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UNIT - 3

RIEMANN INTEGRATION

Sets of measure zero:

Measure zero:

The subset E of R' is said to be measure zero. If for each $\varepsilon > 0$ $\exists$ finite or countable no. of open intervals I_1, I_2 such that $E \subset \bigcup_n I_n$ and

$$\sum_n |I_n| < \varepsilon$$

Thus E is of measure zero if given $\varepsilon > 0$, E can be covered by union of open intervals whose length add upto to less than ε.

Note:

By definition, set consisting at one point as measure zero.

Theorem:

If each of subset $E_1, E_2, \ldots$ of R' is of measure zero then $\bigcup_{n=1}^{\infty} E_n$ is also of measure zero.

Proof:

Let $\varepsilon > 0$, since E_n has measure zero, for each $n \in I$ $\exists$ finite (or) countable no. of open intervals $I_{n_1}, I_{n_2}, \ldots$ such that $E_n \subset \bigcup_{n=1}^{\infty} I_{n_i}$ and $\sum_{n=1}^{\infty} |I_{n_i}| < \varepsilon/2^n$

Thus we have $\bigcup_{n=1}^{\infty} E_n \subset \bigcup_n \bigcup_i I_{n_i}$ and

$$\Sigma(\Sigma|I_{n_i}|) < \sum_{n=1}^{\infty} \varepsilon/2^n$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \cdots$$

$$< \frac{\varepsilon}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)$$

$$< \frac{\varepsilon}{2}\left[1-\frac{1}{2}\right]^{-1}$$

$$< \frac{\varepsilon}{2}(2/1)$$

$$< \varepsilon$$

Hence $\bigcup_{n=1}^{\infty} E_n$ has measure zero.

Corollary:

Every countable subset of R' has measure zero.

Proof:

Since every singleton of measure zero and every countable set of R' is the countable union of singleton sets.

By the above thearem, It has measure zero.

Almost everywhere (a.e)

A statement is said to be hold at almost every point of $[a,b]$ (or) (a.e) in $[a,b]$) if set of points of $[a,b]$ at which statement does not hold is of measure zero.

Thus 'f is continuous at almost every point of $[a,b]$' means same as if E is set of point of $[a,b]$ at which f is not continuous then E is of measure zero.

Exercise:

1. S.T. intervals are not of measure zero.

Soln:

Let I be interval whose length be say S. If we choose $\epsilon < S$ then we cannot find collection of open intervals $\{I_n\}$ whose union is I and $\Sigma |I_n| < \epsilon$.

Hence the intervals are not of measure zero.

2. If A is not of measure zero and $B \subset A$ and if B is of measure zero. P.T. $A-B$ is not of measure zero.

Soln:

Let us assume that $A-B$ is of measure zero.

Since, $A \supset B$, A cannot be written as $A = B \cup [A-B]$, B and $A-B$ are of measure zero.

By theorem

"If each of subset $E_1, E_2, \ldots$ of R^1 is of measure zero then $\bigcup_{n=1}^{\infty} E_n$ is also of measure of zero'.

A is of measure zero which is a contradiction that A is not of measure zero.

$\therefore$ A-B is not measure of zero.

3. S.T. the set of all rational no. is of measure zero.

Proof:

W.K.T.

The set of all rational no. is countable since every countable subset of R' is of measure zero, the set of all rational no. is also of measure zero.

4. S.T. the set of all irrational no. is not of measure zero.

Proof:

W.K.T.

$R' = (-\infty, \infty)$ is not measure zero.

Since Q is set of all rational no. is of measure zero.

By theorem,

'If A is not of measure zero and $B \subset A$ and if B is of measure zero*. Then A-B is not of measure zero'.

The set of all irrational no. which is equal to $R' - Q$ is not measure zero.

Definition of Riemann Integral:

Riemann Integral:

Let $\mathcal{I}$ be any bounded interval of real no. Let f be bounded (real value) function on $\mathcal{I}$ we define

$$M[f, \mathcal{I}], m[f, \mathcal{I}], \omega[f, \mathcal{I}]$$

we have

$$M[f, \mathcal{I}] = \underset{x \in \mathcal{I}}{\text{l.u.b.}} f(x)$$

$$m[f, \mathcal{I}] = \underset{x \in \mathcal{I}}{\text{g.l.b.}} f(x)$$

$$\omega[f, \mathcal{I}] = M[f, \mathcal{I}] - m[f, \mathcal{I}]$$

If a is point of $\mathcal{I}$, we define

$$\omega[f; a] = \text{g.l.b. } \omega[f, \mathcal{I}]$$

where g.l.b. is taken overall from open subinterval of J of $\mathcal{I}$ such that $a \in J$.

Subdivision, Refinement and component:

By a subdivision of closed bounded interval $[a,b]$ we mean a finite subset $\{x_0, x_1, \ldots x_n\}$ of $[a,b]$ such that

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

If σ and τ are two subdivision of closed interval $[a,b]$ we say I is refinement of σ, if $\sigma \subset I$ [i.e., I is refinement of σ means that subdivision I is obtained from subdivision σ by adding more points of subdivision).

If $\sigma = \{x_0, x_1, \ldots x_n\}$ is subdivision of $[a,b]$ then closed interval of $I_1 = \{x_0, x_1\}$, $I_2 = [x_1, x_2] \ldots I_n = [x_{n-1}, x_n]$ are called the component interval of σ.

Lower sum (L) and upper sum (U):

Let f be bounded function on closed bounded interval $[a,b]$ and let σ be any subdivision of $[a,b]$. We define

$$U[f,\sigma] = \sum_{k=1}^{n} M[f, I_k] \cdot |I_k|$$

where $I_1, I_2, \ldots I_n$ are component interval of σ.

Similarly, the lower sum $L[f,\sigma]$ defined as

$$L[f,\sigma] = \sum_{k=1}^{n} m[f, I_k] \cdot |I_k|$$

Clearly, $U[f,\sigma] \geq L[f,\sigma]$.

Remark:

If f is continuous and non-negative value on $[a,b]$ then $U[f,\sigma]$ is sum of area of rectangle and $L[f, \sigma]$ is sum of area of in rectangle.

Lemma:

Let f be bounded function on $[a,b]$ then every upper sum for f is greater than or equal to every lower sum for f.

i.e., if σ and τ are any two subdivision of closed interval $[a,b]$ then $U[f;\sigma] \geq L[f;\tau]$.

Proof:

Let us first show that, σ^* is any refinement of σ.

Then $U[f,\sigma] \geq U[f,\sigma^*] \rightarrow ①$

It is enough to prove that this in the case where σ^* is obtained from σ by adding only one point of subdivision

we can then apply by induction,

Let $I_1, I_2, \ldots I_k \ldots I_m$ be component interval of σ and $I_1, I_2, \ldots I_k^*, I_k^{**}, \ldots I_n$ be component interval of σ^*.

where $I_k = |I_k^*| + |I_k^{**}|$

Since $I_k^* \subset I_k$

we have $M[f, I_k^*] \leq M[f, I_k]$

$|||^{ly}$ $M[f, I_k^{**}] \leq M[f, I_k]$

Thus $U[f, \sigma^*] = \sum_{j=1}^{n} M[f, I_j]|I_j| + M[f, I_k^*]$
$|I_k^*| + M[f, I_k^{**}]|I_k^{**}|$.

$\leq \sum_{j=1}^{n} M[f, I_j]|I_j| + M[f, I_k](|I_k^*| + |I_k^{**}|)$

$\leq \sum_{\substack{j=1 \\ j \neq k}}^{n} M[f, I_j]|I_j| + M[f, I_k]|I_k|$

$U[f, \sigma^*] \leq \sum_{k=1}^{n} M[f, I_k]|I_k|$

$U[f, \sigma^*] \leq U[f; \sigma]$

$|||^{ly}$ If τ^* is any refinement of τ

It can be shown that

$L[f, \tau] \leq L[f, \tau^*] \rightarrow (2)$

But since, $\sigma \cup \tau$ is refinement of both σ & τ

It follows from equ (1) & (2) that

$U[f; \sigma] \geq U[f, \sigma \cup \tau] \geq L[f, \sigma \cup \tau] \geq L[f; \tau]$

$\therefore U[f; \sigma] \geq L[f; \tau]$.

Remark:

From the lemma, we have

g.l.b. $U[f, \sigma] \geq$ l.u.b. $L[f, \sigma]$ where g.l.b. & l.u.b. are taken over all subdivision σ of $[a, b]$.

For if τ is any subdivision of $[a, b]$ then lemma show that $L[f, \tau]$ is a lower bound for set of all upper sum

$U[f,\sigma]$.

Hence $L[f,\tau] \le g.l.b.\ U[f,\sigma]$

For every subdivision τ

But this says that $g.l.b.\ U[f,\sigma]$ is upper bound for set of all lower sum $L[f,\tau]$.

Hence $l.u.b.\ L[f,\tau] \le g.l.b.\ U[f,\sigma]$

Upper integral and lower integral:

Let f be bounded function on closed bounded interval $[a,b]$ we define $\overline{\int_a^b} f(x)dx$ called upper integral of f over $[a,b]$ as

$$\overline{\int_a^b} f(x)\,dx = g.l.b.\ U[f,\sigma]$$

where g.l.b. is taken overall subdivision σ of $[a,b]$.

IIIly we define $\underline{\int_a^b} f(x)dx$ called lower integral of f over $[a,b]$ as

$$\underline{\int_a^b} f(x)\,dx = l.u.b.\ L[f,\sigma]$$

we denote the upper and lower integral of f by $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$ from inequality we can write

$$\underline{\int_a^b} f \le \overline{\int_a^b} f.$$

Riemann Integrable Function:

If f is bounded function on closed bounded interval $[a,b]$ we say that f is Riemann Integrable on $[a,b]$ if $\int_{-a}^{b} f = \int_{a}^{-b} f$.

We denote by $R[a,b]$ the class of all function f which are Riemann integrable on $[a,b]$.

Exercise:

S.T. characteristic function on rational no. in $[0,1]$ is not Riemann Integrable.

Proof:

Let ψ be characteristic function of rational no. in $[0,1]$ then for any interval $I \subset [0,1]$.

$$M[\psi, J] = 1, \quad m[\psi, J] = 0.$$

Hence for any subdivision σ we have

$$U[\psi, \sigma] = 1, \quad L[\psi, \sigma] = 0.$$

$$\therefore \int_{-0}^{1} \psi\, dx = 0, \quad \int_{0}^{-1} \psi\, dx = 1$$

$$\therefore \int_{-0}^{1} \psi\, dx \neq \int_{0}^{-1} \psi\, dx$$

$\therefore \psi$ is not Riemann Integrable.

Note:

Any constant function on closed bounded interval $[a,b]$ is Riemann Integrable on $[a,b]$.

Theorem:

Let f be bounded function on closed bounded interval $[a,b]$ then $f \in R[a,b]$ iff for each $\epsilon > 0$ $\exists$ subdivision σ of $[a,b]$ such that $U[f,\sigma] < L[f,\sigma] + \epsilon$.

Proof:

Suppose first that, given : $\epsilon > 0$ $\exists$ σ such that $U[f,\sigma] < L[f,\sigma] + \epsilon \rightarrow ①$

then since $\int_a^{-b} f \leq U[f,\sigma]$ and

$$\int_{-a}^{b} f \geq L[f,\sigma]$$

$$\Rightarrow \int_a^{-b} f \leq U[f,\sigma] \leq L[f,\sigma] + \epsilon$$

$$\int_a^{-b} f \leq \int_{-a}^{b} f + \epsilon$$

$\because$ ϵ is arbitrary, It follows that

$$\int_a^{-b} f \leq \int_{-a}^{b} f \longrightarrow ②$$

and by $\int_{-a}^{b} f \leq \int_a^{-b} f \rightarrow ③$

From ② & ③

$$\int_{-a}^{b} f = \int_{+a}^{-b} f$$

This proves $f \in R[a,b]$

Conversely,

suppose $f \in R[a,b]$ then

$$\int_{-a}^{b} f = g.l.b.\ U[f,\sigma] = l.u.b.\ L[f;T] = \int_{-a}^{b} f$$

Given : $\epsilon > 0$ choose subdivision τ such that

$$\overline{\int_a^b} f - \frac{\epsilon}{2} < L[f;\tau] \rightarrow (5)$$

$|||^{ly}$ we may choose subdivision τ such that

$$\overline{\int_a^b} f + \frac{\epsilon}{2} > U[f;\sigma] \rightarrow (4)$$

From equ (4) & (5)

$$\overline{\int_a^b} f > U[f;\sigma] - \frac{\epsilon}{2}$$

$$\Rightarrow L[f;\tau] + \frac{\epsilon}{2} > U[f;\sigma] - \frac{\epsilon}{2}$$

By $U[f;\sigma] \geq U[f;\sigma \cup \tau] \geq L[f;\sigma\cup\tau] \geq L[f;\tau]$

we have,

$$L[f;\sigma\cup\tau] + \frac{\epsilon}{2} > U[f;\sigma\cup\tau] - \frac{\epsilon}{2}$$

$$\Rightarrow U[f;\sigma\cup\tau] < L[f;\sigma\cup\tau] + \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\Rightarrow U[f;\sigma\cup\tau] < L[f,\sigma\cup\tau] + \epsilon$$

Replacing $\sigma\cup\tau$ by σ we get

$$U[f;\sigma] < L[f;\sigma] + \epsilon.$$

Problems:

1. Let $f(x) = x$ for $0 \leq x \leq 1$ (a) Let $\sigma = \{0, 1/3, 2/3, 1\}$ compute $U[f;\sigma]$ & $L[f;\sigma]$.

Soln:

$$\sigma = \{0, 1/3, 2/3, 1\}$$

The component interval of σ are

$$I_1 = [0, 1/3], \quad I_2 = [1/3, 2/3], \quad I_3 = [2/3, 1]$$

Then $|I_1| = |I_2| = |I_3| = 1/3$

$M[f, I_1] = 1/3$ $\quad m[f; I_1] = 0$

$M[f, I_2] = 2/3$ $\quad m[f; I_2] = 1/3$

$M[f, I_3] = 1$ $\quad m[f; I_3] = 2/3$

Then,

$$U[f,\sigma] = \sum_{k=1}^{n} M[f, I_k]|I_k|$$

$$U[f,\sigma] = \sum_{k=1}^{3} M[f, I_k]|I_k|$$

$$= M[f, I_1]|I_1| + M[f, I_2]|I_2| + M[f, I_3]|I_3|$$

$$= \frac{1}{3}(1/3) + \frac{2}{3}(1/3) + 1(1/3)$$

$$= \frac{1+2+3}{9}$$

$$= \frac{2}{3}$$

Then,

$$L[f,\sigma] = \sum_{k=1}^{n} m[f, I_k]|I_k|$$

$$L[f,\sigma] = \sum_{k=1}^{3} m[f, I_k]|I_k|$$

$$= m[f, I_1]|I_1| + m[f, I_2]|I_2| + m[f, I_3]|I_3|$$

$$= 0(1/3) + 1/3(1/3) + 2/3(1/3)$$

$$= \frac{1+2}{9}$$

$$= 1/3$$

b) Let $\sigma_n = \{0, 1/n, 2/n, \dots 1\}$ compute $\lim_{n\to\infty} U[f,\sigma_n]$ and $\lim_{n\to\infty} L[f,\sigma_n]$.

Soln:

Let $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right]$; $|I_i| = \frac{1}{n}$

The component interval of σ are $\{0, 1/n, 2/n, \dots 1\}$

$$M[f, I_i] = i/n \quad ; \quad m[f, I_i] = \frac{i-1}{n}$$

$$U[f; \sigma_n] = \sum_{i=1}^{n} M[f, I_i]$$

$$= M[f, I_1]|I_1| + M[f, I_2]|I_2| + \dots M[f, I_n]|I_n|$$

$$= \frac{1}{n}(1/n) + \frac{2}{n}(1/n) + \dots + (n/n)(1/n)$$

$$= \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{n^2(1+1/n)}{n^2 \cdot 2}$$

$$\lim_{n\to\infty} U[f; \sigma_n] = \lim_{n\to\infty} \frac{(1+1/n)}{2} = \frac{1}{2}$$

$$L[f; \sigma_n] = \sum_{i=1}^{n} m[f, I_i]|I_i|$$

$$= m[f, I_i]|I_1| + m[f, I_2]|I_2| + \dots + m[f, I_n]|I_n|$$

$$= 0(1/n) + 1/n(1/n) + 2/n(1/n) + \dots + \frac{n-1}{n}(1/n)$$

$$= \frac{1}{n^2}[1 + 2 + \dots + n-1]$$

$$= \frac{1}{n^2}\left[\frac{n(n-1)}{2}\right]$$

$$= \frac{n^2(1-1/n)}{n^2 \cdot 2} = \frac{1-1/n}{2}$$

$$\lim_{n\to\infty} L[f, \sigma_n] = \lim_{n\to\infty} \frac{1-1/n}{2} = 1/2.$$

2. For each $n \in I$ and let σ_n be subdivision $\{0, 1/n, 2/n, \dots n/n\}$ compute $\lim_{n\to\infty} L[f, \sigma_n]$ for the

function $f(x) = x^2$, $[0 \leq x \leq 1]$.

Soln:

Let $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right]$, $|I_i| = 1/n$

The component interval of σ_n are

$\{0, 1/n, 2/n, \dots n/n\}$.

$$m[f, I_i] = \left(\frac{i-1}{n}\right)^2$$

$$L[f; \sigma_n] = \sum_{i=1}^{n} m[f, I_i]$$

$$= m[f, I_1]|I_1| + m[f, I_2]|I_2| + \dots + m[f, I_n]|I_n|.$$

$$= 0(1/n) + (1/n)^2(1/n) + (2/n)^2(1/n) + \dots + (n-1/n)^2(1/n)$$

$$= \frac{1}{n^3}[1 + 2^2 + \dots + (n-1)^2]$$

$$= \frac{1}{n^3}\left[\frac{n(n-1)(2n-1)}{6}\right]$$

$$= \frac{n^3}{n} \frac{(1-1/n)(2-1/n)}{6}$$

$$\lim_{n\to\infty} L[f; \sigma] = \lim_{n\to\infty} \frac{(1-1/n)(2-1/n)}{6}$$

$$= \frac{2}{6}$$

$$\lim_{n\to\infty} L[f; \sigma] = 1/3$$

Give example of bounded function which is not Rieman Integral over $[0,1]$. (or)

Let f be ~~two~~ function defined on $[0,1]$ by

$$f(x) = \begin{cases} 0 & ; \text{ when } x \text{ is irrational} \\ 1 & ; \text{ when } x \text{ is rational} \end{cases}$$

Calculate $\int_{-0}^{1} f$ and $\int_{0}^{-1} f$ and hence show that $f \notin R[0,1]$.

Soln:

W.K.T.

f is bounded, for all $x \in [0,1]$

If σ be any subdivision then $I_r = [x_{r-1}, x_r]$ and we have $m[f, I_1] = 0$, $M[f, I_r] = 1$

$$\Rightarrow L[f, \sigma] = 0 \quad , \quad U[f, \sigma] = 1$$

$$\int_{-0}^{1} f = \underset{\sigma}{\text{l.u.b}}\ L[f; \sigma] = 0 \quad \&$$

$$\int_{0}^{-1} f = \underset{\sigma}{\text{g.l.b.}}\ U[f; \sigma] = 1$$

$$\int_{-0}^{1} f \neq \int_{0}^{-1} f$$

$$f \notin R[0,1].$$

For $f(x) = \sin x$ $(0 \le x \le \pi/2)$ and $\sigma_n = \{0, \pi/2n, 2\pi/2n, \dots n\pi/2n\}$ compute $U[f, \sigma_n]$ and P.T. $\lim_{n \to \infty} U[f, \sigma_n] = 1$.

Soln:

Let $I_1 = [0, \pi/2n]$, $I_2 = [\pi/2n, 2\pi/2n] \dots$

$$I_n = \left[\frac{(n-1)\pi}{2n}, \frac{n\pi}{2n} \right]$$

and $|I_1| = |I_2| = \cdots |I_n| = \pi/2n$ is length of interval.

$$M[f;\sigma_i] = \frac{\sin i\pi}{2n}$$

$$U[f;\sigma_n] = \sum_{i=1}^{n} M[f;I_i]|I_i|$$

$$= \sum_{i=1}^{n}\left(\frac{\sin i\pi}{2n}\right)\left(\frac{\pi}{2n}\right)$$

$$= \frac{\pi}{2n}\left[\frac{\sin\pi}{2n} + \frac{\sin 2\pi}{2n} + \cdots + \frac{\sin n\pi}{2n}\right]$$

Let us use identity, $\sin x + \sin 2x + \cdots + \sin nx = \frac{1}{2\sin x/2}\left[\cos\frac{x}{2} - \cos\frac{(2n+1)}{2}x\right]$.

Put $x = \pi/2n$.

$$U[f;\sigma_n] = \frac{\pi/2n}{2\sin\frac{\pi}{4n}}\left[\cos\frac{\pi}{4} - \cos\frac{(2n+1)}{2}\cdot\frac{\pi}{2n}\right]$$

$$= \frac{\pi}{4n\sin\frac{\pi}{4n}}\left[\cos\frac{\pi}{4} - \cos\left(\frac{\pi}{2}+\frac{\pi}{4n}\right)\right]$$

$$\lim_{n\to\infty} U[f,\sigma_n] = \lim_{n\to\infty}\frac{\pi}{4\pi\sin\frac{\pi}{4n}}\left[\cos\frac{\pi}{4n} - \cos\left(\frac{\pi}{2}+\frac{\pi}{4n}\right)\right]$$

$$\lim_{n\to\infty} U[f,\sigma_n] = 1.$$

Properties of Riemann Integral:

If $f\in R[a,b]$ and $a<c<b$ then $f\in R[a,c]$, $f\in R[c,b]$ and integral $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof:

Let $f\in R[a,b]$.

By theorem,

'Let f be bounded function on closed bounded interval $[a,b]$ then $f \in R[a,b]$ iff f is continuous atmost every point in $[a,b]$'.

The set E of points in $[a,b]$ at which f is not continuous is of measure zero. obviously then $E \cap [a,c]$ is of measure zero. and so $f \in R[a,c]$.

$|||^{ly}$ $f \in R[c,b]$

If σ is any subdivision of $[a,c]$ & τ is any subdivision $[c,b]$ then $\sigma \cup \tau$ is a subdivision $[a,b]$ whose component intervals are those of σ togather with those of τ.

Hence $L[f,\sigma] + L[f,\tau] = L[f,[\sigma\cup\tau]] \leq \underline{\int_a^b} f$

and $L[f,\sigma] + L[f,\tau] \leq \int_a^b f$

By taking l.u.b. on left over all σ [keeping τ fixed for moment]. we get

$$\int_a^c f + \int_c^b f \leq \int_a^b f \rightarrow ①$$

Going back to original σ & τ we also have $U[f,\sigma] + U[f;\tau] = U[f,\sigma\cup\tau] \geq \overline{\int_a^b} f$

so that, $U[f;\sigma] + U[f;\tau] \geq \int_a^b f$

Taking g.l.b. in first part we get

$$\int_a^c f + \int_c^b f \geq \int_a^b f \rightarrow ②$$

From ① & ②

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Theorem:

If $f \in R[a,b]$ and λ is any real no. then $\lambda f \in R[a,b]$ & $\int_a^b \lambda f = \lambda \int_a^b f$.

Proof:

If $\lambda = 0$ the theorem is obvious.

Suppose $\lambda > 0$ since λf is continuous at every point where f is continuous, it is clear that $\lambda f \in R[a,b]$.

Since $\lambda > 0$, if J is any interval contained in $[a,b]$ then $M[\lambda f; J] = \lambda M[f; J]$ and so for any subdivision σ of $[a,b]$.

$$U[\lambda f, \sigma] = \lambda U[f, \sigma]$$

Taking g.l.b. of B.S. [over all σ], we get

$$\int_a^b \lambda f = \lambda \int_a^b f \quad (\lambda > 0) \rightarrow ①$$

when $\lambda \leq 0$, we proceed as follows

Now for any J we also have

$$M[-f, J] = -m[f; J]$$

Hence,

$$\int_a^b (-f) = \underset{\sigma}{\text{g.l.b.}}\, U[-f, \sigma] = \underset{\sigma}{\text{g.l.b.}} \{-L(f, \sigma)\}$$

$$= - \text{l.u.b.} L[f,\sigma]$$

$$\int_a^b (-f) = -\int_a^b f \rightarrow ②$$

If $\mu < 0$ then $\lambda = -\mu > 0$ and so by equ②

$$\int_a^b \mu f = \int_a^b -(\lambda f)$$

$$= -\int_a^b \lambda f$$

$$= -\lambda \int_a^b f$$

$$\int_a^b \mu f = \mu \int_a^b f.$$

Theorem:

If $f \in R[a,b]$, $g \in R[a,b]$ then $f+g \in R[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Proof:

Since f and $g \in R[a,b]$, the set E_f & E_g of points at which f and g respectively, are not continuous, are both of measure 0.

Hence the set $E_f \cup E_g$ is of measure zero. But if $x \in [a,b] - [E_f \cup E_g]$, then f, g & hence $f+g$ are continuous at x.

Thus $f+g$ is continuous atmost every point in $[a,b]$ & so $f+g \in R[a;b]$.

If J is any interval contained in $[a,b]$ & if $y \in J$ we have,

$f(y) + g(y) \le M[f, J] + M[g; J]$

$f(y) + g(y) \le M[f; J] + M[g; J]$

Hence $M[f+g; J] \le M[f; J] + M[g; J]$ then we have for any subdivision σ then we have

$$\int_a^b (f+g) \le U[f+g; \sigma] \le U[f; \sigma] + U[g; \sigma] \rightarrow ①$$

But given $\varepsilon > 0$ By theorem

Let f be bounded function on closed bounded interval $[a, b]$ then $f \in R[a, b]$ iff for each $\varepsilon > 0$, $\exists$ subdivision $\sigma[a, b]$ such that $U[f; \sigma] < L[f; \sigma] + \varepsilon$.

A subdivision $\sigma_1[a, b]$ such that

$$U[f, \sigma_1] < L[f, \sigma_1] + \varepsilon/2$$

$$U[f, \sigma_1] < \int_a^b f + \varepsilon/2$$

also there is subdivision $\sigma_2[a, b]$ such that

$$U[g; \sigma_2] < L[g; \sigma_2] + \varepsilon/2$$

$$U[g; \sigma_2] < \int_a^b g + \varepsilon/2$$

If $\sigma = \sigma_1 \cup \sigma_2$ then by equ① we have

$$\int_a^b (f+g) \le U[f; \sigma] + U[g; \sigma]$$

$$< \int_a^b f + \varepsilon/2 + \int_a^b g + \varepsilon/2$$

$$\int_a^b (f+g) < \int_a^b f + \int_a^b g + \varepsilon$$

Since ε was arbitrary. This proves

$$\int_a^b (f+g) \leq \int_a^b f + \int_a^b g \rightarrow ②$$

Since f and g were any riemann integral function we can sub. $-f, -g$ for f, g in equ ②

Hence $\int_a^b (-f-g) \leq \int_a^b (-f) + \int_a^b (-g)$

$$-\int_a^b (f+g) \leq -\left[\int_a^b f + \int_a^b g\right]$$

$$\int_a^b (f+g) \geq \int_a^b f + \int_a^b g$$

From equ ② & ③

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

Lemma:

If $f \in R[a,b]$ and if $f(x) \geq 0$ almost everywhere $(a \leq x \leq b)$ then $\int_a^b f \geq 0$.

Proof:

If m and M are g.l.b. and l.u.b. of f on $[a,b]$ then,

$$m[b-a] \leq \int_a^b f \leq M(b-a)$$

Since, $f(x) \geq 0, \forall x \in [a,b]$

$\Rightarrow m \geq 0$ also $b-a \geq 0$

$$\int_a^b f \geq m(b-a) \geq 0$$

i.e., $\int_a^b f \geq 0$.

Caroллary:

If $f \in R[a,b]$, $g \in R[a,b]$ and if $f(x) \leq g(x)$ almost everywhere $(a \leq x \leq b)$ then

$\int_a^b f \leq \int_a^b g$.

Proof:

By theorem ④, ⑤

The function $-f$ and $g-f$ are Riemann integrable.

Since $f(x) = g(x)$ almost everywhere

$\Rightarrow g(x) - f(x) > 0$

Hence by lemma

if $f(x) \geq 0 \Rightarrow \int_a^b f(x) \geq 0$ and above theorem we get

$$0 \leq \int_a^b (g-f)$$

$$0 \leq \int_a^b [g+(-f)]$$

$$\leq \int_a^b g + \int_a^b (-f)$$

$$\leq \int_a^b g - \int_a^b f$$

$$\int_a^b f \leq \int_a^b g.$$

Carollary:

If $f \in R[a,b)$ then $|f| \in R[a,b]$ and $|\int_a^b f| \leq \int_a^b |f|$.

Proof: Since $|f|$ will be continuous at every paint where f is continuous & $|f| \in R[a,b]$.

Now, since $f(x) \leq |f(x)| \leq |f|(x)$ $\forall x \in [a,b]$

By the above caroallary,

$$\int_a^b f \leq \int_a^b |f| \rightarrow ①$$

Since $-f(x)$ is also less than we get $-\int_a^b f \leq \int_a^b |f| \rightarrow ②$

From equ ① & ② we get,

$$|\int_a^b f| \leq \int_a^b |f|.$$

1. Let $x \in R[a,b]$ then function f defined on $[a,b]$ by $F(x) = \int_a^x f(t)dt$ is continuous on $[a,b]$.

Soln: Since $f \in [a,b]$, f is bounded on $[a,b]$

Suppose $|f(t) \leq M$, $\forall t \in [a,b]$.

Let $a \leq x \leq y \leq b$ then

$$|F(y) - F(x)| = |\int_a^y f(t)dt - \int_a^x f(t)dt|$$

$$\leq |\int_a^x f(t)dt + \int_x^y f(t)dt - \int_a^x f(t)dt|$$

$$\le \left|\int_x^y f(t)dt\right|$$

$$\le \int_x^y |f(t)|dt$$

$$\le M\,[t]_x^y$$

$$|F(y)-F(x)| \le M\,|(y-x)| \rightarrow ①$$

Let $\epsilon > 0$ be given then if $|y-x| < \epsilon/M$

$$① \Rightarrow |F(y)-F(x)| \le M \cdot \frac{\epsilon}{M} < \epsilon$$

This proves that $F(x)$ is continuous on $[a,b]$.

2. If f is continuous on closed bounded interval $[a,b]$ if $f(x) \ge 0$ $[a \le x \le b]$ and if $f(c) > 0$ for some $c \in [a,b]$. P.T. $\int_a^b f(x)dx > 0$.

Soln:

Since f is continuous on $[a,b]$ it is continuous at c and so given $\epsilon > 0$ we can find $\delta > 0$ such that

$$|f(x)-f(t)| < \epsilon \ \forall x \text{ with } |x-c| < \delta$$

Take $\epsilon = \frac{f(c)}{2}$

$$-\epsilon < f(x)-f(c) < \epsilon$$

$$f(c)-\epsilon < f(x) < f(c)+\epsilon$$

$$f(c)-\frac{f(c)}{2} < f(x) < f(c)+\frac{f(c)}{2}$$

$$\frac{f(c)}{2} < f(x) \le \frac{3f(c)}{2}$$

Now $\int_a^b f = \int_a^{c-\delta} f + \int_{c-\delta}^{c+\delta} f + \int_{c+\delta}^{b} f$

$\int_a^b f > \int_{c-\delta}^{c+\delta} f$

Since $f(x) \geq 0$ on $[a,b]$ and f is continuous on $[a,b]$.

$\int_a^{c-\delta} f \geq 0$ and $\int_{c+\delta}^{b} f \geq 0$

$\therefore \int_a^b f(x)dx \geq \int_{c-\delta}^{c+\delta} f(x)dx > \frac{f(c)}{2} \int_{c-\delta}^{c+\delta} dx$

$> \frac{f(c)}{2} [x]_{c-\delta}^{c+\delta}$

$> \frac{f(c)}{2} [c+\delta-c+\delta]$

$> \frac{f(c)}{2} 2\delta$

$> \delta f(c)$

$\therefore \int_a^b f(x)dx > 0.$

3. If f is continuous on $[a,b]$ and $f(x) \geq 0$, $(a \leq x \leq b)$ and if $\int_a^b f(x)dx = 0$. P.T. f is identically zero on $[a,b]$.

Soln:

Suppose f is not identically zero on $[a,b]$.

Then, let c be point $[a,b]$ such that $f(c) > 0$.

$\Rightarrow \int_a^b f(x)dx > 0.$

which is a contradiction to

hypathesis $\int_a^b f(x)dx = 0.$

so no such c exist in $[a,b]$ with

$f(c)>0$. In athwards,

f is identically zero on $[a,b]$.

Derivatives:

Let f be real valued function on interval $J \subset R'$ if $c \in J$ we say that f has derivative at c if

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} \text{ exists}$$

If this limit exists we denated it by

$$f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}.$$

Theorem:

If real valued fun f as derivative at paint $c \in R'$ then f is continuous at c.

Proof:

For $x \neq c$ we have

$$f(x)-f(c) = \left[\frac{f(x)-f(c)}{x-c}\right] x-c \rightarrow (1)$$

Since $\lim_{x \to c}\left[\frac{f(x)-f(c)}{x-c}\right] = f'(c)$

and $\lim_{x \to c} x - c = 0$ then from equ ①

taking $\lim_{x \to c}$ on B.S.

$$\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot x - c$$

$$= f'(c) \cdot 0$$

$$= 0$$

$$\lim_{x \to c} [f(x) - f(c)] = 0 \to ②$$

Hence, since $f(x) = f(c) + [f(x) - f(c)]$

Taking $\lim_{x \to c}$ on B.S. we get

$$\lim_{x \to c} f(x) = \lim_{x \to c} [f(c) + [f(x) - f(c)]]$$

$$= \lim_{x \to c} f(c) + \lim_{x \to c} [f(x) - f(c)]$$

$$= f(c) + 0 \quad [\because \text{by ①}]$$

$$\lim_{x \to c} f(x) = f(c).$$

Hence, if f has derivative at point $c \in R'$ then f is continuous at c.

Theorem:

If f and g both has derivative at $c \in R'$ then so $f+g$, $f-g$, fg and

a) $(f+g)'(c) = f'(c) + g'(c)$,

b) $(f-g)'(c) = f'(c) - g'(c)$,

c) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

Futhermore if $g'(c) \neq 0$ then fun has derivative at c and $\left(\frac{f}{g}\right)' c = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$.

Proof:

Let $c \in R'$ then for $x \neq c$.

a) If $h = f+g \Rightarrow h(x) - h(c)$

$= (f+g)(x) - (f+g)(c)$

$= f(x) + g(x) - f(c) - g(c)$

$\div \; x-c$

$$\frac{h(x)-h(c)}{x-c} = \frac{f(x)-f(c)}{x-c} + \frac{g(x)-g(c)}{x-c} \rightarrow ②$$

Since f and g have derivative at c

i.e., $\lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right] = f'(c)$ &

$\lim_{x\to c}\left[\frac{g(x)-g(c)}{x-c}\right] = g'(c)$

Taking $\lim_{x\to c}$ on B.S. in equ ①

$$\lim_{x\to c}\left[\frac{h(x)-h(c)}{x-c}\right] = \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right] + \lim_{x\to c}\left[\frac{g(x)-g(c)}{x-c}\right]$$

$h'(c) = f'(c) + g'(c)$

$(f+g)'(c) = f'(c) + g'(c)$.

b) Let $h = f-g$

$\Rightarrow h(x) - h(c) = (f-g)(x) - (f-g)(c)$

$= f(x) - g(x) - f(c) + g(c)$

$= f(x) - g(c) - [g(x) - f(c)]$

$= f(x) - f(c) - [g(x) - g(c)]$

$\div\ x - c$

$$\frac{h(x) - h(c)}{x-c} = \frac{f(x) - f(c)}{x-c} - \frac{g(x) - g(c)}{x-c}$$

Since f and g have derivative at c

i.e., $\lim_{x \to c} \left[\frac{f(x) - f(c)}{x-c}\right] = f'(c)$

$\lim_{x \to c} \left[\frac{g(x) - g(c)}{x-c}\right] = g'(c).$

$$\lim_{x \to c} \left[\frac{h(x) - h(c)}{x-c}\right] = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x-c}\right] - \lim_{x \to c} \left[\frac{g(x) - g(c)}{x-c}\right]$$

$h'(c) = f'(c) - g'(c)$

$(f-g)'(c) = f'(c) - g'(c).$

c) $(fg)'(c) = f(c)g'(c) + g(c)f'(c).$

Let $h = fg$

$h(x) - h(c) = fg(x) - fg(c)$

$= f(x)g(x) - f(c)g(c)$

$= f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)$

$= g(x)[f(x) - f(c)] + f(c)[g(x) - g(c)]$

$$\frac{h(x) - h(c)}{x-c} = g(x)\left[\frac{f(x) - f(c)}{x-c}\right] + f(c)\left[\frac{g(x) - g(c)}{x-c}\right]$$

Taking $\lim_{x \to c}$ on B.S.

$$\lim_{x\to c} \frac{h(x)-h(c)}{x-c} = \lim_{x\to c}\left[\frac{f(x)-f(c)}{x-c}\right]\cdot g(x) + \lim_{x\to c} f(c)\cdot\left[\frac{g(x)-g(c)}{x-c}\right]$$

$$h'(c) = f'(c)g(c) + f(c)g'(c)$$

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

iv) Let $h = \frac{f}{g}$ and $c \in R'$,

$f'(c)$ and $g'(c)$ exist with $g'(c) \neq 0$.

$$h(x) - h(c) = \left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(c)$$

$$= \frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}$$

$$= \frac{f(x)g(c) - g(x)f(c)}{g(x)g(c)}$$

$$= \frac{f(x)g(c) + f(c)g(c) - f(c)g(c) - g(x)f(c)}{g(x)g(c)}$$

$$= \frac{[f(x)-f(c)]g(c) - f(c)[g(x)-g(c)]}{g(x)g(c)}$$

$$h(x) - h(c) = \frac{[f(x)-f(c)]g(c) - f(c)[g(x)-g(c)]}{g(x)g(c)}$$

$\div\ x-c$

$$\frac{h(x)-h(c)}{x-c} = \frac{1}{g(x)g(c)}\left[\frac{[f(x)-f(c)]g(c)}{x-c} - \right.$$

$$\left.\frac{f(c)[g(x)-g(c)]}{x-c}\right]$$

Taking $\lim_{x\to c}$ on B.S.

$$\lim_{x\to c}\frac{h(x)-h(c)}{x-c} = \lim_{x\to c}\left\{\frac{\left[\frac{f(x)-f(c)}{x-c}\right]g(c) - \left[\frac{g(x)-g(c)}{x-c}\right]f(c)}{g(x).g(c)}\right\}$$

$$h'(c) = \frac{f'(c)g(c)-g'(c)f(c)}{g(c).g(c)}$$

$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c)-g'(c)f(c)}{[g(c)]^2}$$

Chain rule for derivative:

Theorem:

Suppose f has derivative at c and that g has derivative at $f(c)$ then $\phi = g \circ f$ has derivative at c and $\phi'(c) = g'[f(c)]f'(c)$.

Proof:

Let $k(h) = f(c+h) - f(c) \rightarrow ①$

then $\lim_{h\to 0}\frac{k(h)}{h} = f'(c) \rightarrow ②$

Also $\lim_{h\to 0} k(h) = 0 \rightarrow ③$

Case i)

Let $f'(c) \neq 0$

From equ ① for sufficiently small $h \neq 0$, $\frac{K(h)}{h} \neq$

and $K(h) \neq 0$.

For such h,

$$\frac{\phi(c+h)-\phi(h)}{h} = \frac{\phi[f(c+h)] - g[f(c)]}{h}$$

$$\frac{\phi(c+h)-\phi(c)}{h} = \frac{\phi[K(h)+f(c)] - g[f(c)]}{h\, K(h)} \cdot K(h) \rightarrow ④$$

Taking $\lim_{h \to 0}$ on B.S.

$$\lim_{h \to 0} \frac{\phi(c+h)-\phi(c)}{h} = \lim_{h \to 0} \frac{\phi[K(h)+f(c)] - g[f(c)]}{K(h)}$$

$$\lim_{h \to 0} \frac{K(h)}{h}$$

$$\phi'(c) = g'[f(c)] \cdot f'(c).$$

Case ii)

If $f'(c) = 0$ there it is possible that $K(h) = 0$ for h arbitrary small.

If $K(h) = 0$ then $f(c+h) = f(c)$

So, $$\frac{\phi(c+h)-\phi(c)}{h} = \frac{g[f(c+h)] - g[f(c)]}{h} = 0 \rightarrow$$

if $K(h) \neq 0$ then $f'(c) = \lim_{h \to 0} \frac{K(h)}{h} = 0$, the quantity $\frac{K(h)}{h}$ will be close to zero.

when h is sufficiently small.

Also if $K(h) \neq 0$ the quantity.

$g[f(c)+k(h)] - g[f(c)]$ will be close to $g'[f(c)]$ when h is small hence by equ (5)

$$\frac{\phi(c+h)-\phi(c)}{h} = 0, \text{ if } k(h) = 0 \text{ it}$$

follows from that $f'(c) = 0$.

$$\therefore \lim_{h\to 0} \frac{\phi(c+h)-\phi(c)}{h} = 0 = g'[f(c)]\, f'(c).$$

$$\therefore \phi'(c) = g'[f(c)]\, f'(c).$$

Theorem :

Let f be one-one real valued fun on interval J. Let ϕ be inverse function for f. If f is continuous at c $(c \in J)$ and if ϕ has derivative at $d = f(c)$ with $\phi'(c) \neq 0$ then $f'(c)$ exist and $f'(c) = \dfrac{1}{\phi'(d)}$.

Proof :

For $h \neq 0$,

Let $k(h) = f(c+h) - f(c)$. since f is one-one, $k(h) \neq 0$ iff $h \neq 0$.

Then $d + k(h) = f(c) + k(h) = f(c+h)$

hence $\phi(d+k(h)) = \phi[f(c+h)] = c+h$

we have
$$\frac{f(c+h)-f(c)}{h} = \frac{d+k(h)-d}{c+h-c}$$

$$= \frac{k(h)}{\phi[d+k(h)] - \phi(d)}$$

$$= \frac{1}{\frac{\phi[d+k(h)]-\phi(d)}{k(h)}}$$

But $\lim_{h\to 0} k(h) = 0 \longrightarrow ①$

Since by hypothesis, f is continuous at c. Thus f approaches zero, the RHS of equ ① approaches $\lim \frac{1}{\phi'(d)}$.

Higher order derivatives:

Suppose $f'(x)$ exists for every x in some interval J. If $c \in J$ and if $\lim_{x\to c} \frac{f'(x)-f'(c)}{x-c}$ exists we say that f has second derivative at c i.e.,

$$f''(c) = \lim_{x\to c} \frac{f'(x)-f'(c)}{x-c}$$

III^ly

n^{th} derivative of f at c is defined as

$$f^{(n)}(c) = \lim_{x\to c} \frac{f^{(n-1)}(x)-f^{(n-1)}(c)}{x-c}.$$

provides $f^{(n-1)}(x)$ exists $\forall x$ in interval containing c & provided that limit exist.