Definition :

For real or complex value of z the functions $\frac{e^{z}-e^{-z}}{2}$ and $\frac{e^{z}+e^{-z}}{2}$ are defined as a sine hyperbolic function and cosine hyperbolic function of z respectively.

It is denoted by $\sinh z$ & $\cosh z$.

$$\therefore \sinh z = \frac{e^{z}-e^{-z}}{2}, \quad \cosh z = \frac{e^{z}+e^{-z}}{2}, \quad \tanh z = \frac{e^{z}-e^{-z}}{e^{z}+e^{-z}}$$

Note :

i) $e^{z} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots$

ii) $e^{-z} = 1 - \frac{z}{1!} + \frac{z^2}{2!} - \dots$

iii) $\frac{e^{z}-e^{-z}}{2} = \sinh z = \frac{z}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$

iv) $\frac{e^{z}+e^{-z}}{2} = \cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} \dots$

Relation between circular and hyperbolic function.

i) $\sin(ix) = i\sinh x$

ii) $\cos(ix) = \cosh x$

iii) $\tan ix = i\tanh x$.

Addition formula for hyperbolic function:

i) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

ii) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

iii) $\tanh(x+y) = \frac{\tanh x + \tanh y}{1+\tanh x \tanh y}$.

iv) $\sinh(2x) = 2\sinh x \cosh x$.

v) $\cosh^2 x - \sinh^2 x = 1$.

$\cosh^2 x + \sinh^2 x = \cosh 2x$.

$\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

$\operatorname{sech}^2 x = 1 - \tanh^2 x$

$\operatorname{cosech}^2 x = \coth^2 x - 1$

Periods of hyperbolic function:

The periods of hyperbolic sine is $2\pi i$.

The periods of hyperbolic cosine is $2\pi i$

The periods of hyperbolic tangent is πi

Inverse hyperbolic function:

Inverse ~~hyp~~ circular functions are $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, ...

If $y = \sinh x$, x is called the inverse hyperbolic sine of y. It can be return as $x = \sinh^{-1} y$

III'ly We can define $\cosh^{-1}y$, $\tanh^{-1}y$, $\operatorname{cosech}^{-1}y$, $\operatorname{sech}^{-1}y$ and $\coth^{-1}y$.

Inverse functions interms of logarthiemic functions

i) $\sinh^{-1}x = \log_e(x + \sqrt{x^2+1})$

ii) $\cosh^{-1}x = \log_e(x + \sqrt{x^2-1})$

iii) $\tanh^{-1}x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$

Problems :

1. Prove that $\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$.

Soln :

W.K.T
$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}.$$

Put $x = ix$.

$$\cos 2(ix) = \frac{1-\tan^2(ix)}{1+\tan^2(ix)}$$

$$\cosh 2x = \frac{1-i^2\tanh^2 x}{1+i^2\tanh^2 x}.$$

$$\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x}$$

$$\therefore \cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x}.$$

2. Prove that $\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$.

Sol :

W.K.T
$$\tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}.$$

Put $x = ix$.

$$\tan 3(ix) = \frac{3\tan(ix) - \tan^3(ix)}{1-3\tan^2(ix)}$$

$$i\tanh 3x = \frac{i3\tanh x - i^3\tanh^3 x}{1-i^2 3\tanh^2 x}.$$

$$= \frac{i3\tanh x + i\tanh^3 x}{1+3\tanh^2 x}.$$

$$\cancel{i}\tanh 3x = \frac{\cancel{i}(3\tanh x + \tanh^3 x)}{1+3\tanh^2 x}$$

$$\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}.$$

$$\therefore \tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}.$$

3. Prove that $\cosh 2x + \sinh 2x = \frac{1+\tanh x}{1-\tanh x}.$

soln:

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}.$$

Put $x = ix$.

$$\cos 2(ix) = \frac{1-\tan^2(ix)}{1+\tan^2(ix)}$$

$$\cosh 2x = \frac{1-i^2\tanh^2 x}{1+i^2\tanh^2 x}$$

$$\cosh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x} \quad \text{——} ①.$$

Also

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}.$$

Put $x = ix$.

$$\sin 2(ix) = \frac{2\tan(ix)}{1+\tan^2(ix)}$$

$$i\sinh 2x = \frac{i2\tanh x}{1+i^2\tanh^2 x}$$

$$\cancel{i}\sinh 2x = \frac{\cancel{i}2\tanh x}{1-\tanh^2 x}$$

$$\sinh 2x = \frac{2\tanh x}{1-\tanh^2 x} \quad \text{——} ②$$

Adding ① and ②

$$\cosh 2x + \sinh 2x = \frac{1+\tanh^2 x}{1-\tanh^2 x} + \frac{2\tanh x}{1-\tanh^2 x}$$

$$= \frac{1+\tanh^2 x + 2\tanh x}{1-\tanh^2 x}$$

$$= \frac{(1+\tanh x)^2}{1-\tanh^2 x.}$$

$$= \frac{(1+\tanh x)^2}{(1+\tanh x)(1-\tanh x)}$$

$$= \frac{1+\tanh x}{1-\tanh x.}$$

$$\cosh 2x + \sinh 2x = \frac{1+\tanh x}{1-\tanh x.}$$

Hence proved.

4. If $\tan\frac{x}{2} = \tanh y/_2$ Prove that i) $\sinh y = \tan x$.

ii) $y = \log_e \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$.

Solu !

i) W.K.T

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}.$$

$$\tan x = \frac{2\tan x/_2}{1-\tan^2 x/_2}.$$

$$\tan x = \frac{2\tan h\, y/_2}{1-\tanh^2 y/_2}.$$

$$= \frac{\frac{2\sinh y/2}{\cosh y/2}}{1 - \frac{\sinh^2 y/2}{\cosh^2 y/2}}$$

$$= \frac{\frac{2\sinh y/2}{\cosh y/2}}{\frac{\cosh^2 y/2 - \sinh^2 y/2}{\cosh^2 y/2}}$$

$$= \frac{2\sinh y/2 \cosh y/2}{\cosh^2 y/2 - \sinh^2 y/2}$$

$$= \frac{\sinh 2y/2}{1}$$

$$= \sinh y.$$

$\therefore \sinh y = \tan x$

Hence proved.

ii)

$\tan x/2 = \tanh y/2$.

$\tanh^{-1}(\tan x/2) = \frac{y}{2}$

$$y/2 = \frac{1}{2}\log_e\left(\frac{1+\tan x/2}{1-\tan\frac{x}{2}}\right).$$

$$= \frac{1}{2}\log_e\left(\frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}}\right)$$

$$\frac{y}{2} = \frac{1}{2}\log_e \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$y = \log_e \tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

Hence proved.

5. If $\tan\frac{x}{2} = \tanh\frac{x}{2}$ show that $\cos x \cdot \cosh x = 1$.

Solu:

W.K.T

$$\cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x}$$

By Given.

$$x = x/2 \Rightarrow \cos x = \frac{1-\tan^2 x/2}{1+\tan^2 x/2} = \frac{1-\tanh^2 x/2}{1+\tanh^2 x/2}$$

$$= \frac{1-\frac{\sinh^2 x/2}{\cosh^2 x/2}}{1+\frac{\sinh^2 x/2}{\cosh^2 x/2}}$$

$$= \frac{\cosh^2 x/2 - \sinh^2 x/2 \,/\cos^2 x/2}{\cosh^2 x/2 + \sinh^2 x/2 \,/\cos^2 x/2}$$

$$= \frac{1}{\cos 2h\, x/2}$$

$$\cos x = \frac{1}{\cosh x}.$$

$$\cos x \cdot \cosh x = 1.$$

Hence proved.

6. If $\sin(A+iB) = x+iy$ then i) $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$

$\cos(x+iy) = u+iv.$ ii) $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$

Solu: $\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 y} = 1.$

W.K.T $\frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x} = 1$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B.$$

$$\sin(A+iB) = \sin A\cos(iB) + \cos A\sin(iB)$$

$$x+iy = \sin A\cosh B + i\cos A\sinh B.$$

Equating the real and Imaginary part.

$$x = \sin A \cosh B \quad \text{--- (1)}$$

$$y = \cos A \sinh B \quad \text{--- (2)}$$

i. (1) ⇒ $\dfrac{x}{\cosh B} = \sin A$

squaring on both sides

$$\frac{x^2}{\cosh^2 B} = \sin^2 A \quad \text{--- (3)}$$

(2) ⇒ $\dfrac{y}{\sinh B} = \cos A$

squaring on both sides.

$$\frac{y^2}{\sinh^2 B} = \cos^2 A \quad \text{--- (4)}$$

Adding (3) & (4)

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \cos^2 A + \sin^2 A.$$

$$= 1.$$

$$\therefore \frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1.$$

Hence proved.

ii. (1) ⇒ $\dfrac{x}{\sin A} = \cosh B$

squaring on both sides

$$\frac{x^2}{\sin^2 A} = \cosh^2 B \quad \text{--- (5)}$$

② ⇒ $\frac{y}{\cos A} = \sinh B$.

Squaring on b.s.

$$\frac{y^2}{\cos^2 A} = \sinh^2 B \quad — ⑥.$$

subtracting ⑤ & ⑥

$$\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B$$

$$= 1.$$

7. Show that i) $\sinh 3x = 3\sinh x + 4\sinh^3 x$.

Solu:

W.K.T

$$\sin 3x = 3\sin x - 4\sin^3 x.$$

Put $x = ix$.

$$\sin 3(ix) = 3\sin(ix) - 4\sin^3(ix)$$

$$i\sinh 3x = i3\sinh x + i4\sinh^3 x.$$

$$i(\sinh 3x) = i(3\sinh x + 4\sinh^3 x)$$

$$\sinh 3x = 3\sinh x + 4\sinh^3 x.$$

Hence proved.

ii) $2\cosh^2 x - 1 = \cosh^2 x + \sinh^2 x$.

Solu:

$$2\cos^2 x - 1 = \cos^2 x - \sin^2 x.$$

Put $x = ix$.

$$2\cos^2(ix) - 1 = \cos^2(ix) - \sin^2(ix)$$

$$2\cosh^2 x - 1 = \cosh^2 x + \sinh^2 x.$$

Hence proved.

Show that i) $\cosh 2x = 1 + 2\sinh^2 x$.

solu:

W.K.T $\cos 2x = 1 - 2\sin^2 x$.

Put $x = ix$

$\cos 2(ix) = 1 - 2\sin^2(ix)$

$\cosh 2x = 1 + 2\sinh^2 x$.

Hence proved.

ii) $\cosh 2x = 2\cosh^2 x - 1$

solu:

W.K.T $\cos 2x = 2\cos^2 x - 1$

Put $x = ix$.

$\cos 2(ix) = 2\cos^2(ix) - 1$

$\cosh 2x = 2\cosh^2 x - 1$.

Hence proved.

9. Subrate real and imaginary part $\tan(x+iy)$.

solu:

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)}$$

Multiply & Divided by 2.

$$\tan(x+iy) = \frac{2\sin(x+iy)}{2\cos(x+iy)}$$

$$= \frac{2\sin(x+iy)}{2\cos(x+iy)} \times \frac{\cos(x-iy)}{\cos(x-iy)}$$

Let $A = x+iy$, $B = x-iy$.

$$= \frac{2\sin A \cos B}{2\cos A \cos B}$$

W.K.T

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$

$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$

$$\tan(x+iy) = \frac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)}$$

$$= \frac{\sin(x+iy+x-iy)+\sin(x+iy-x+iy)}{\cos(x+iy+x-iy)+\cos(x+iy-x+iy)}$$

$$= \frac{\sin 2x+\sin(2iy)}{\cos 2x+\cos(2iy)}$$

$$= \frac{\sin 2x + i\sinh 2y}{\cos 2x+\cosh 2y}.$$

$$= \frac{\sin 2x}{\cos 2x+\cosh 2y} + i\frac{\sinh 2y}{\cos 2x+\cosh 2y}.$$

$\therefore$ Real part is $\dfrac{\sin 2x}{\cos 2x+\cosh 2y}$.

Imaginary part is $\dfrac{\sinh 2y}{\cos 2x+\cosh 2y}$

10. If $\tan(x+iy) = u+iv$ Show that $\dfrac{u}{v} = \dfrac{\sin 2x}{\sinh 2y}$.

solu :

$$\tan(x+iy) = u+iv.$$

$$\frac{\sin(x+iy)}{\cos(x+iy)} = u+iv.$$

$$\sin(x+iy) = (u+iv)\cos(x+iy) \quad \text{--- (1)}$$

Put $i = -i$

$$\sin(x-iy) = (u-iv)\cos(x-iy) \quad \text{--- (2)}$$

$$\sin 2x = \sin[(x+iy)+(x-iy)] \quad (x+iy+x-iy=2x)$$

$$= \sin(x+iy)\cos(x-iy) + \cos(x+iy)\sin(x-iy)$$

$$= (u+iv)\cos(x+iy)\cos(x-iy) + \cos(x+iy)(u-iv)\cos(x-iy)$$

$$= \cos(x+iy)\cos(x-iy)[u+\cancel{iv}+u-\cancel{iv}].$$

$$\sin 2x = 2u\cos(x+iy)\cos(x-iy) \quad \text{--- (3)}$$

$$\sin 2(iy) = \sin[(x+iy)-(x-iy)]$$

$$i\sinh 2y = \sin(x+iy)\cos(x-iy) - \cos(x+iy)\sin(x-iy)$$

$$i\sinh 2y = (u+iv)\cos(x+iy)\cos(x-iy) - \cos(x+iy)(u-iv)\cos(x-iy)$$

$$= \cos(x+iy)\cos(x-iy)[u+iv-u+iv]$$

$$i\sinh 2y = 2iv\cos(x+iy)\cos(x-iy)$$

$$\sinh 2y = 2v\cos(x+iy)\cos(x-iy) \quad \text{--- (4)}$$

$$\frac{(3)}{(4)} \Rightarrow \frac{\sin 2x}{\sinh 2y} = \frac{\cancel{2}u\cancel{\cos(x+iy)}\cos(x-iy)}{\cancel{2}v\cancel{\cos(x+iy)}\cancel{\cos(x-iy)}}$$

$$= \frac{u}{v}.$$

$$\therefore \frac{u}{v} = \frac{\sin 2x}{\sinh 2y}.$$

Hence proved.

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11. If $\cos(x+iy) = u+iv$ then show that

i) $\dfrac{u^2}{\cosh^2 y} + \dfrac{v^2}{\sinh^2 y} = 1$

ii) $\dfrac{u^2}{\cos^2 x} - \dfrac{v^2}{\sin^2 x} = 1.$

Solu :

W.K.T

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy)$$

$$u+iv = \cos x \cosh y - i \sin x \sinh y.$$

Equating real and imaginary terms.

$$u = \cos x \cosh y \quad \text{—— (1)}$$

$$v = -\sin x \sinh y \quad \text{—— (2)}$$

i) (1) ⇒ $\dfrac{u}{\cosh y} = \cos x.$

Squaring on both sides.

$$\frac{u^2}{\cosh^2 y} = \cos^2 x. \quad \text{—— (3)}$$

(2) ⇒ $\dfrac{v}{\sinh x} = -\sin x.$

Squaring on b.s

$$\frac{v^2}{\sinh^2 x} = \sin^2 x \quad \text{—— (4)}$$

Adding (3) & (4)

$$\frac{u^2}{\cosh^2 y} + \frac{v^2}{\sinh^2 x} = \cos^2 x + \sin^2 x$$

$$= 1$$

ii) (1) ⇒ $\dfrac{u}{\cos x} = \cosh y.$

Squaring

$$\frac{u^2}{\cos^2 x} = \cosh^2 y \quad \text{—— (5)}$$

(2) ⇒ $\dfrac{v}{-\sin x} = \sinh y$

Squaring

$$\frac{v^2}{\sin^2 x} = \sinh^2 y \quad \text{—— (6)}$$

(5) − (6)

$$\frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x} = \cosh^2 y - \sinh^2 y$$

$$= 1.$$

If $\tan y = \tan\alpha \tanh\beta$, $\tan z = \cot\alpha \tanh\beta$ then prove that $\tan(y+z) = \sinh 2\beta \operatorname{cosec} 2\alpha$.

Solu:

Given: $\tan y = \tan\alpha \tanh\beta$.

$\tan z = \cot\alpha \tanh\beta$

To prove:

$\tan(y+z) = \sinh 2\beta \operatorname{cosec} 2\alpha$.

$$\tan(y+z) = \frac{\tan y + \tan z}{1 - \tan y \tan z}$$

$$= \frac{\tan\alpha \tanh\beta + \cot\alpha \tanh\beta}{1 - (\tan\alpha \tanh\beta)(\cot\alpha \tanh\beta)}$$

$$= \frac{\tanh\beta\left(\tan\alpha + \frac{1}{\tan\alpha}\right)}{1 - (\cancel{\tan\alpha}\tanh\beta)\left(\frac{1}{\cancel{\tan\alpha}}\right)\tanh\beta}$$

$$= \frac{\tanh\beta\left(\tan\alpha + \frac{1}{\tan\alpha}\right)}{1 - \tanh^2\beta}$$

$$= \frac{\tanh\beta\left(\frac{\tan^2\alpha + 1}{\tan\alpha}\right)}{1 - \tanh^2\beta}$$

$$= \frac{\tanh\beta(\tan^2\alpha + 1)}{(1 - \tanh^2\beta)(\tan\alpha)}$$

$$= \frac{\tanh\beta}{1 - \tanh^2\beta}\left[\frac{\tan^2\alpha + 1}{\tan\alpha}\right].$$

Multiply and divided by 2.

$$= \frac{2\tanh\beta}{1 - \tanh^2\beta}\left[\frac{\tan^2\alpha + 1}{2\tan\alpha}\right].$$

$$= \frac{2\tanh\beta}{\operatorname{sech}^2\beta}\left[\frac{\tan^2\alpha+1}{2\tan\alpha}\right]$$

$$= \frac{2\sinh\beta}{\cosh\beta} \times \frac{\cosh^2\beta}{1}\left[\frac{\tan^2\alpha+1}{2\tan\alpha}\right].$$

$$= 2\sinh\beta\cosh\beta\left[\frac{\tan^2\alpha+1}{2\tan\alpha}\right]$$

$$\tan(y+z) = \sinh 2\beta \operatorname{cosec} 2\alpha.$$

Hence proved.

Sepprate into real and imaginary part in $\tanh(x+iy)$ "$(1+i)$"

Solu:

$$\tan(ix) = i\tanh x.$$

Multiply & divided by i on b.s.

$$i\tan(ix) = -\tanh x.$$

Multiply by $(-)$ on b.s.

$$-i\tan(ix) = \tanh x.$$

Put $x = x+iy$.

$$\tanh(x+iy) = -i\tan(i(x+iy))$$

$$= -i\tan(ix-y)$$

$$= -i\left[\frac{\sin(ix-y)}{\cos(ix-y)}\right].$$

Multiply & divided by 2

$$= -i\left[\frac{2\sin(ix-y)}{2\cos(ix-y)}\right].$$

$$= -i\left[\frac{2\sin(ix-y)}{2\cos(ix-y)} \times \frac{\cos(-ix-y)}{\cos(-ix-y)}\right].$$

$$= -i\left[\frac{2\sin(ix-y)\cos(-ix-y)}{2\cos(ix-y)\cos(-ix-y)}\right].$$

$$= -i\left[\frac{\sin(ix-y-ix-y)+\sin(ix-\cancel{y}+\cancel{y}+ix)}{\cos(ix-y-y-ix)+\cos(ix-\cancel{y}+\cancel{y}+ix)}\right]$$

$$= -i\left[\frac{\sin(-2y)+\sin(2ix)}{\cos(-2y)+\cos(2ix)}\right].$$

$\cos(-x)=\cos x$

$$= -i\left[\frac{-\sin 2y + i\sinh 2x}{\cos 2y + \cosh 2x}\right].$$

$$= -i\left[\frac{i^2\sin 2y + i\sinh 2x}{\cos 2y + \cosh 2x}\right].$$

$$= \frac{-i^2\left[i\sin 2y + \sinh 2x\right]}{\cos 2y + \cosh 2x}$$

$$= \frac{i\sin 2y}{\cos 2y + \cosh 2x} + \frac{\sinh 2x}{\cos 2y + \cosh 2x}.$$

$$\text{Real} = \frac{\sinh 2x}{\cos 2y + \cosh 2x}$$

$$\text{Imaginary} = \frac{\sin 2y}{\cos 2y + \cosh 2x}.$$

If $\tan(\alpha+i\beta) = x+iy$ then prove i) $1-x^2-y^2-2x\cot 2\alpha=0$

ii) $1+x^2+y^2-2y\coth 2\beta=0$, iii) $x\cot 2\alpha + y\coth 2\beta = 1$.

Given:

$\tan(\alpha+i\beta) = x+iy$

i) To prove : $1-x^2-y^2-2x\cot 2\alpha=0$

$$\tan(\alpha+i\beta) = x+iy$$

Taking conjugate

$$\tan(\alpha-i\beta) = x-iy$$

$$\tan 2\alpha = \tan[(\alpha+i\beta)+(\alpha-i\beta)]$$

$$= \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$$

$$= \frac{x+\cancel{iy}+x-\cancel{iy}}{1-(x+iy)(x-iy)}$$

$$= \frac{2x}{1-(x^2-i^2y^2)}$$

$$\tan 2\alpha = \frac{2x}{1-(x^2+y^2)}$$

$$1-x^2-y^2 = \frac{2x}{\tan 2\alpha}$$

$$= 2x\cot 2\alpha.$$

$$1-x^2-y^2-2x\cot 2\alpha = 0 \quad \text{——— (1)}.$$

ii) $\tan(\alpha+i\beta) = x+iy$.

Taking conjugate

$$\tan(\alpha-i\beta) = x-iy.$$

$$\tan 2i\beta = \tan((\alpha+i\beta)-(\alpha-i\beta))$$

$$\tan 2i\beta = \frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$$

$$i\tanh 2\beta = \frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$$

$$= \frac{2iy}{1+(x^2-i^2y^2)}$$

$$\cancel{i}\tanh 2\beta = \frac{2\cancel{i}y}{1+(x^2+y^2)}$$

$$1+x^2+y^2 = \frac{2y}{\tanh 2\beta}$$

$$1+x^2+y^2 = 2y\coth 2\beta$$

$$1+x^2+y^2-2y\coth 2\beta = 0. \quad \text{—— (2)}$$

iii. Solve (1) & (2).

$$1+x^2+y^2-2y\coth 2\beta = 0$$

$$1-x^2-y^2-2x\cot 2\alpha = 0$$

$$2-2x\cot 2\alpha-2y\coth 2\beta = 0$$

$$2(1-x\cot 2\alpha-y\coth 2\beta) = 0$$

$$x\cot 2\alpha+y\coth 2\beta = 1$$

Hence proved.

Sepperate into real and imaginary part in $\tan(1+i)$.

Solu:

$$\tan(ix) = i\tanh x.$$

Multiply by i on b.s.

$$i\tan(ix) = -\tanh x.$$

$$-i\tan(ix) = \tanh x$$

Put $x = 1+i$

$$\tanh(1+i) = -i\tan(i(1+i))$$

$$= -i\tan(i-1)$$

$$= -i\left[\frac{\sin(i-1)}{\cos(i-1)}\right].$$

Multiply and divided by 2.

$$= -i\left[\frac{2\sin(i-1)}{2\cos(i-1)}\right]$$

$$= -i\left[\frac{2\sin(i-1)}{2\cos(i-1)} \times \frac{\cos(-i-1)}{\cos(-i-1)}\right]$$

$$= -i\left[\frac{\sin(i-1-i-1) + \sin(i-1+i+1)}{\cos(i-1-i-1) + \cos(i-1+i+1)}\right]$$

$$= -i\left[\frac{\sin(-2) + \sin(2i)}{\cos(-2) + \cos(2i)}\right]$$

$$= -i\left[\frac{-\sin(2) + i\sinh 2}{\cos(2) + \cosh 2}\right].$$

$$= -i\left[\frac{+i^2 \sin(2) + i\sin 2h}{\cos(2) + \cos 2h}\right]$$

$$= \frac{i\sin(2) + \sin 2h}{\cos(2) + \cos 2h}$$

$$= \frac{i\sin(2)}{\cos(2) + \cos 2h} + \frac{\sin 2h}{\cos(2) + \cos 2h}$$

$\therefore$ Real $= \dfrac{\sin 2h}{\cos(2) + \cos 2h}$

Imaginary $= \dfrac{\sin(2)}{\cos(2) + \cos 2h}$.

If $\tan(x+iy) = u+iv$ show that $\dfrac{u}{v} = \dfrac{\sin 2x}{}$

If $\sin(\theta+i\phi) = \tan(x+iy)$ show that $\dfrac{\tan\theta}{\tanh\phi} = \dfrac{\sin 2x}{\sinh 2y}$

Solu:

Given: $\sin(\theta+i\phi) = \tan(x+iy)$

To prove:

$$\frac{\tan\theta}{\tanh\phi} = \frac{\sin 2x}{\sinh 2y}$$

Let $\tan(x+iy) = \dfrac{\sin(x+iy)}{\cos(x+iy)}$

Multiply and divided by 2.

$$= \frac{2\sin(x+iy)}{2\cos(x+iy)} \times \frac{\cos(x-iy)}{\cos(x-iy)}$$

$$\tan(x+iy) = \frac{\sin 2x + \sin(2iy)}{\cos 2x + \cos(2iy)} \quad \text{——— (1)}$$

$$\sin(\theta + i\phi) = \tan(x+iy)$$

$$\sin\theta \cos i\phi + \cos\theta \sin i\phi = \frac{\sin 2x + \sin(2iy)}{\cos 2x + \cos 2iy}$$

$$= \frac{\sin 2x + i\sinh 2y}{\cos 2x + \cosh 2y}$$

$$\sin\theta\cosh\phi + i\cos\theta\sinh\phi = \frac{\sin 2x}{\cos 2x + \cosh 2y} + \frac{i \sinh 2y}{\cos 2x + \cosh 2y}$$

Real part

$$\sin\theta\cosh\phi = \frac{\sin 2x}{\cos 2x + \cosh 2y} \quad \text{——— (2)}$$

Imaginary part

$$\cos\theta\sinh\phi = \frac{\sinh 2y}{\cos 2x + \cosh 2y} \quad \text{——— (3)}$$

(2) ÷ (3)

$$\frac{\sin\theta\cosh\phi}{\cos\theta\sinh\phi} = \frac{\sin 2x}{\cos 2x + \cosh 2y} \times \frac{\cos 2x + \cosh 2y}{\sinh 2y}$$

$$\frac{\tan\theta}{\tanh\phi} = \frac{\sin 2x}{\sinh 2y}$$

Hence proved.

Prove that $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

Solu:

W.K.T

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

Put $x = ix$, $y = iy$.

$$\sin(ix+iy) = \sin(ix)\cos(iy) + \cos(ix)\sin(iy)$$

$$\sin i(x+y) = i\sinh x \cosh y + i\cosh x \sinh y.$$

$$i\sinh(x+y) = i[\sinh x \cosh y + \cosh x \sinh y].$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$$

Hence proved.

Show that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

Solu:

W.K.T

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Put $x = ix$, $y = iy$.

$$\cos(ix+iy) = \cos ix \cos iy - \sin ix \sin iy$$

$$\cos i(x+y) = \cosh x \cosh y - i^2 \sinh x \sinh y.$$

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

$\therefore$ Hence proved.

Show that $\sinh 2x = \dfrac{2\tanh x}{1-\tanh^2 x}$.

solu,

W.K.T

$$\sin 2x = \frac{2\tan x}{1+\tan^2 x}.$$

Put $x = ix$.

$$\sin 2ix = \frac{2\tan ix}{1+\tan^2 ix}$$

$$i\sinh 2x = \frac{i2\tanh x}{1+i^2\tanh^2 x}$$

$$i\sinh 2x = \frac{i2\tanh x}{1-\tanh^2 x}$$

$$\sinh 2x = \frac{2\tanh x}{1-\tanh^2 x}.$$

Hence proved.

If $\tan(\alpha+i\beta) = x+iy$ Prove that $x^2+y^2+2x\cot 2\alpha = 1$

solu!

$\tan(\alpha+i\beta) = x+iy$

Taking Conjugate

$\tan(\alpha-i\beta) = x-iy$

$$\tan 2\alpha = \tan[(\alpha+i\beta)+(\alpha-i\beta)]$$

$$= \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$$

$$= \frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$= \frac{2x}{1-(x^2-i^2y^2)}$$

$$\tan 2\alpha = \frac{2x}{1-x^2-y^2}$$

$$1-x^2-y^2 = \frac{2x}{\tan 2\alpha}$$

$$1-x^2-y^2 = 2x\cot 2\alpha$$

$$x^2+y^2+2x\cot 2\alpha = 1.$$

$$\therefore x^2+y^2+2x\cot 2\alpha = 1.$$

Hence proved.

Show that $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}.$

Soln:

W.K.T

$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

Put $x = ix$

$$\tan 2ix = \frac{2\tan ix}{1-\tan^2 ix}.$$

$$i\tanh 2x = \frac{i2\tanh x}{1-i^2\tanh^2 x}$$

$$i\tanh 2x = \frac{i2\tanh x}{1+\tanh^2 x}$$

$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}.$$

$$\therefore \tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}.$$

Hence proved.

Show that $16\cosh^5\theta = \cosh 5\theta + 5\cosh 3\theta + 10\cosh\theta$.

Solu:

W.K.T $\cosh\theta = \dfrac{e^{\theta}+e^{-\theta}}{2}$

$2\cosh\theta = e^{\theta}+e^{-\theta}$

$(\cosh\theta)^5 = \dfrac{(e^{\theta}+e^{-\theta})^5}{2^5}$

$\cosh^5\theta = \dfrac{1}{32}\left[e^{\theta}+e^{-\theta}\right]^5$

$(x+y)^n = x^n + nc_1 x^{n-1}y + nc_2 x^{n-2}y^2$

$= \dfrac{1}{32}\Big[(e^{\theta})^5 + 5c_1(e^{\theta})^4(e^{-\theta}) + 5c_2(e^{\theta})^3 e^{-2\theta} + 5c_3(e^{\theta})^2(e^{-3\theta}) + 5c_4 e^{\theta}e^{-4\theta} + 5c_5 e^{-5\theta}\Big]$

$= \dfrac{1}{32}\Big[e^{5\theta} + 5e^{3\theta} + 10e^{\theta} + 10e^{-\theta} + 5e^{-3\theta} + e^{-5\theta}\Big]$

$= \dfrac{1}{32}\Big[(e^{5\theta}+e^{-5\theta}) + 5(e^{3\theta}+e^{-3\theta}) + 10(e^{\theta}+e^{-\theta})\Big]$

$= \dfrac{1}{32}\Big[2\cosh 5\theta + 5(2\cosh 3\theta) + 10(2\cosh\theta)\Big]$

$= \dfrac{1}{16}\Big[\cosh 5\theta + 5\cosh 3\theta + 10\cosh\theta\Big]$

$16\cosh^5\theta = \cosh 5\theta + 5\cosh 3\theta + 10\cosh\theta$.

Hence proved.

Show that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$.

Sepparate into real and imaginary part $\tan^{-1}(x+iy)$

Solu:

Let $\tan^{-1}(x+iy) = A+iB$.

$$x+iy = \tan(A+iB)$$

$$\tan^{-1}(x-iy) = A-iB$$

$$x-iy = \tan(A-iB)$$

$$\tan 2A = \tan[(A+iB)+(A-iB)]$$

$$= \frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$= \frac{x+iy+x-iy}{1-(x+iy)(x-iy)}$$

$$\tan 2A = \frac{2x}{1-(x^2+y^2)}$$

$$A = \frac{1}{2}\tan^{-1}\left[\frac{2x}{1-(x^2+y^2)}\right]$$

$$\tan 2iB = \tan[(A+iB)-(A-iB)]$$

$$= \frac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)}$$

$$= \frac{x+iy-x+iy}{1+(x^2+y^2)}$$

$$= \frac{2iy}{1+x^2+y^2}$$

$$i\tan h2B = \frac{2iy}{1+x^2+y^2}$$

$$B = \frac{1}{2}\tanh^{-1}\left[\frac{2y}{1+x^2+y^2}\right]$$

$\therefore$ Real part is $A = \frac{1}{2}\tan^{-1}\left[\frac{2x}{1-(x^2+y^2)}\right]$

Imaginary part is $B = \frac{1}{2}\tanh^{-1}\left[\frac{2y}{1+x^2+y^2}\right]$.

If $\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$ Then prove that

$$y = \frac{1}{2}\log_e\left(\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right)$$

Solu:

Given: $\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$.

To prove: $y = \frac{1}{2}\log_e\left(\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right)$.

$\cos(x+iy) = r(\cos\alpha + i\sin\alpha)$

$\cos x\cos iy - \sin x\sin iy = r\cos\alpha + ir\sin\alpha$

$\cos x\cosh y - i\sin x\sinh y = r\cos\alpha + ir\sin\alpha$

Equating real and imaginary

$\cos x\cosh y = r\cos\alpha$ — (1)

$-\sin x\sinh y = r\sin\alpha$ — (2)

$② \div ①$ $\quad \dfrac{-\sin x \sinh y}{\cos x \cosh y} = \dfrac{r\sin\alpha}{r\cos\alpha}$.

$$-\tan x \tanh y = \tan\alpha .$$

$$\tanh y = -\frac{\tan\alpha}{\tan x}$$

$$y = \tanh^{-1}\left(\frac{-\tan\alpha}{\tan x}\right)$$

$\tanh^{-1} x = \frac{1}{2}\log_e\left(\frac{1+x}{1-x}\right)$

$$y = \frac{1}{2}\log_e\left(\frac{1+\left(-\frac{\tan\alpha}{\tan x}\right)}{1-\left(-\frac{\tan\alpha}{\tan x}\right)}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{\frac{\tan x - \tan\alpha}{\tan x}}{\frac{\tan x + \tan\alpha}{\tan x}}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{\tan x - \tan\alpha}{\tan x + \tan\alpha}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{\frac{\sin x}{\cos x} - \frac{\sin\alpha}{\cos\alpha}}{\frac{\sin x}{\cos x} + \frac{\sin\alpha}{\cos\alpha}}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{\frac{\sin x\cos\alpha - \sin\alpha\cos x}{\cos x\cos\alpha}}{\frac{\sin x\cos\alpha + \sin\alpha\cos x}{\cos x\cos\alpha}}\right)$$

$$y = \frac{1}{2}\log_e\left(\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right)$$

If $\tan(x+iy) = u+iv$ show that $\dfrac{1-u^2-v^2}{1+u^2+v^2} = \dfrac{\cos 2x}{\cosh 2y}$.

Soln:

Given: $\tan(x+iy) = u+iv$.

To prove: $\dfrac{1-u^2-v^2}{1+u^2+v^2} = \dfrac{\cos 2x}{\cosh 2y}$.

$$\frac{\sin(x+iy)}{\cos(x+iy)} = (u+iv)$$

$$\cos(x+iy) = \frac{\sin(x+iy)}{u+iv} \quad \text{--- (1)}$$

$$\cos(x-iy) = \frac{\sin(x-iy)}{u-iv} \quad \text{--- (2)}$$

$$\cos 2x = \cos\left[(x+iy)+(x-iy)\right]$$

$$= \cos(x+iy)\cos(x-iy) - \sin(x+iy)\sin(x-iy)$$

$$= \frac{\sin(x+iy)}{u+iv}\,\frac{\sin(x-iy)}{u-iv} - \sin(x+iy)\sin(x-iy)$$

$$= \sin(x+iy)\sin(x-iy)\left[\frac{1}{u^2+v^2} - 1\right]$$

$$\cos 2x = \sin(x+iy)\sin(x-iy)\left[\frac{1-u^2-v^2}{u^2+v^2}\right] \quad \text{--- (3)}.$$

$$\cos 2iy = \cos\left[(x+iy)-(x-iy)\right]$$

$$\cosh 2y = \cos(x+iy)\cos(x-iy) + \sin(x+iy)\sin(x-iy)$$

$$= \frac{\sin(x+iy)}{u+iv}\,\frac{\sin(x-iy)}{u-iv} + \sin(x+iy)\sin(x-iy)$$

$$= \sin(x+iy)\sin(x-iy)\left[\frac{1}{u^2+v^2}+1\right]$$

$$\cosh 2y = \sin(x+iy)\sin(x-iy)\left[\frac{1+u^2+v^2}{u^2+v^2}\right] \quad \text{--- } (4)$$

$(3) \div (4)$

$$\frac{\cos 2x}{\cosh 2y} = \frac{\sin(x+iy)\sin(x-iy)\left[\frac{1-u^2-v^2}{u^2+v^2}\right]}{\sin(x+iy)\sin(x-iy)\left[\frac{1+u^2+v^2}{u^2+v^2}\right]}$$

$$\frac{\cos 2x}{\cosh 2y} = \frac{1-u^2-v^2}{1+u^2+v^2}.$$

If $\tan(x+iy) = \tan\alpha + i\sec\alpha$ Show that $e^{2y} = \pm\cot\frac{\alpha}{2}$.

Soln:

Given: $\tan(x+iy) = \tan\alpha + i\sec\alpha$

To prove: $e^{2y} = \pm\cot\frac{\alpha}{2}$.

$$\tan(x+iy) = \tan\alpha + i\sec\alpha \quad \text{--- } (1)$$

$$\tan(x-iy) = \tan\alpha - i\sec\alpha$$

$$\tan 2x = \tan\left[(x+iy)+(x-iy)\right]$$

$$= \frac{\tan(x+iy)+\tan(x-iy)}{1-\tan(x+iy)\tan(x-iy)}$$

$$= \frac{\tan\alpha + i\sec\alpha + \tan\alpha - i\sec\alpha}{1-(\tan\alpha+i\sec\alpha)(\tan\alpha - i\sec\alpha)}$$

$$\tan 2x = \frac{2\tan\alpha}{1-(\tan^2\alpha+\sec^2\alpha)}$$

$$\tan 2iy = \tan\left[(x+iy)-(x-iy)\right]$$

$$= \frac{\tan(x+iy)-\tan(x-iy)}{1+\tan(x+iy)\tan(x-iy)}$$

$$= \frac{\tan\alpha + i\sec\alpha - \tan\alpha + i\sec\alpha}{1+(\tan\alpha + i\sec\alpha)(\tan\alpha - i\sec\alpha)}$$

$$i\tanh 2y = \frac{2i\sec\alpha}{1+(\tan^2\alpha+\sec^2\alpha)}$$

$s^2-1=t^2$

$\sqrt{s^2-1}=t$

$$\tanh 2y = \frac{2\sec\alpha}{1+\tan^2\alpha+\sec^2\alpha}$$

$$2y = \tanh^{-1}\left[\frac{2\sec\alpha}{1+\tan^2\alpha+\sec^2\alpha}\right]$$

$$2y = \cancel{\tfrac{1}{2}}\tanh^{-1}\left[\frac{2\sec\alpha}{1+\tan^2\alpha+\sec^2\alpha}\right]$$

$$2y = \frac{1}{2}\log_e\left(\frac{1+\left(\frac{2\sec\alpha}{1+\tan^2\alpha+\sec^2\alpha}\right)}{1-\left(\frac{2\sec\alpha}{1+\tan^2\alpha+\sec^2\alpha}\right)}\right)$$

$$2y = \frac{1}{2}\log_e\left(\frac{\frac{1+\tan^2\alpha+\sec^2\alpha+2\sec\alpha}{\cancel{1+\tan^2\alpha+\sec^2\alpha}}}{\frac{1+\tan^2\alpha+\sec^2\alpha-2\sec\alpha}{\cancel{1+\tan^2\alpha+\sec^2\alpha}}}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{2\sec^2\alpha+2\sec\alpha}{2\sec^2\alpha-2\sec\alpha}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{\cancel{2\sec\alpha}(\sec\alpha+1)}{\cancel{2\sec\alpha}(\sec\alpha-1)}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{\sec\alpha+1}{\sec\alpha-1}\right).$$

$$= \frac{1}{2} \log_e \left(\frac{\frac{1}{\cos\alpha} + 1}{\frac{1}{\cos\alpha} - 1} \right)$$

$$= \frac{1}{2} \log_e \left(\frac{\frac{1+\cos\alpha}{\cos\alpha}}{\frac{1-\cos\alpha}{\cos\alpha}} \right)$$

$$= \frac{1}{2} \log_e \left(\frac{\cancel{2}\cos^2 \alpha/2}{\cancel{2}\sin^2 \alpha/2} \right)$$

$$2y = \frac{1}{2} \log_e \cot^2 \alpha/2$$

$$4y = \log_e \cot^2 \alpha/2$$

$$e^{4y} = \cot^2 \alpha/2$$

$$e^{2y} = \pm \cot \alpha/2 .$$

Hence proved.

If $\sin(\theta + i\phi)\sin(\alpha + i\beta) = 1$ then show that

i) $\tanh^2 \phi \cosh^2 \beta = \cos^2 \alpha$ ii) $\cosh^2 \phi \tanh^2 \beta = \cos^2 \theta$.

solu:

$$\sin(\theta + i\phi)\sin(\alpha + i\beta) = 1$$

$$\sin(\theta + i\phi) = \frac{1}{\sin(\alpha + i\beta)}$$

$$\sin(\theta + i\phi) = \operatorname{cosec}(\alpha + i\beta) \quad \text{—— (1)}$$

~~Similarly~~ Taking conjugate .

$$\sin(\theta - i\phi) = \operatorname{cosec}(\alpha - i\beta) \quad \text{—— (2)}$$

$$\cos(\theta + i\phi) = \sqrt{1 - \sin^2(\theta + i\phi)}$$

$$= \sqrt{1-\operatorname{cosec}^2(\alpha+i\beta)}$$

$$= \sqrt{1-\cot^2(\alpha+i\beta)}$$

$$= \sqrt{i^2\cot^2(\alpha+i\beta)}$$

$$\cos(\theta+i\phi) = i\cot(\alpha+i\beta) \quad \text{—— (3)}$$

Taking conjugate

$$\cos(\theta-i\phi) = -i\cot(\alpha-i\beta) \quad \text{—— (4)}$$

$$\cos 2i\phi = \cos[(\theta+i\phi)-(\theta-i\phi)]$$

$$= \cos(\theta+i\phi)\cos(\theta-i\phi) + \sin(\theta+i\phi)\sin(\theta-i\phi)$$

$$\cosh 2\phi = i\cot(\alpha+i\beta)(-i\cot(\alpha-i\beta)) + \operatorname{cosec}(\alpha+i\beta)\operatorname{cosec}(\alpha-i\beta)$$

$$= -i^2(\cot(\alpha+i\beta)\cot(\alpha-i\beta)) + \operatorname{cosec}(\alpha+i\beta)\operatorname{cosec}(\alpha-i\beta)$$

$$= \frac{\cos(\alpha+i\beta)}{\sin(\alpha+i\beta)}\frac{\cos(\alpha-i\beta)}{\sin(\alpha-i\beta)} + \frac{1}{\sin(\alpha+i\beta)}\frac{1}{\sin(\alpha-i\beta)}$$

$$= \frac{\cos(\alpha+i\beta)\cos(\alpha-i\beta)+1}{\sin(\alpha+i\beta)\sin(\alpha-i\beta)} \quad \text{—— (5)}$$

Consider

$$\frac{\cosh 2\phi-1}{\cosh 2\phi+1} = \frac{\dfrac{1+\cos(\alpha+i\beta)\cos(\alpha-i\beta)}{\sin(\alpha+i\beta)\sin(\alpha-i\beta)}-1}{\dfrac{1+\cos(\alpha+i\beta)\cos(\alpha-i\beta)}{\sin(\alpha+i\beta)\sin(\alpha-i\beta)}+1}$$

$$\frac{\cancel{2}\sinh^2\phi}{\cancel{2}\cosh^2\phi} = \frac{1+\cos(\alpha+i\beta)\cos(\alpha-i\beta)-\sin(\alpha+i\beta)\sin(\alpha-i\beta)}{1+\cos(\alpha+i\beta)\cos(\alpha-i\beta)+\sin(\alpha+i\beta)\sin(\alpha-i\beta)}$$

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$$= \frac{1+\cos[(\alpha+i\beta)+(\alpha-i\beta)]}{1+\cos[(\alpha+i\beta)-(\alpha-i\beta)]}$$

$$= \frac{1+\cos 2\alpha}{1+\cos 2i\beta}$$

$$\tanh^2\phi = \frac{2\cos^2\alpha}{2\cosh^2\beta}$$

$$\cosh^2\beta \tanh^2\phi = \cos^2\alpha.$$

ii) $\sin(\theta+i\phi)\sin(\alpha+i\beta) = 1$

$$\sin(\alpha+i\beta) = \frac{1}{\sin(\theta+i\phi)}$$

$$\sin(\alpha+i\beta) = \operatorname{cosec}(\theta+i\phi) \quad \text{——} (6)$$

Taking conjugate

$$\sin(\alpha-i\beta) = \operatorname{cosec}(\theta-i\phi) \quad \text{——} (7)$$

$$\cos(\alpha+i\beta) = \sqrt{1-\sin^2(\alpha+i\beta)}$$

$$= \sqrt{1-\operatorname{cosec}^2(\theta+i\phi)}$$

$$= \sqrt{-\cot^2(\theta+i\phi)}$$

$$= \sqrt{i^2\cot^2(\theta+i\phi)}$$

$$\cos(\alpha+i\beta) = i\cot(\theta+i\phi) \quad \text{——} (8)$$

Taking conjugate

$$\cos(\alpha-i\beta) = -i\cot(\theta-i\phi) \quad \text{——} (9)$$

$$\cos 2i\beta = \cos[(\alpha+i\beta)-(\alpha-i\beta)]$$

$$= \cos(\alpha+i\beta)\cos(\alpha-i\beta) + \sin(\alpha+i\beta)\sin(\alpha-i\beta)$$

$$= -i^2\cot(\theta+i\phi)\cot(\theta-i\phi) + \operatorname{cosec}(\theta+i\phi)\operatorname{cosec}(\theta-i\phi)$$

$$= \cot(\theta+i\phi)\cot(\theta-i\phi) + \operatorname{cosec}(\theta+i\phi)\operatorname{cosec}(\theta-i\phi)$$

$$= \frac{\cos(\theta+i\phi)}{\sin(\theta+i\phi)}\frac{\cos(\theta-i\phi)}{\sin(\theta-i\phi)} + \frac{1}{\sin(\theta+i\phi)}\frac{1}{\sin(\theta-i\phi)}$$

$$\cosh 2\beta = \frac{\cos(\theta+i\phi)\cos(\theta-i\phi)+1}{\sin(\theta+i\phi)\sin(\theta-i\phi)} \quad \text{—— } (10)$$

Consider

$$\frac{\cosh 2\beta-1}{\cosh 2\beta+1} = \frac{1+\cos(\theta+i\phi)\cos(\theta-i\phi)-\sin(\theta+i\phi)\sin(\theta-i\phi)}{1+\cos(\theta+i\phi)\cos(\theta-i\phi)+\sin(\theta+i\phi)\sin(\theta-i\phi)}$$

$$\frac{2\sinh^2\beta}{2\cosh^2\beta} = \frac{1+\cos[(\theta+i\phi)+(\theta-i\phi)]}{1+\cos[(\theta+i\phi)-(\theta-i\phi)]}$$

$$\tanh^2\beta = \frac{1+\cos 2\theta}{1+\cos 2i\phi}$$

$$\tanh^2\beta = \frac{1+\cos 2\theta}{1+\cosh 2\phi}$$

$$\tanh^2\beta = \frac{\cancel{2}\cos^2\theta}{\cancel{2}\cosh^2\phi}$$

$$\tanh^2\beta\cosh^2\phi = \cos^2\theta$$

If $x = 2\cos\alpha\cosh\beta$ and $y = 2\sin\alpha\sinh\beta$ show that

i) $\sec(\alpha+i\beta)+\sec(\alpha-i\beta) = \frac{4x}{x^2+y^2}$ ii) $\sec(\alpha+i\beta)-\sec(\alpha-i\beta) = \frac{4iy}{x^2+y^2}$

Solu:

Given: $x = 2\cos\alpha\cosh\beta$

$y = 2\sin\alpha\sinh\beta$.

$$\sec(\alpha+i\beta) = \frac{1}{\cos(\alpha+i\beta)} \times \frac{\cos(\alpha-i\beta)}{\cos(\alpha-i\beta)}$$

$$\sec(\alpha+i\beta) = \frac{\cos(\alpha-i\beta)}{\cos(\alpha+i\beta)\cos(\alpha-i\beta)} \quad \text{—— } (1)$$

Taking conjugate

$$\sec(\alpha-i\beta) = \frac{\cos(\alpha+i\beta)}{\cos(\alpha-i\beta)\cos(\alpha+i\beta)} \quad \text{—— } (2)$$

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① + ②

$$\sec(\alpha+i\beta)+\sec(\alpha-i\beta) = \frac{\cos(\alpha-i\beta)}{\cos(\alpha+i\beta)\cos(\alpha-i\beta)} + \frac{\cos(\alpha+i\beta)}{\cos(\alpha+i\beta)\cos(\alpha-i\beta)}$$

$$= \frac{\cos(\alpha-i\beta)+\cos(\alpha+i\beta)}{\cos(\alpha+i\beta)\cos(\alpha-i\beta)}$$

$$= \frac{2\cos\alpha\cos i\beta}{\frac{1}{2}\left[\cos[(\alpha+i\beta)+(\alpha-i\beta)]+\cos[(\alpha+i\beta)-(\alpha-i\beta)]\right]}$$

$$= \frac{4\cos\alpha\cosh\beta}{\cos 2\alpha+\cos 2i\beta}$$

$$= \frac{4\cos\alpha\cosh\beta}{\cos 2\alpha+\cosh 2\beta}$$

$$= \frac{2x}{\cos 2\alpha+\cosh 2\beta} \quad \text{—— ③}$$

Consider

$$x^2+y^2 = (2\cos\alpha\cosh\beta)^2+(2\sin\alpha\sinh\beta)^2$$

$$= 4\cos^2\alpha\cosh^2\beta + 4\sin^2\alpha\sinh^2\beta$$

$$= \cancel{4}\left(\frac{1+\cos 2\alpha}{\cancel{2}}\right)\left(\frac{1+\cosh 2\beta}{\cancel{2}}\right)+\cancel{4}\left(\frac{1-\cos 2\alpha}{\cancel{2}}\right)\left(\frac{\cosh 2\beta-1}{\cancel{2}}\right)$$

$$= (1+\cos 2\alpha)(1+\cosh 2\beta)+(1-\cos 2\alpha)(\cosh 2\beta-1)$$

$$= \cancel{1}+\cosh 2\beta+\cos 2\alpha+\cancel{\cos 2\alpha\cosh 2\beta}+\cosh 2\beta\cancel{-1}$$
$$-\cancel{\cos 2\alpha\cosh 2\beta}+\cos 2\alpha$$

$$= 2\cos 2\alpha+2\cosh 2\beta$$

$$x^2+y^2 = 2(\cos 2\alpha+\cosh 2\beta)$$

$$\frac{x^2+y^2}{2} = \cos 2\alpha+\cosh 2\beta \quad \text{—— ④}$$

Sub ④ in ③

$$\sec(\alpha+i\beta)+\sec(\alpha-i\beta) = \frac{2x}{\frac{x^2+y^2}{2}}$$

$$= \frac{4x}{x^2+y^2}$$

ii). ① − ②

$$\sec(\alpha+i\beta) - \sec(\alpha-i\beta) = \frac{\cos(\alpha-i\beta)}{\cos(\alpha+i\beta)\cos(\alpha-i\beta)} - \frac{\cos(\alpha+i\beta)}{\cos(\alpha-i\beta)\cos(\alpha+i\beta)}$$

$$= \frac{\cos(\alpha-i\beta) - \cos(\alpha+i\beta)}{\cos(\alpha+i\beta)\cos(\alpha-i\beta)}$$

$$= \frac{\cos\alpha\cos i\beta + \sin\alpha\sin i\beta - (\cos\alpha\cos i\beta - \sin\alpha\sin i\beta)}{\cos(\alpha+i\beta)\cos(\alpha-i\beta)}$$

$$= \frac{2\sin\alpha\sin i\beta}{\cos(\alpha+i\beta)\cos(\alpha-i\beta)}$$

$$= \frac{i2\sin\alpha\sinh\beta}{\frac{1}{2}\left[\cos[(\alpha+i\beta)+(\alpha-i\beta)] + \cos[(\alpha+i\beta)-(\alpha-i\beta)]\right]}$$

$$= \frac{i4\sin\alpha\sinh\beta}{\cos[(\alpha+i\beta)+(\alpha-i\beta)] + \cos[(\alpha+i\beta)-(\alpha-i\beta)]}$$

$$= \frac{i4\sin\alpha\sinh\beta}{\cos 2\alpha + \cos 2i\beta} \quad \text{—— ⑤}$$

Sub ④ in 5

$$\sec(\alpha+i\beta) - \sec(\alpha-i\beta) = \frac{2iy}{\frac{x^2+y^2}{2}}$$

$$= \frac{4iy}{x^2+y^2}$$

If $\tan(\alpha+i\beta)=i$ where $\alpha+\beta$ are real P.T α is indeterminate β is infinite.

Solu:

$$\tan(\alpha+i\beta)=i$$

Taking conjugate

$$\tan(\alpha-i\beta)=-i$$

$$\tan 2\alpha = \tan[(\alpha+i\beta)+(\alpha-i\beta)]$$

$$= \frac{\tan(\alpha+i\beta)+\tan(\alpha-i\beta)}{1-\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$$

$$= \frac{i-i}{1-(i)(-i)}$$

$$= \frac{0}{1-1}$$

$$\tan 2\alpha = 0$$

$\therefore \alpha$ is indeterminate.

$$\tan 2i\beta = \tan[(\alpha+i\beta)-(\alpha-i\beta)]$$

$$= \frac{\tan(\alpha+i\beta)-\tan(\alpha-i\beta)}{1+\tan(\alpha+i\beta)\tan(\alpha-i\beta)}$$

$$= \frac{i+i}{1+(i)(-i)}$$

$$= \frac{2i}{2}$$

$$i\tanh 2\beta = i$$

$$2\beta = \tanh^{-1}(1)$$

$$= \frac{1}{2}\log_e\left(\frac{1+1}{1-1}\right)$$

$$2\beta = \infty$$

$\therefore \beta$ is infinite.

If $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$ S.T i) $\theta = \frac{n\pi}{2} + \frac{\pi}{4}$.

ii) $\phi = \frac{1}{2}\log\left(\tan\frac{\pi}{2} + \frac{\alpha}{2}\right)$.

solu:

Gn: $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$.

T.P $\theta = \frac{n\pi}{2} + \frac{\pi}{4}$

$\phi = \frac{1}{2}\log\left(\tan\frac{\pi}{2} + \frac{\alpha}{2}\right)$.

i) $\tan(\theta+i\phi) = \cos\alpha + i\sin\alpha$ — (1)

Taking Conjugate

$\tan(\theta-i\phi) = \cos\alpha - i\sin\alpha$ — (2)

$$\tan 2\theta = \tan\left[(\theta+i\phi)+(\theta-i\phi)\right]$$

$$= \frac{\tan(\theta+i\phi)+\tan(\theta-i\phi)}{1-\tan(\theta+i\phi)\tan(\theta-i\phi)}$$

$$= \frac{\cos\alpha + \cancel{i\sin\alpha} + \cos\alpha - \cancel{i\sin\alpha}}{1-(\cos\alpha+i\sin\alpha)(\cos\alpha-i\sin\alpha)}$$

$\cos^2\alpha - i\cos\alpha\sin\alpha + i\sin\alpha\cos\alpha - i^2$

$$= \frac{2\cos\alpha}{1-(\cos^2\alpha - i^2\sin^2\alpha)} = \frac{2\cos\alpha}{1-(\cos^2\alpha+\sin^2\alpha)}$$

$$\tan 2\theta = \frac{2\cos\alpha}{1-1} = \infty$$

$$2\theta = \tan^{-1}(\infty)$$

$$2\theta = n\pi + \frac{\pi}{2}$$

$$\theta = \frac{1}{2}\left(n\pi + \frac{\pi}{2}\right)$$

$$\theta = \frac{n\pi}{2} + \frac{\pi}{4}$$

ii) $\tan 2i\phi = \tan[(\theta + i\phi) - (\theta - i\phi)]$

$$= \frac{\tan(\theta + i\phi) - \tan(\theta - i\phi)}{1 + \tan(\theta + i\phi)\tan(\theta - i\phi)}$$

$$\tan 2i\phi = \frac{\cos\alpha + i\sin\alpha - \cos\alpha + i\sin\alpha}{1 + (\cos\alpha + i\sin\alpha)(\cos\alpha + i\sin\alpha)}$$

$$= \frac{2i\sin\alpha}{1 + (\cos^2\alpha + \sin^2\alpha)}$$

$$\tan 2i\phi = \frac{\cancel{2}i\sin\alpha}{\cancel{2}}$$

$$i\tanh 2\phi = \cancel{i}\sin\alpha$$

$$2\phi = \tanh^{-1}(\sin\alpha)$$

$$2\phi = \frac{1}{2}\log_e\left(\frac{1+\sin\alpha}{1-\sin\alpha}\right)$$

$$= \frac{1}{2}\log\left[\frac{1 + \frac{2\tan\alpha/2}{1+\tan^2\alpha/2}}{1 - \frac{2\tan\alpha/2}{1+\tan^2\alpha/2}}\right]$$

$$= \frac{1}{2}\log\left[\frac{1 + \tan^2\alpha/2 + 2\tan\alpha/2}{1 + \tan^2\alpha/2 - 2\tan\alpha/2}\right]$$

$$= \frac{1}{2}\log\left[\frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}}\right]^2$$

$$2\phi = \frac{2}{2}\log\left[\frac{1 + \tan\alpha/2}{1 - \tan\alpha/2}\right] = \log\left[\frac{\tan\pi/4 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{\alpha}{2}}\right]$$

$$\phi = \frac{1}{2}\log\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right).$$

If ϕ & ψ are define the relation by $\tan(x+iy) = \phi + i\psi$. P.T $\phi^2+\psi^2 = \dfrac{\cosh^2 y - \cos^2 x}{\cosh^2 y - \sin^2 x}$.

Solu:

$$\phi + i\psi = \tan(x+iy)$$

Taking conjugate

$$\phi - i\psi = \tan(x-iy)$$

$$(\phi+i\psi)(\phi-i\psi) = \tan(x+iy)\tan(x-iy)$$

$$\phi^2 - i^2\psi^2 = \frac{\tan x + \tan iy}{1-\tan x \tan iy} \cdot \frac{\tan x - \tan iy}{1+\tan x \tan iy}$$

$$\phi^2+\psi^2 = \frac{\tan x + i\tanh y}{1-i\tan x \tanh y} \cdot \frac{\tan x - i\tanh y}{1+i\tan x \tanh y}$$

$$= \frac{\tan^2 x - i^2\tanh^2 y}{1^2 - i^2\tan^2 x \tanh^2 y}$$

$$= \frac{\dfrac{\sin^2 x}{\cos^2 x} + \dfrac{\sinh^2 y}{\cosh^2 y}}{1 + \dfrac{\sin^2 x}{\cos^2 x}\cdot\dfrac{\sinh^2 y}{\cosh^2 y}}$$

$$= \frac{\sin^2 x\cosh^2 y + \sinh^2 y\cos^2 x}{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y}$$

$$= \frac{(1-\cos^2 x)\cosh^2 y + (\cosh^2 y - 1)\cos^2 x}{\cos^2 x\cosh^2 y + \sin^2 x(\cosh^2 y - 1)}$$

$$= \frac{\cosh^2 y - \cancel{\cos^2 x\cosh^2 y} + \cancel{\cos^2 x\cosh^2 y} - \cos^2 x}{\cos^2 x\cosh^2 y + \sin^2 x\cosh^2 y - \sin^2 x}$$

$$= \frac{\cosh^2 y - \cos^2 x}{\cosh^2 y(\cos^2 x + \sin^2 x) - \sin^2 x}$$

Scanned with CamScanner

$$\phi^2+\varphi^2 = \frac{\cosh^2 y - \cos^2 x}{\cosh^2 y - \sin^2 x}$$

Hence proved.

Q. If $u = \log\tan\left[\frac{\pi}{4}+\frac{\theta}{2}\right]$. Show that i) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$

ii) $\theta = -i\log\tan\left(\frac{\pi}{4}+\frac{iu}{2}\right)$.

solu:

$$\tanh\frac{u}{2} = \tan\frac{\theta}{2}$$

$$u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

Taking e on b.s

$$e^u = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$= \frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}$$

$$= \frac{1+\tan\theta/2}{1-\tan\theta/2}$$

i) Consider

$$\frac{e^u-1}{e^u+1} = \frac{\frac{1+\tan\theta/2}{1-\tan\theta/2}-1}{\frac{1+\tan\theta/2}{1-\tan\theta/2}+1}$$

$$= \frac{\cancel{1}+\tan\theta/2-\cancel{1}+\tan\theta/2}{1+\cancel{\tan\theta/2}+1-\cancel{\tan\theta/2}}$$

$$= \frac{\cancel{2}\tan\theta/2}{\cancel{2}}$$

$$= \tan\theta/2$$

$$\frac{e^{u/2}(e^{u/2}-e^{-u/2})}{e^{u/2}(e^{u/2}+e^{-u/2})} = \tan\theta/2$$

$$\tan h\frac{u}{2} = \tan\frac{\theta}{2}$$ $\because \tan x = \dfrac{e^x(e^x - e^{-x})}{e^x(e^x + e^{-x})}$

iii) $\theta = -i \log \tan\left(\frac{\pi}{4} + \frac{iu}{2}\right)$

Consider

$$\tan h\frac{u}{2} = \tan\frac{\theta}{2}$$

x' ly $\div$ by i in L.H.S

$$\frac{i\tan hu/_2}{i} = \tan\frac{\theta}{2}$$

$$\tan\frac{iu}{2} = i\tan\frac{\theta}{2}$$

Consider

$$\frac{1+\tan iu/_2}{1-\tan iu/_2} = \frac{1+i\tan\theta/_2}{1-i\tan\theta/_2}$$

$$= \frac{1+i\left(\frac{\sin\theta/_2}{\cos\theta/_2}\right)}{1-i\left(\frac{\sin\theta/_2}{\cos\theta/_2}\right)}$$

$$= \frac{\cos\theta/_2 + i\sin\theta/_2}{\cos\theta/_2 - i\sin\theta/_2}$$

$$= \frac{\cos\theta/_2 + i\sin\theta/_2}{\cos\theta/_2 - i\sin\theta/_2} \times \frac{\cos\theta/_2 + i\sin\theta/_2}{\cos\theta/_2 + i\sin\theta/_2}$$

$$= \frac{(\cos\theta/_2 + i\sin\theta/_2)^2}{\cos^2\theta/_2 + \sin^2\theta/_2}$$

$$= \frac{(\cos\theta/_2 + i\sin\theta/_2)^2}{1}$$

By Demoivres theorem

$$= \cos^2\theta/_2 + i\sin^2\theta/_2$$

$$\frac{1+\tan iu/2}{1-\tan iu/2} = \cos\theta + i\sin\theta = e^{i\theta}$$

$$\frac{\tan\frac{\pi}{4} + \tan\frac{iu}{2}}{1-\tan\frac{\pi}{4}\tan\frac{iu}{2}} = e^{i\theta}$$

$$\tan\left(\frac{\pi}{4} + \frac{iu}{2}\right) = e^{i\theta}$$

Apply log on b·s

$$\log\tan\left(\frac{\pi}{4} + \frac{iu}{2}\right) = i\theta$$

Apply i on b·s

$$i\log\tan\left(\frac{\pi}{4} + \frac{iu}{2}\right) = i^2\theta$$

$$-i\log\tan\left(\frac{\pi}{4} + \frac{iu}{2}\right) = \theta.$$

INVERSE HYPERBOLIC FUNCTION:

1. P.T $\sinh^{-1}x = \log_e\left(x \pm \sqrt{x^2+1}\right)$

Solu:

Let $y = \sinh^{-1}x$.

$$\sinh y = x$$

$$\frac{e^y - e^{-y}}{2} = x.$$

$$e^y - e^{-y} = 2x$$

$$e^y - \frac{1}{e^y} = 2x$$

$$\frac{e^{2y}-1}{e^y} = 2x.$$

$$e^{2y} - 1 = e^y 2x$$

$$e^{2y} - 2xe^y - 1 = 0.$$

This is 2^{nd} degree eqn in terms of e^y.

$a = 1,\ b = -2x,\ c = -1.$

$$e^y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2x \pm \sqrt{4x^2+4}}{2}$$

$$= \frac{2x \pm \sqrt{4(x^2+1)}}{2}$$

$$= \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

$$e^y = x \pm \sqrt{x^2+1}$$

Taking log on b.s

$$\log e^y = \log_e \left(x \pm \sqrt{x^2+1}\right)$$

$$y = \log_e \left(x \pm \sqrt{x^2+1}\right)$$

$$\sinh^{-1} x = \log_e \left(x \pm \sqrt{x^2+1}\right).$$

2. P.T $\cosh^{-1} x = \log_e \left(x \pm \sqrt{x^2-1}\right).$

Solu:

Let $y = \cosh^{-1} x.$

$$x = \cosh y.$$

$$\frac{e^y + e^{-y}}{2} = x.$$

$$e^y + e^{-y} = 2x.$$

$$e^y + \frac{1}{e^y} = 2x.$$

$$e^{2y} + 1 = e^y 2x.$$

$$e^{2y} - 2xe^{y} + 1 = 0.$$

$$e^{y} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2x \pm \sqrt{4x^2 - 4(1)(1)}}{2}$$

$$= \frac{2x \pm \sqrt{4x^2 - 4}}{2}$$

$$e^{y} = x \pm \sqrt{x^2 - 1}$$

Taking log on b.s

$$\log e^{y} = \log\left(x \pm \sqrt{x^2 - 1}\right)$$

$$y = \log\left(x \pm \sqrt{x^2 - 1}\right)$$

$$\cosh^{-1} x = \log\left(x \pm \sqrt{x^2 - 1}\right).$$

Hence proved.

(x) If $\cos^{-1}(u+iv) = \alpha + i\beta$ Show that $\cos^2\alpha$ & $\cosh^2\beta$ are roots of equation $x^2 - x(1+u^2+v^2)+u^2 = 0$.

Solu:

Given: $\cos^{-1}(u+iv) = \alpha+i\beta$.

$x^2 - x(1+u^2+v^2)+u^2=0.$

$x^2 - x$ (sum of roots) + product of roots $= 0$

$$\cos^{-1}(u+iv) = \alpha+i\beta$$

$$u+iv = \cos(\alpha+i\beta)$$

$$= \cos\alpha \cos i\beta - \sin\alpha \sin i\beta$$

$$u+iv = \cos\alpha \cosh\beta - i\sin\alpha \sinh\beta.$$

Equating real & imaginary

$$u = \cos\alpha \cosh\beta$$

$$v = -\sin\alpha \sinh\beta$$

Sum of roots $= 1+u^2+v^2$

$$= 1+(\cos^2\alpha \cosh^2\beta) + \sin^2\alpha \sinh^2\beta$$

$$= 1+\cos^2\alpha\cosh^2\beta + (1-\cos^2\alpha)(\cosh^2\beta - 1)$$

$$= 1+\cos^2\alpha\cosh^2\beta + \cosh^2\beta - 1 - \cos^2\alpha\cosh^2\beta + \cos^2\alpha$$

$$= \cos^2\alpha + \cosh^2\beta$$

Product of roots $= u^2$

$$= \cos^2\alpha \cosh^2\beta.$$

3. P.T $\tanh^{-1} x = \frac{1}{2} \log_e \left(\frac{1+x}{1-x} \right)$

Sol :

Let $\tanh^{-1} x = y$

$$\tanh y = x$$

$$\frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} = x$$

$$\frac{e^{y} - \frac{1}{e^{y}}}{e^{y} + \frac{1}{e^{y}}} = x$$

$$\frac{\frac{e^{2y} - 1}{\cancel{e^{y}}}}{\frac{e^{2y} + 1}{\cancel{e^{y}}}} = x$$

$$\frac{e^{2y} - 1}{e^{2y} + 1} = x$$

$$e^{2y} - 1 = x(e^{2y} + 1)$$

$$e^{2y} - 1 = xe^{2y} + x$$

$$-xe^{2y} + e^{2y} = +1 + x$$

$$e^{2y}(1 - x) = 1 + x$$

$$e^{2y} = \frac{1+x}{1-x}$$

Taking log on b.s

$$\log e^{2y} = \log \left(\frac{1+x}{1-x} \right)$$

$$2y = \log \left(\frac{1+x}{1-x} \right)$$

$$y = \frac{1}{2} \log \left(\frac{1+x}{1-x} \right)$$

$$\tanh^{-1} x = \frac{1}{2} \log_e \left(\frac{1+x}{1-x} \right)$$

Hence proved.