

By 11<sup>th</sup> axis them,

$$\begin{aligned}
 \text{abt a tangent line} &= M \cdot \frac{1}{3} \text{abt } O\alpha + Ma^2 \\
 &= \frac{2Ma^2}{3} + Ma^2 \\
 &= \frac{5Ma^2}{3}
 \end{aligned}$$

UNIT - IV (1)

CENTRAL FORCE & CENTRAL ORBIT

Central force: - If a particle is subject to the action of a force which is always either towards (or) away from a fixed pt, the particle is said to be under the action of central force.

i.e) A central force is a force whose line of action always passes through a fixed pt.

Centre of force: - A central force is a whose line of action always passes through a fixed pt. The fixed pt is called the centre of force.

Notation: - Central force per unit mass by  $\phi(r)\hat{r}$ , where  $\hat{r}$  is the unit vector in the radial direction.

Attractive central force =  $\phi(r) = -F = -\frac{F}{r^2}$   
 $\phi(r) \cdot \hat{r} = -F$

Central Orbit:- The path described by a particle under a central force is called a central orbit.

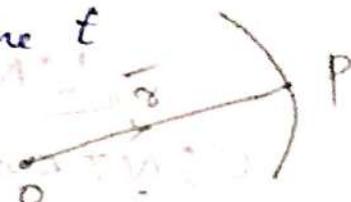
BOOKWORK:- Show that a central orbit is a plane curve. (2)

Soln:- Let us make the following assumption

O: Centre of force.

P: position of the particle at time t

$\vec{r} = \vec{OP}$  ( $r = OP$ )  
 $\hat{r} =$  unit vector along OP  
 $m =$  Mass of the particle.



$\phi(r) \cdot \hat{r}$ : central force per unit mass

Then, the eqn of motion of the particle is

$$m \ddot{\vec{r}} = m \phi(r) \hat{r} \Rightarrow \boxed{\ddot{\vec{r}} - \phi(r) \hat{r}} \rightarrow (1)$$

Let us consider,

Eqn of Orbit

$$\frac{d}{dt} (\vec{r} \times \dot{\vec{r}}) = \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}}$$

variation in increment on

$$= 0 + \vec{r} \times \phi(r) \hat{r} \text{ (by (1))}$$

full derivative in (2m)

$\Rightarrow \vec{r} \times \dot{\vec{r}}$  is a constant vector (say  $\vec{c}$ )

Then  $\vec{r}$  (or)  $\vec{OP}$  is always  $\perp$  to  $\vec{c}$ .

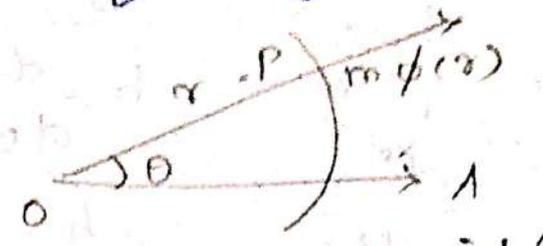
$\therefore \vec{P}$  is always in the plane through O &  $\perp$  to  $\vec{c}$ . Hence the motion of P is coplanar, and the orbit is a plane curve.

Differential eqn of a central orbit:-

BOOKWORK:- Find the diff. eqn of a central orbit in polar co-ordinates.

Proof: Let us make the following assumption

- $O$ : centre of force
- $P$ : pole
- $\phi$ : Initial line



$(r, \theta)$ : position of the particle at time  $t$ .

$\phi(r) = -F$ : Central force per unit mass in the direction of  $r$ .

$m$ : mass of the particle.

The motion is coplanar motion

$\therefore$  the eqn of motion corresponding to the radial & transverse directions.

w.k.t: Radial acceleration components  $= \ddot{r} - r\dot{\theta}^2$

Transverse acceleration components  $= \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta})$

The force components in these directions are

$(m\phi(r), 0)$   
 $\therefore$  the eqn of motion are,

$$m(\ddot{r} - r\dot{\theta}^2) = m\phi(r)$$

$$\ddot{r} - r\dot{\theta}^2 = \phi(r) \rightarrow \textcircled{1}$$

and  $m \left( \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) \right) = 0$

$$\frac{1}{r} \left( \frac{d}{dt} (r^2 \dot{\theta}) \right) = 0$$

$$\frac{d}{dt} (r^2 \dot{\theta}) = 0$$

Now integrating,  $\boxed{r^2 \dot{\theta} = h}$  (Constant)

$$\dot{\theta} = \frac{h}{r^2}$$

Angular velocity is constant in central orbit

$$\dot{\theta} = hu' \rightarrow \textcircled{2} \text{ (take } u = 1/r)$$

Diff:  $r = 1/u$  w.r.t.  $t$

$$\frac{dr}{dt} = -1/u^2 \frac{du}{dt} \Rightarrow -1/u^2 \frac{du}{d\theta} \frac{d\theta}{dt}$$

$$= -\frac{1}{u} \frac{du}{dt} (hu^2) \quad \left[ \because \dot{\theta} = \frac{d\theta}{dt} = hu^2 \right]$$

$$\frac{\ddot{r}}{r} = -h \frac{du}{dt}$$

$$\begin{aligned} \frac{d^2 r}{dt^2} &= -h \frac{d}{dt} \left( \frac{du}{dt} \right) = -h \frac{d}{do} \left( \frac{du}{do} \right) \frac{do}{dt} \\ &= -h \frac{d^2 u}{do^2} \frac{do}{dt} = -h \frac{d^2 u}{do^2} \frac{do}{dt} \end{aligned}$$

$$= -h \frac{d^2 u}{do^2} (hu^2) \quad \left[ \because \text{by } \textcircled{2} \right]$$

$$\frac{\ddot{r}}{r} = -h^2 u^3 \frac{d^2 u}{do^2}$$

Subs  $\textcircled{2}$  &  $\textcircled{3}$  in  $\textcircled{1}$ ,

$$\textcircled{1} = -h^2 u^3 \frac{d^2 u}{do^2} - r(hu^2)^2 = \phi(r) = -F$$

$$= -h^2 u^3 \frac{d^2 u}{do^2} - ru^2 = \frac{-F}{h^2 u^2}$$

$$\Rightarrow \frac{d^2 u}{do^2} + ru^2 = \frac{F}{h^2 u^2}$$

$$\Rightarrow \frac{d^2 u}{do^2} + \frac{1}{u} \cdot u^2 = \frac{F}{h^2 u^2} \quad \left[ \because u = \frac{1}{r} \right]$$

$\textcircled{1}$  diff. eqn attractive force

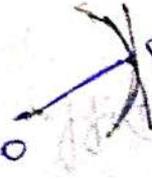
$$\textcircled{2m} \quad \frac{d^2 u}{do^2} + u = \frac{F}{h^2 u^2}$$

$$\Rightarrow \frac{d^2 u}{do^2} + u = \frac{-\phi(r)}{h^2 u^2}$$

This is the eqn of the central orbit in polar form

APSE:-

By  $O$  is the pole &  $P$  is a point on a curve such that  $OP$  is  $\perp$  to the tangent at  $P$  then  $P$  is an apse. By  $P$  is an apse  $OP$  is the maximum (or) minimum of  $r$ . For eg: in an ellipse if  $S$  is the pole then the ends of the major axis are apses.



Since  $r \dot{\theta}$  is a constant in a central orbit the angular velocity of the particle about  $O$  is a maximum (or) minimum. According to the radius vector  $r$  is a minimum (or) maximum. At an apse, the angular velocity of the particle is either a maximum (or) minimum. The apses are given by  $\frac{du}{d\theta} = 0$ . (5)  $u = \frac{1}{r}$

Force per unit mass :- The force per unit mass is  $\phi(r) \hat{r} = -h^2 u^2 \left( \frac{d^2 u}{d\theta^2} + u \right) \hat{r}$ . (from the diff. eqn of central orbit)

Velocity  $v$  at a point  $P$  in a central orbit :- In a coplanar motion the velocity component along the radial & transverse directions are  $\dot{r}$ ,  $r\dot{\theta}$ .

$$\begin{aligned}
 \text{Velocity } v^2 &= (\dot{r})^2 + (r\dot{\theta})^2 \\
 &= h^2 \left( \frac{du}{d\theta} \right)^2 + \frac{1}{2} u (hu^2)^2 \\
 &= h^2 \left( \frac{du}{d\theta} \right)^2 + h^2 u^2 \\
 v^2 &= h^2 \left[ \left( \frac{du}{d\theta} \right)^2 + u^2 \right] \\
 v &= h \sqrt{\left( \frac{du}{d\theta} \right)^2 + u^2}
 \end{aligned}$$

Areal velocity :-

Derive a formula for areal velocity of a particle exhibiting central orbit.

Soln:- Let the initial position of the particle be  $Q$  the position at time  $t$  be  $P(r, \theta)$ . Let the area  $OQP$  swept by the radius vector moving from  $OQ$  to  $OP$  be  $A$ .  $v = \frac{dA}{dt}$

Then  $\frac{dA}{dt}$  is called the areal velocity of the particle.

Let  $P$  be the position of the particle at time  $t + \Delta t$

Area  $\Delta A = \frac{1}{2} r^2 \dot{\theta} \Delta t$  (area of sector)  $\approx \frac{1}{2} r^2 \dot{\theta} \Delta t$  (since  $\Delta \theta$  is small)

$$\frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \dot{\theta}$$

$\therefore$  The areal velocity of P is  $\frac{1}{2} r^2 \dot{\theta}$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

In a central orbit  $r^2 \dot{\theta}$  is a constant which we have denoted by  $h$ .

$\therefore$  In a central orbit the areal velocity  $\frac{1}{2} r^2 \dot{\theta}$  is a constant.

(10) areal velocity =  $\frac{1}{2} r v$   $v$  is the velocity of the

particle at P.  $2 \times \text{areal velocity} = r v$

BOOKWORK - Find the diff. eqn of a central orbit in

co-ordinates (10) P.T.  $h^2/p^3 \frac{dp}{ds} = F$

Proof - For any P on the orbit, the radius vector  $r = OP$  & the  $\perp$  distance  $p$  of O from the tangent at P are the p & r co-ordinates of P.

From pedal eqn,

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

Diff. w.r.t.  $\theta$ ,

$$-2/p^3 \frac{dp}{d\theta} = 2u \frac{du}{d\theta} + 2 \frac{du}{d\theta} \cdot \frac{d^2u}{d\theta^2}$$

$$= 2 \frac{du}{d\theta} \left[ u + \frac{d^2u}{d\theta^2} \right] \rightarrow (1)$$

If the central force (per unit mass) in the radial direction is  $\phi(r)$

then, the diff. eqn of a central orbit.

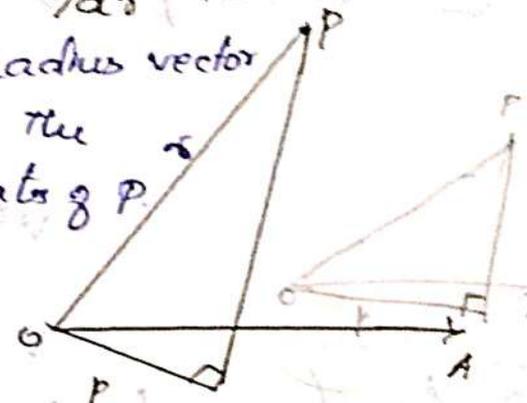
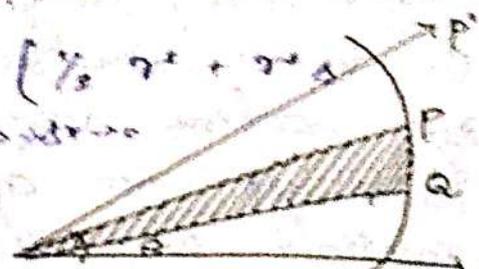
$$\frac{d^2u}{d\theta^2} + u = \frac{-\phi(r)}{h^2 u^2} \rightarrow (2)$$

Subs (2) in (1),

$$-2/p^3 \frac{dp}{d\theta} = 2 \frac{du}{d\theta} \left[ \frac{-\phi(r)}{h^2 u^2} \right]$$

$$\Rightarrow \frac{1}{p^3} \frac{dp}{d\theta} = \frac{\phi(r)}{h^2 u^2} \frac{d(1/r)}{d\theta}$$

$$\frac{1}{p^3} \frac{dp}{dr} \frac{dr}{d\theta} = \frac{\phi(r)}{h^2 r^2} \left( -\frac{1}{r^2} \frac{dr}{d\theta} \right)$$



$$= \frac{-\phi(r)}{h^2} \cdot \frac{dr}{d\theta} \quad [\because u = 1/r, u' = -1/r^2]$$

$$\Rightarrow \frac{h^2}{p^3} \frac{dp}{dr} = -\phi(r) \Rightarrow h^2/p^3 \frac{dp}{d\theta} = F$$

This eqn is in terms of  $p$  &  $\theta$ .  
 This eqn is the  $p$ - $\theta$  eqn of the orbit. This is also called pedal eqn of the orbit.  
 For an attractive central force  $\phi(r) = -F$ .  
 $\therefore$  The  $p$ - $\theta$  eqn of the orbit is  $h^2/p^3 \frac{dp}{d\theta} = F$ .

Law of a Central force:

Book work: Inverse square law  
 Find the orbit of a particle moving under an attractive central force varying inversely as the square of the distance. (or) A particle moves under a force of attraction inversely proportional to the square of its distance from a fixed pt. Show the path of the orbit is conic section having the fixed pt as focus.

Soln: Let the pole be the centre of force. Since force  $\propto 1/r^2$   
 hypothesis, let the force per unit mass be  $\mu/r^2$ .  
 Then from the diff. eqn of orbit in  $p$ - $\theta$  co-ordinates,

a)  $h^2/p^3 \frac{dp}{d\theta} = F$

$\Rightarrow h^2/p^3 \frac{dp}{d\theta} = \mu/r^2$

$\Rightarrow h^2 \frac{dp}{p^3} = \mu \cdot dr/r^2$

$\Rightarrow h^2 \left[ \frac{p^{-3+1}}{-3+1} \right] = \mu \left[ \frac{r^{-2+1}}{-2+1} \right]$

$\Rightarrow h^2 \left[ \frac{p^{-2}}{-2} \right] = \mu \left[ \frac{r^{-1}}{-1} \right]$

$h^2 \left[ -1/2 p^2 \right] = \mu (-1/r) + A$

$\Rightarrow h^2/p^2 = 2\mu/r + A$  ①

w.k.t the p-r eqn of an ellipse a parabola & a hyperbola are,

$$\left. \begin{aligned} b^2/p^2 &= 2a/r_0 - 1 \\ p^2 &= ar \\ b^2/p^2 &= 2a/r_0 + 1 \end{aligned} \right\} \text{general derived eqn} \rightarrow \textcircled{2} \quad h^2/p^2 = \frac{2\mu}{r} + \textcircled{1}$$

from  $\textcircled{1}$  &  $\textcircled{2}$ ,  
The orbit is an ellipse if  $\frac{h^2}{p^2} < 0$   
parabola if  $\frac{h^2}{p^2} = 0$   
hyperbola if  $\frac{h^2}{p^2} > 0$  }  $\rightarrow \textcircled{3}$

Also w.k.t  $h = p v$   
Then eqn  $\textcircled{1}$  becomes,

$$2 \times A \cdot v = p v \text{ Constant } \textcircled{4}$$

$$\frac{p^2 v^2}{p^2} = \frac{2\mu}{r} + \textcircled{1}$$

$$v^2 = \frac{2\mu}{r} + \textcircled{1} = v^2 - \frac{2\mu}{r}$$

Thus the condition  $\textcircled{3}$  becomes that the orbit is an ellipse if  $v < \sqrt{\frac{2\mu}{r}}$

Parabola if  $v = \sqrt{\frac{2\mu}{r}}$ ; hyperbola if  $v > \sqrt{\frac{2\mu}{r}}$ .

BOOKWORK:- find the orbit of a particle moving under an attractive force varying as the distance.

Proof:- Let the position vector of the particle of mass  $m$  at time  $t$  be  $\vec{r}$ .

Given, force  $\propto r$

Let  $n^2 r$  is the force per unit mass  $= \gamma F = n^2 r$

Then the eqn of motion is  $m \ddot{\vec{r}} = m \gamma F = m n^2 r \hat{r} = m \ddot{\vec{r}}$

$$\ddot{\vec{r}} = -n^2 \vec{r} \rightarrow \textcircled{1}$$

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$\dot{\vec{r}} = \dot{x}\vec{i} + \dot{y}\vec{j}$$

$$\ddot{\vec{r}} = \ddot{x}\vec{i} + \ddot{y}\vec{j} \rightarrow \textcircled{2}$$

From  $\textcircled{1}$  &  $\textcircled{2}$ ,

$$\ddot{x}\vec{i} + \ddot{y}\vec{j} = -n^2 (x\vec{i} + y\vec{j}) \Rightarrow \ddot{x} + n^2 x = 0 \quad ; \quad \ddot{y} + n^2 y = 0$$

solns of these diff. eqn are  $m^2 + n^2 = 0$   
 $m = -n$   
 $m = ni$   
 $x = A \cos nt + B \sin nt$   
 $y = C \cos nt + D \sin nt$

where the constants A, B, C, D depends upon the initial conditions.

Solving these two eqns for  $\cos nt, \sin nt$

$$\begin{vmatrix} B & -x \\ D & -y \end{vmatrix} = \frac{-\sin nt}{\begin{vmatrix} A & x \\ C & y \end{vmatrix}} = \frac{1}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}}$$

$$\text{ii) } \frac{\cos nt}{-By + Dx} = \frac{-\sin nt}{Ay - Cx} = \frac{1}{AD - BC}$$

$\therefore \cos nt = \frac{Dx - By}{AD - BC}$  ;  $\sin nt = \frac{Cx - Ay}{AD - BC}$

Squaring & Adding the above eqn.

$$1 = \frac{(Dx - By)^2}{(AD - BC)^2} + \frac{(Cx - Ay)^2}{(AD - BC)^2}$$

$$(Cx - Ay)^2 + (Dx - By)^2 = (AD - BC)^2$$

which being a 2nd degree eqn represents a conic & it satisfies  $b^2 - 4ac < 0$ .

Hence it represents an ellipse since  $(-x, -y)$  satisfies the eqn of the ellipse is symmetrical about the origin. So, the centre of force is at the centre of ellipse.

Example:-

S.T the force towards the pole under which a particle describes the curve  $r^n = a^n \cos n\theta$  varies inversely as the  $(2n+3)^{\text{th}}$  power of the distance of the particle from the pole.

Soln: w.k.t.  $r = \gamma u \Rightarrow r^n = \gamma^n u^n$

$\Rightarrow r^n \cos n\theta = 1$

$\because r^n = a^n \cos n\theta$

To order to reduce power  
Converting into the power n

$\Rightarrow \log(a^n u^n \cos n\theta) = \log 1$

$\Rightarrow \log a^n + \log u^n + \log \cos n\theta = 0$

$\Rightarrow n \log a + n \log u + \log \cos n\theta = 0$

Diff. w.r.t.  $\theta$

To derive  $du/d\theta$

$0 + n \cdot \gamma u \frac{du}{d\theta} + \frac{1}{\cos n\theta} (-\sin n\theta) \cdot n = 0$

$\Rightarrow n \left( \gamma u \frac{du}{d\theta} - \frac{\sin n\theta}{\cos n\theta} \right) = 0$

$\Rightarrow \gamma u \frac{du}{d\theta} = \tan n\theta$

(10)

Again, diff. w.r.t.  $\theta$ ,  $\Rightarrow \frac{d^2u}{d\theta^2} = u \tan n\theta$

$\Rightarrow \frac{d^2u}{d\theta^2} = u \tan^2 n\theta + u n \sec^2 n\theta \rightarrow (1)$

w.k.t. the diff. eqn of the central orbit in polar co-ordinates is,

$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2 u^2} \rightarrow (2)$

Subs (1) in (2),

$u \tan^2 n\theta + u n \sec^2 n\theta + u = \frac{F}{h^2 u^2} \rightarrow (3)$

$u (n \sec^2 n\theta + \sec^2 n\theta) = \frac{F}{h^2 u^2}$   
 $u \sec^2 n\theta (n+1) = \frac{F}{h^2 u^2}$

$r^n = a^n \cos n\theta$

$\frac{r^n}{a^n} = \cos n\theta$

$\sec n\theta = \frac{a^n}{r^n}$

$F = h^2 u^3 \sec^2 n\theta (n+1)$

$F = h^2 \gamma^3 \left( \frac{a^{2n}}{r^{2n}} \right) (n+1)$

$F = \frac{h^2 a^{2n} (n+1)}{r^{2n+3}}$

$F = h^2 \left( \frac{a^{2n}}{r^{2n+3}} \right) (n+1)$

$F \propto \frac{1}{r^{2n+3}}$

This is the central attractive force which varies inversely as  $r^{2n+3}$ .

the position vector of a particle at time  $t$  is  $\vec{a} \cos nt + \vec{b} \sin nt$ , where  $\vec{a}$  &  $\vec{b}$  are constant vectors constant. S.T. the particle is moving under a attractive force varying as the distance.

Qn.  $\vec{r} = \vec{a} \cos nt + \vec{b} \sin nt$

$\dot{\vec{r}} = -n\vec{a} \sin nt + n\vec{b} \cos nt$

$\ddot{\vec{r}} = -n^2\vec{a} \cos nt - n^2\vec{b} \sin nt$

$\ddot{\vec{r}} = -n^2(\vec{a} \cos nt + \vec{b} \sin nt)$

$F = m \ddot{\vec{r}} = -m n^2 \vec{r}$

$F = m \ddot{\vec{r}} = m \phi(r) \hat{r}$   
 $\hat{r} = \frac{\vec{r}}{r}$   
 $\phi(r) = -n^2 r$   
 $F = -m n^2 \vec{r}$

$\therefore$  The particle is moving under a central force varying as the distance.

particle moves along the path  $r = e^{-\theta}$  under a force. S.T. the force is  $\frac{2mb^2}{r^3}$  & speed of the particle is  $\frac{h}{r} \sqrt{2}$ .

Soln: Qn.  $r = e^{-\theta} \Rightarrow \frac{1}{u} = e^{-\theta} \Rightarrow u = e^{-\theta}$

Diff. w.r.t.  $\theta$ ,

$\frac{du}{d\theta} = -e^{-\theta} = -u$

$\Rightarrow \frac{d^2u}{d\theta^2} = -e^{-\theta}(-1) = e^{-\theta}$

$\Rightarrow \frac{d^2u}{d\theta^2} = u \rightarrow \textcircled{1}$

with the diff. eqn for an attractive central force towards zero is,

$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2 u^2}$

$\Rightarrow F = h^2 u^3 \left( \frac{d^2u}{d\theta^2} + u \right) \Rightarrow \text{away from } 0$

$F$  is the central acceleration.

$\therefore$  force towards 0 =  $M \times \text{acc.}$   
 $= m b^2 u^3 \left( \frac{d^2u}{d\theta^2} + u \right)$

$$= mb^2 u^2 (u + u)$$

$$= 2mb^2 u^3$$

$$= 2mb^2 / r^3$$

$$V = h \sqrt{(du/d\theta)^2 + u^2} = h \sqrt{(-u)^2 + u^2} = h \sqrt{2u^2}$$

$$= hu\sqrt{2}$$

$$v = h/r \sqrt{2}$$

(12)



In an orbit described under a force to a centre, the velocity at any point to it is inversely proportional to the distance of the point from the centre of force. If the path is an equiangular spiral.

Soln:- Given velocity  $\propto$  distance

$$\Rightarrow v = k/r = ku$$

$$\Rightarrow v^2 = k^2 u^2 \rightarrow \text{--- (1)}$$

w.k.t,

$$v^2 = h^2 \left[ \left( \frac{du}{d\theta} \right)^2 + u^2 \right]$$

$$\Rightarrow h^2 \left[ \left( \frac{du}{d\theta} \right)^2 + u^2 \right] = k^2 u^2$$

$$\Rightarrow h^2 \left[ \frac{du}{d\theta} \right]^2 = k^2 u^2 - h^2 u^2 \quad \text{Constant}$$

$$\Rightarrow \left( \frac{du}{d\theta} \right)^2 = (k^2 - h^2) \cdot \frac{u^2}{h^2}$$

$$= \lambda^2 u^2 \quad \left[ \because \lambda = \frac{k^2 - h^2}{h^2} \right]$$

$$\Rightarrow \frac{du}{d\theta} = \lambda u$$

$$\Rightarrow \frac{du}{u} = \lambda d\theta$$

Intg.  $\log u = \lambda \theta + \log A$

$$\frac{u}{A} = e^{\lambda \theta} \Rightarrow u = A e^{\lambda \theta}$$

$$\frac{1}{r} = A e^{\lambda \theta}$$

$$r = A e^{-\lambda \theta}$$

$$\boxed{r = A e^{-\lambda \theta}}$$

$\therefore$  The path is an equiangular spiral //

CONIC AS A CENTRAL ORBIT

BOOK WORK:- When a central orbit is a conic with the centre of the force at one focus then find the law of force & the speed of the particle.

Soln:-

$$\frac{l}{r} = 1 + e \cos \theta$$

radius of circle  $\rightarrow$  conic

$$\Rightarrow u l = 1 + e \cos \theta$$

$$\Rightarrow u = \frac{1}{l} + \frac{e}{l} \cos \theta \quad ; \quad (l \text{ is the semi-latus rectum})$$

differ. w.r.t.  $\theta$

$$\frac{du}{d\theta} = 0 + \frac{e}{l} (-\sin \theta) = -\frac{e}{l} \sin \theta$$

Again, differ. w.r.t.  $\theta$

$$\frac{d^2u}{d\theta^2} = -\frac{e}{l} \cos \theta$$

The differ. eqn of the central orbit of polar co-ordinates

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2 u^2} \Rightarrow F = h^2 u^2 \left[ \frac{d^2u}{d\theta^2} + u \right]$$

$$\Rightarrow F = h^2 u^2 \left[ -\frac{e}{l} \cos \theta + \frac{1}{l} + \frac{e}{l} \cos \theta \right]$$

at  $\theta = 0$

$$F = \frac{h^2 u^2}{l} = \frac{h^2}{l a^2} \rightarrow \text{distance variable}$$

Law of force:-

The force per unit mass in the radial direction towards the pole is  $\frac{h^2}{l a^2}$

which is inversely proportional to the square of the distance from the pole.

This rule of force is called inverse square law.

Velocity:-

$$v^2 = h^2 \left[ \left( \frac{du}{d\theta} \right)^2 + u^2 \right]$$

$$= h^2 \left[ \left( \frac{e}{l} \sin \theta \right)^2 + \left( \frac{1}{l} + \frac{e}{l} \cos \theta \right)^2 \right]$$

$$= \frac{h^2}{l^2} \left[ e^2 \sin^2 \theta + 1 + 2e \cos \theta + e^2 \cos^2 \theta \right]$$

$$\begin{aligned}
 &= h^2/\mu^2 (e^2 + 1 + 2e \cos \theta) \\
 &= h^2/\mu^2 (e^2 + 1 + 2(1/a - 1)) \left( \frac{h^2}{\mu} = 2ae \cos \theta \right) \\
 &= h^2/\mu^2 (e^2 + 1 + \frac{2e}{a} - 2) \\
 &= h^2/\mu^2 (e^2 - 1 + 2/a) \\
 &= h^2/\mu^2 \left[ \frac{e^2 - 1}{a} + 2/a \right] = h^2/\mu^2 \left[ \frac{e^2 - 1}{a} + 2/a \right] \\
 &= \mu \left[ \frac{2}{a} + \frac{e^2 - 1}{a} \right] \rightarrow \text{①} \quad \left[ \text{where } \mu = \frac{b^2}{a} \right]
 \end{aligned}$$

If the orbit is a parabola:  
 If the path is a parabola,  $e = 1$   
 $\therefore v^2 = \mu \cdot 2/a$  (latus section)

Ellipse:- If the path is an ellipse,  $l = b^2/a$   
 $\downarrow$  ①  $b^2 = a^2(1 - e^2)$   
 $\therefore v^2 = \mu \left[ \frac{2}{a} + \frac{e^2 - 1}{b^2/a} \right]$   
 $= \mu \left[ \frac{2}{a} + \frac{a^2(e^2 - 1)}{b^2} \right]$   
 $= \mu \left[ \frac{2}{a} + \frac{a^2(e^2 - 1)}{ab^2} \right]$   
 $= \mu \left[ \frac{2}{a} - \frac{b^2}{ab^2} \right] \quad [b^2 = a^2(1 - e^2)]$   
 $v^2 = \mu \left[ \frac{2}{a} - \frac{1}{a} \right]$

Hyperbola:- If the path is a hyperbola,  $l = b^2/a, b^2 = a^2(e^2 - 1)$   
 $\downarrow$   $b^2 = a^2(e^2 - 1)$   $v^2 = \mu \left[ \frac{2}{a} + \frac{e^2 - 1}{b^2/a} \right]$   
 $= \mu \left[ \frac{2}{a} + \frac{a^2(e^2 - 1)}{ab^2} \right]$   
 $= \mu \left[ \frac{2}{a} + \frac{b^2}{ab^2} \right] = \mu \left[ \frac{2}{a} + \frac{1}{a} \right]$

Note:- i) If the path is the branch of the hyperbola not nearer to the centre of force its eqn is,  
 $1/a = -1 + e \cos \theta$

ii) periodic time for ellipse.  
 periodic time =  $\frac{\text{Total area}}{\text{Areal velocity}} = \frac{\pi ab}{h/2} = \frac{2\pi ab}{h/2(\sqrt{\mu})}$

Kepler's Laws of Planetary Motion (11)

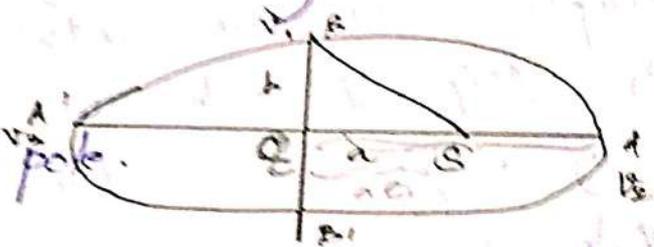
The planets describe ellipses about the sun as focus. The radius vectors drawn from the sun to a planet sweep out equal areas in equal times. The squares of the periods of the planets are proportional to the cubes of the semi-major axes of their respective orbits.

A particle describes an elliptic orbit under a force towards one focus. If  $v$  is the speed at point B of the minor axis &  $v_1, v_2$  the speed at points A, A' of the major axis. S.T.  $v^2 = v_1 v_2$  (10)  
 A particle describes an elliptic orbit under a force toward one focus. S.T. the ratio of minimum & maximum speed of the particle is  $1 \pm e$  where  $e$  is the eccentricity.

AA' - major axis

BB' - minor axis

S - force takes S as pole.



$v_1$  = velocity at B

$v_2$  = velocity at A.

$v_3$  - velocity at A'

$SA = CA - CS = a - ae = a(1-e)$

$SB = a$  (as  $CB = a$ )

for ellipse,  $v^2 = \mu \left( \frac{1}{r} - \frac{1}{a} \right)$

$v_1^2 = \mu \left( \frac{1}{SB} - \frac{1}{a} \right)$

$v_1^2 = \mu \left[ \frac{1}{a} - \frac{1}{a} \right] = \mu/a \rightarrow (11)$

$$v_1^2 = \mu \left( \frac{1}{a} - \frac{1}{r} \right) = \mu \left( \frac{1}{SA} - \frac{1}{r} \right)$$

$$v_1^2 = \mu \left( \frac{1+e}{a(1-e)} \right) \rightarrow \text{--- (2)}$$

$$v_2^2 = \mu \left( \frac{1}{SA} - \frac{1}{a} \right) = \mu \left( \frac{1}{a(1+e)} - \frac{1}{a} \right) \rightarrow \text{--- (3)}$$

$$v_2^2 = \mu \left( \frac{1-e}{a(1+e)} \right) \rightarrow \text{--- (3)}$$

from (2) & (3)

$$v_1^2 \cdot v_2^2 = \mu \left( \frac{1+e}{a(1-e)} \right) \mu \left( \frac{1-e}{a(1+e)} \right)$$

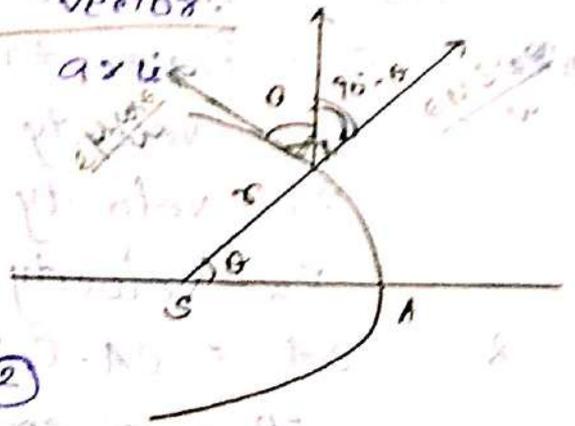
$$v_1^2 \cdot v_2^2 = \mu^2 \left( \frac{(1+e)(1-e)}{a^2(1-e)(1+e)} \right) = \frac{\mu^2}{a^2} (v_1^2)^2$$

Min. SA =  $v_2$   
 Max. SA =  $v_1$

$$\frac{v_2^2}{v_1^2} = \frac{\mu \left( \frac{1-e}{a(1+e)} \right)}{\mu \left( \frac{1+e}{a(1-e)} \right)} = \frac{(1-e)^2}{(1+e)^2}$$

$$v_2 : v_1 = (1-e) : (1+e)$$

S.T. the velocity of a particle moving in an ellipse abt the centre of force at a given instant is  
 i)  $\mu/h \perp$  to the radius vector.  
 ii)  $e\mu/h \perp$  to the major axis.



Soln: The eqn of ellipse is  
 $l/r = 1 + e \cos \theta \rightarrow \text{--- (1)}$

$$\text{w.t.t. } \mu = \frac{h^2}{l} \cos^2 \theta \quad l = \frac{h^2}{\mu} \rightarrow \text{--- (2)}$$

$$r^2 \dot{\theta} = h \rightarrow \text{--- (3)}$$

The velocity components in the radial & transverse directions are  $\dot{r}$  &  $r\dot{\theta}$

Now. Diff. w.r.t. to t.

$$-\frac{2}{r^2} r \dot{r} = -e \sin \theta \frac{d\theta}{dt}$$

$$-\frac{2}{r^2} \dot{r} = -e \sin \theta \dot{\theta}$$

$$r^2 \dot{\theta} = \text{constant} \quad (4^a)$$

$$\dot{r} = \frac{e h \beta \sin \theta}{h} \quad (by 3)$$

$$\dot{r} = \frac{e h \beta \sin \theta}{h} \rightarrow (4) \quad (by 3)$$

let  $r \dot{\theta} = \frac{r^2 \dot{\theta}}{r} = \frac{1}{2} \frac{d}{dt} (r^2 \dot{\theta}) = \frac{1}{2} \frac{d}{dt} (2 \text{constant}) = \text{constant}$

$$r \dot{\theta} = \frac{h}{m} (1 + \cos \theta) = \frac{h}{m} + \frac{e h \beta \cos \theta}{h} \quad (5)$$

from (4) & (5), we can say that the velocity of the particle is composed of three velocities namely,

$\frac{e h \beta}{h} \sin \theta$  in the radial direction  $\rightarrow (6)$

$\frac{e h \beta}{h} \cos \theta$  in the transverse direction  $\rightarrow (7)$

$\frac{h}{m}$  in the transverse direction  $\rightarrow (8)$

from (6), (7) & (8), it is clear that the radial & transverse directions are inclined to the  $\perp^2$  by the major axis at an angle  $\theta$ .  
 $\therefore$  velocity  $v^2 = (\text{radial})^2 + (\text{transverse})^2$

$$= \left( \frac{e h \beta}{h} \sin \theta \right)^2 + \left( \frac{e h \beta}{h} \cos \theta \right)^2$$

$$v^2 = \left( \frac{e h \beta}{h} \right)^2$$

$$\Rightarrow v = \frac{e h \beta}{h}$$

velocity  $\frac{e h \beta}{h} \perp^2$  to the major axis

the angular velocity of a particle moving in a plane curve is constant about fixed origin.  $\therefore$  its transverse acceleration is proportional to its radial velocity.

Given, Angular velocity = k (constant) velocity

$$\dot{\theta} = k$$

$$\text{Transverse acceleration} = \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = \frac{1}{r} \frac{d}{dt} (r^2 k)$$

$$= \frac{k}{r} \frac{d}{dt} r^2$$

$$= \frac{k}{r} 2 r \dot{r} = 2k \frac{\dot{r}}{r}$$

Transverse acc. =  $2k$  (radial velocity) in radial direction  
 Transverse acc.  $\propto$  (radial velocity)

velocity of a particle along  $\perp$  to the radius vector from a fixed origin  $O$  is  $a$  &  $b$ . Find the path of the acc. along  $\perp$  to the radius vector.

Let the radial velocity  $v_r = a \rightarrow (1)$   
 Transverse velocity  $v_t = b \rightarrow (2)$   
 from (1) & (2)  $\frac{v_r}{v_t} = \frac{a}{b} = \frac{dr/dt}{r d\theta/dt} = \frac{a}{b}$   
 $\Rightarrow \frac{1}{r} dr/d\theta = \frac{a}{b} \Rightarrow dr/r = \frac{a}{b} d\theta$   
 Integ.  $\log r = \frac{a}{b} \theta + \log c$   
 $\log r - \log c = \frac{a}{b} \theta \Rightarrow \log(r/c) = \frac{a}{b} \theta$  (18)  
 $\Rightarrow r/c = e^{a\theta/b}$   
 $r = ce^{a\theta/b}$

This is the eq. path of the orbit, from  $O$ ,  
 $\dot{r} = a$   
 $\Rightarrow \ddot{r} = 0$

Radial acc.  $= \ddot{r} - r\dot{\theta}^2 = 0 - r(b/r)^2 = -ab^2/r^2$   
 $= -b^2/r \rightarrow (3)$

Transverse acc.  $= \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta})$   
 $= \frac{1}{r} \frac{d}{dt}(r^2 \cdot b/r) = b/r \frac{dr}{dt}$   
 $= b/r \cdot a = ab/r \rightarrow (4)$

Eqn (3) & (4) describe the acc. along  $\perp$  to the radius vector

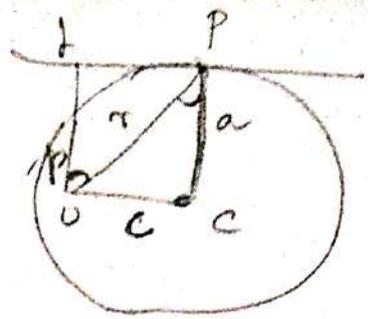
Q. Find the law of force to an internal pt under which a body will describe a circle.

Ans: Let  $c$  be the centre.  
 $a$  - radius  
 $OC = c$

Let  $p$  be any pt on the circle &  $OC$  be the  $\perp$  from  $O$  on the tangent at  $p$ .  
 $OP = r$  &  $OT = p$

From  $\triangle OPC$ ,

$$\begin{aligned} c^2 &= r^2 + a^2 - 2ra \cos \angle OPC \\ &= r^2 + a^2 - 2ra \cos \angle POL \\ &= r^2 + a^2 - 2ra \cdot p/r \\ &= r^2 + a^2 - 2ap \end{aligned} \rightarrow (1)$$



(1) which is the pedal eqn of the circle for a general position of the pole.

From (1), Diff. w.r.t.  $r$

$$\begin{aligned} 0 &= 2r - 2a \frac{dp}{dr} \\ \frac{dp}{dr} &= r/a \end{aligned}$$

$$\begin{aligned} \frac{1}{r^2} &= \frac{du}{dr} \\ F &= \frac{h^2}{p^3} \frac{dp}{dr} \end{aligned}$$

w.b.t, the diff. eqn of a central orbit in  $p-r$  co-ordinates is,

$$F = \frac{h^2}{p^3} \cdot \frac{dp}{dr} = \frac{h^2}{p^3} \cdot \frac{r}{a} \quad (\text{by (1)}) \rightarrow (2)$$

From (1),  $2ap = r^2 + a^2 - c^2$

$$p = \frac{1}{2a} (r^2 + a^2 - c^2)$$

$$\therefore (2) = \frac{h^2 r}{\frac{1}{2a} (r^2 + a^2 - c^2)^3 \cdot a}$$

$$F = \frac{8h^2 a^2 \cdot r}{(r^2 + a^2 - c^2)^3}$$

This is the req. law of force