

Subject : OPERATIONS RESEARCH

Subject code : CST 53

Unit : II

Unit-2

Simplex method

Definition [General linear programming problem]

The linear programming involving more than 2 variables may be expressed as follows

$$\text{In maximum or minimum } Z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

Subject to constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 = \alpha_1 \geq \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 = \alpha_2 \geq \\ a_{31}x_1 + a_{32}x_2 + \dots + a_{3n}x_n &\leq b_3 = \alpha_3 \geq \\ &\vdots && \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m = \alpha_m \geq \end{aligned}$$

and the non negativity restrictions

$$x_1, x_2, x_3, \dots, x_n \geq 0$$

Definition (solution)

A set of values  $x_1, x_2, \dots, x_n$  in which satisfies the constraints of the LPP is called its solution

Feasible solution: Any solution to LPP which satisfies all the non-negativity restrictions of the LPP is called the feasible solution.

Optimum Solution / optimal solution

Any feasible solution which optimizes (maximize / minimize) the objective function of the LPP is called its optimum solution.

### ⑧ Slack variables

If the constraints of a general LPP be

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \text{ where } i = 1, 2, \dots, k$$

Then the non-negative variable  $s_i$  which are introduced to convert the inequalities to the equalities

$$\text{i.e., } \sum_{j=1}^n a_{ij} x_j + s_i = b_i$$

are called the slack variables.

If the constraints of the general LPP  
be

$$\sum_{j=1}^n a_{ij} x_j \geq b_i \quad (i = k, k+1, k+2, \dots)$$

then the non-negative variables  $s_i$  which are introduced to convert the inequality to the equality

i.e.,  $\sum_{j=1}^n a_{ij} x_j - s_i = b_i \quad (i = k, k+1, \dots)$

are called the surplus variable.

canonical form of LPP

Maximize  $Z = c_1 x_1 + c_2 x_2 + c_3 x_3 + \dots + c_n x_n$

subject to constraints

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \leq b_2$$

$\vdots \quad \vdots \quad \vdots$

$\vdots \quad \vdots \quad \vdots$

$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \leq b_m$$

and the non-negativity restrictions

$$x_1, x_2, \dots, x_n \geq 0$$

This form of LPP is called the  
canonical form of LPP

characteristic of the canonical form;

\* The objective function is of maximization type.

\* All constants are  $\leq$  type.

\* All variables  $x_i$  are non-negative.

2m\*

Definition: standard form of LPP

The general linear programming problem in the form

$$\text{maximize } Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

and  $x_1, x_2, \dots, x_n \geq 0$  is known

as standard form of LPP.

characteristics of standard form

\* The objective function is of maximization type.

\* All constraints are expressed as equations

\* Right hand side of each constraint is non-negative

\* All variables are non-negative

basic solution:

A system of  $m$  linear equations with  $n$  variables ( $m < n$ ) : the solution obtained by setting  $(n-m)$  variables equal to zero and solving for the remaining  $m$  variables is called basic solution.

Non-basic solution:

The  $m$  variable are called basic variables and they form the basic solution. The  $(n-m)$  variables which are put to zero are called as non-basic solution.

Non-degenerate basic solution:

A basic solution is said to be <sup>non-</sup> degenerate basic solution if none of the basic variable is zero.

Degenerate basic solution:

A basic solution is said to be degenerate basic solution if one or more of the basic variables are zero.

Basic Feasible solution:

A feasible solution which is also basic is called a basic feasible solution.

## Simplex method

Solve the following L.P.P by using  
Simplex method

$$\text{maximize } z = 4x_1 + 10x_2$$

Subject to

$$2x_1 + x_2 \leq 50$$

$$2x_1 + 5x_2 \leq 100$$

$$2x_1 + 3x_2 \leq 90$$

and  $x_1, x_2 \geq 0$

Solution:

By introducing the slack variable  
 $s_1, s_2$  and  $s_3$  the problem in standard  
form becomes

$$\text{maximize } Z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

$$2x_1 + x_2 + 1s_1 + 0s_2 + 0s_3 = 50$$

$$2x_1 + 5x_2 + 0s_1 + 1s_2 + 0s_3 = 100$$

$$2x_1 + 3x_2 + 0s_1 + 0s_2 + 1s_3 = 90$$

and  $x_1, x_2, s_1, s_2, s_3 \geq 0$

$x_1, x_2$  are variable

$s_1, s_2, s_3$  are slack variable

Since there are 3 equations with 5 variables

The initial basic feasible solution is

$$(5-3) = 2$$

$$\therefore S_1 = 50$$

$$S_2 = 100$$

$$S_3 = 90$$

Pivot element = 1

The initial simplex table is given by

	$C_j$	4	10	0	0	0		
$C_B$	Basic	$x_B$	$x_1$	$x_2$	$S_1$	$S_2$	$S_3$	B
	$y_B$							Ratio
0	$S_1$	2	1	10	0	50		$\frac{50}{1} = 50$
0	<u><math>S_2</math></u>	2	5	0	1	6	100	<u><math>\frac{100}{5} = 20</math></u>
0	$S_3$	2	3	0	0	1	90	$\frac{90}{3} = 30$
	$Z_j$		0	0	0	0	0	

$$Z_j - C_j \quad -4 \quad -10 \quad 0 \quad 0 \quad 0$$

↑  
min

Leaving variable  $\rightarrow S_2$

Entering variable  $\rightarrow x_2$

Pivot element  $\rightarrow 5$

New  $x_2$

$$\frac{2}{5} \quad \frac{5}{5} \quad \frac{0}{5} \quad \frac{1}{5} \quad \frac{0}{5} \quad \frac{100}{5}$$

old  $S_1$  - new  $S_2$

$$\begin{array}{cccccc}
 2 & 1 & 1 & 0 & 0 & 50 \\
 \frac{2}{5} & 1 & 0 & \frac{1}{5} & 0 & 20 \\
 \hline
 \frac{8}{5} & 0 & 1 & -\frac{1}{5} & 0 & 30
 \end{array}$$

old  $S_3$  - 3 new  $S_2$

$$\begin{array}{cccccc}
 2 & 3 & 0 & 0 & 1 & 90 \\
 \frac{6}{5} & 3 & 0 & \frac{3}{5} & 0 & 60 \\
 \hline
 \frac{4}{5} & 0 & 0 & -\frac{3}{5} & 1 & 30
 \end{array}$$

$C_B$	$c_j$	H	10	0	0	0	B	Ratio
$Y_B$	BASIC	$x_1$	$x_2$	$S_1$	$S_2$	$S_3$		
0	$S_1$	$\frac{8}{5}$	0	1	$-\frac{1}{5}$	0	30	
10	$x_2$	$\frac{20}{5}$ $\frac{2}{5}$	1	0	$\frac{1}{5}$	0	20	
0	$S_3$	$\frac{4}{5}$	0	0	$-\frac{3}{5}$	1	30	
	$Z_j^*$	4	10	0	2	0	200	
	$Z_j^* - c_j$	0	0	0	2	0		

Since all  $Z_j^* - c_j \geq 0$

Therefore the basic feasible solution  
is optimal.

Therefore the optimal solution is

$$\max z = 800$$

$$x_1 = 0, x_2 = 20$$

2) use simplex method to

$$\text{minimize } I = x_2 - 3x_3 + 2x_5$$

subject to constraints

$$8x_2 - x_3 + 2x_5 \leq 7$$

$$-2x_2 + 4x_3 \leq 12$$

$$-4x_2 + 3x_3 + 8x_5 \leq 10$$

$$x_2 \geq 0, x_3 \geq 0, x_5 \geq 0$$

Solution :

Since the given objective function  
is of minimisation type, we shall  
convert it into a maximisation  
type

$$\text{Therefore } \max(-z) = \max z^*$$

$$z^* = -x_2 + 3x_3 - 2x_5$$

Subject to

$$8x_2 - x_3 + 2x_5 \leq 7$$

$$-2x_2 + 4x_3 \leq 12$$

$$-4x_2 + 3x_3 + 8x_5 \leq 10$$

by introducing non-negative  
slack variables  $s_1, s_2$  and  $s_3$ ,  
therefore the standard form of  
the LPP becomes

$$\text{maximize } z^* = -x_2 + 3x_3 - 2x_4 + 0x_5 + 0x_6$$

subject to

$$3x_2 + x_3 + 2x_4 + 1x_5 + 0x_6 + 0x_7 = 7$$

$$+ 2x_2 + 4x_3 + 0x_4 + 1x_5 + 0x_6 = 12$$

$$- 11x_2 + 9x_3 + 8x_4 + 0x_5 + 0x_6 + 1x_7 = 10$$

and

$$x_2, x_3, x_4, s_1, s_2, s_3 \geq 0$$

$\therefore$  The initial basic feasible solution  
is given by,

$$s_1 = 7$$

$$s_2 = 12$$

$$s_3 = 10$$

		$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$s_3$	Ratio
CB	$y_B$	0	1	0	1	0	0	-1
0	$s_1$	3	-1	2	0	0	0	-1
0	$s_2$	-2	4	0	0	1	0	12
0	$s_3$	-11	9	8	0	0	1	10
		$x_2^*$	0	0	0	0	0	
		$x_3^*$	1	-3	2	0	0	
		$x_4^*$	0	0	0	0	0	

since  $(z^* - c_2) < 0$  the current basic feasible solution is not optimal

the non basic  $x_3$  enters into the basis and the basic variable  $s_2$  leaves the basis.

$c_B$	$y_B$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	Ratio
0	$s_1$	$\frac{5}{2}$ <sup>Pivot</sup>	0	2	1	$\frac{1}{4}$	0
3	$s_3$	$-\frac{1}{2}$	1	0	0	$\frac{1}{4}$	0
0	$s_3$	$-\frac{5}{2}$	0	8	0	$-\frac{3}{4}$	1
	$z_j$	$-\frac{5}{2}$	3	0	0	$\frac{3}{4}$	0
	$z_j - c_j$	$-\frac{1}{2}$	0	2	0	$\frac{3}{4}$	0

old  $s_3 - 3x$  new  $x_3$

$$\begin{array}{ccccccc}
 & -4 & 3 & 8 & 0 & 0 & 1 & 10 \\
 \text{e)} & -\frac{3}{2} & 3 & 0 & 0 & \frac{3}{4} & 0 & 9 \\
 & \hline
 & -\frac{5}{2} & 0 & 8 & 0 & -\frac{3}{4} & 1 & 1
 \end{array}$$

Since  $(z^* - c_1) < 0$  the current basic feasible solution is not optimal

The non-basic variable  $x_2$  enters the basis and the basic variable  $x_1$  leaves the basis.

old  $\bar{z}_2 + \frac{1}{2}$  new  $\bar{z}_1$

$$-\frac{1}{2} \quad 1 \quad 0 \quad 0 \quad -\frac{1}{4} \quad 0 \quad 3$$

$$\frac{1}{2} \quad 0 \quad \frac{2}{5} \quad \frac{1}{5} \quad \frac{1}{20} \quad 0 \quad 2$$

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$$0 \quad 1 \quad \frac{2}{5} \quad \frac{1}{5} \quad \frac{3}{10} \quad 0 \quad 5$$

CB	Basic	-1	3	-2	0	0	0
	YB	$x_2$	$x_3$	$x_5$	$s_1$	$s_2$	$s_3$

$$-1 \quad x_2 \quad 1 \quad 0 \quad \frac{4}{5} \quad \frac{2}{5} \quad \frac{1}{10} \quad 0 \quad 4$$

$$3 \quad x_3 \quad 0 \quad 1 \quad \frac{2}{5} \quad \frac{1}{5} \quad \frac{3}{10} \quad 0 \quad 5$$

$$0 \quad s_3 \quad 0 \quad 0 \quad 10 \quad -\frac{1}{2} \quad 1 \quad 1$$

$$z_j^* - c_j \quad -1 \quad 3 \quad \frac{2}{5} \quad \frac{1}{5} \quad \frac{4}{5} \quad 0$$

$$S_1 \times \frac{2}{5}$$

$$z_j^* - c_j \quad 0 \quad 2 \times \frac{1}{5} \quad 1 \times \frac{2}{5} \quad \frac{1}{5} \times \frac{1}{5} \quad 0$$

$$\frac{2}{10} \times \frac{1}{5}$$

Since all  $z_j^* - c_j \geq 0$

The current basic feasible solution is optimal solution is given by

maximize  $Z^* = 11$ ,  $a_2 = 4$ ,  $a_3 = 5$ ,  $a_5 = 0$

but minimize  $Z = -\max Z^*$   
 $= -11$

minimize  $Z = -11$ ,  $a_2 = 4$ ,  $a_3 = 5$ ,  $a_5 = 0$

3) A gear manufacturing company received an order for 3 specific types of gears for regular supply. The management is considering to devote the available excess capacity to 1 or more of the three type say A, B and C. The available capacity on the machines which might limit output and the no. of machine hours required for each unit of the respective gear is also given below

Machine type	Available machine hours/week	productivity hours/unit		
		Gear A	Gear B	Gear C
Gear hobbing	250	8	2	3
Gear shaping	150	4	3	0
Face grinding	50	2	-	1

The unit profit would be Rs 20,  
Rs 6 and Rs 8 respectively for  
the gears A, B and C.

Find how much of each gear  
the company should produce in  
order to maximize profit.

Solution:

Let  $x_1$ ,  $x_2$  and  $x_3$  be the no. of  
units of gears A, B and C

Therefore the mathematical formulation  
of LPP is given by

$$\text{max } z = 20x_1 + 6x_2 + 8x_3$$

Subject to constraints

$$8x_1 + 2x_2 + 3x_3 \leq 250$$

$$4x_1 + 3x_2 \leq 150$$

$$2x_1 + 7x_3 \leq 50$$

and  $x_1, x_2, x_3 \geq 0$

Therefore the slack variables are  
 $s_1$ ,  $s_2$  and  $s_3$

Therefore the standard form of  
 the LPP

$$\text{maximize } Z = 20x_1 + 6x_2 + 8x_3 + 0s_1 + \\ 0s_2 + 0s_3$$

subject to constraints

$$8x_1 + 2x_2 + 3x_3 + s_1 + 0s_2 + 0s_3 = 250 \\ 4x_1 + 3x_2 + 0s_1 + s_2 + 0s_3 = 150 \\ 2x_1 + 7x_3 + 0s_1 + 0s_2 + s_3 = 50$$

$$\text{and } x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

$c_j$	20	6	8	0	0	0		
$c_B$	$y_B$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	Ratio
0	$s_1$	B	2	3	1	0	0	250
0	$s_2$	4	3	0	0	1	0	150
0	$s_3$	2	0	-7	0	0	1	50
	$z_j$	0	0	0	0	0	0	
	$Z_j$	-20	-6	-8	0	0	0	
	$c_j$							

↑

$$\frac{25}{2} \quad \frac{9}{2} \quad \frac{3}{2} \quad \frac{0}{2} \quad \frac{9}{2} \quad \frac{1}{2}$$

$C_j$	20	6	8	0	0	0		
CB	$Y_B$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	B
0	$s_1$	0	2	-1	1	0	-4	$\frac{50}{2}$
0	$s_2$	0	3	-2	0	0	-2	$\frac{50}{2}$
20	$x_1$	1	0	$y_2$	0	0	$\frac{1}{2}$	$\frac{25}{2}$
$x_1^*$	00	0	10	0	0	10		
$x_1^*$	0	-6	2	0	0	10		
$c_j$								

$C_j$	20	6	8	0	0	0	Ratio	
CB	$Y_B$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$	B
0	$s_1$	0	0	$\frac{1}{3}$	1	$-\frac{2}{3}$	$-\frac{8}{3}$	$\frac{50}{3}$
6	$x_2$	0	1	$-\frac{2}{3}$	0	0	$-\frac{2}{3}$	$\frac{50}{3}$
20	$x_1$	1	0	$\frac{1}{2}$	0	0	$\frac{1}{2}$	25
$x_1^*$	20	6	6	0	2	17		
$x_1^*$	0	0	-2	0	2	17		
$c_j$								

Leaving -  $x_1$ , Entering  $\rightarrow x_3$

CB	$Y_B$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	$s_3$
0	$s_1$	$-\frac{2}{3}$	0	0	1	$-\frac{2}{3}$	-3
6	$x_2$	$-\frac{4}{3}$	1	0	0	$\frac{1}{3}$	0
8	$x_3$	2	0	1	0	0	1
$x_3^*$	24	6	8	0	2	8	

### Artificial variables:

To solve a LPP by simplex by simplex method we have to start with the initial basic feasible solution and construct the initial simplex table in the previous problem, we see that the slack variables provided the initial basic feasible solutions. However in some problems the slack variables cannot provide the initial basic feasible solution. In this problems atleast one of the constraints is of  $=$  or  $\geq$  type to solve such a linear programming problem, there are 2 methods available.

1) The Big M - method (or)

M - technique (or) the method of penalties

2) the two phase method.

problem

big M - method

Solve the following problem by simplex method

maximize  $Z = x_1 + 2x_2 + 3x_3 - x_4$

subject to

$$x_1 + 2x_2 + 3x_3 = 15 \quad \text{with } x_1 \geq 0$$

$$2x_1 + x_2 + 5x_3 \geq 20$$

$$x_1 + 2x_2 + x_3 + x_4 \geq 10 \quad \text{surplus } (-)$$

$$x_1, x_2, x_3, x_4 \geq 0$$

solution

By introducing the non-negative surplus variable  $s_1$  and  $s_2$

$$\therefore \text{maximize } Z = x_1 + 2x_2 + 3x_3 - x_4 + 0s_1 + 0s_2$$

subject to

$$x_1 + 2x_2 + 3x_3 + 0s_1 + 0s_2 = 15$$

$$2x_1 + x_2 + 5x_3 - s_1 + 0s_2 = 20$$

$$x_1 + 2x_2 + x_3 + x_4 + 0s_1 - s_2 = 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

To get the basic feasible solution

the artificial variable  $A_1, A_2, A_3$

to the left hand side of the constraints

eqn which does not passes the slack

variables and assign  $-M$  to the

-function

"The LPP becomes

$$\text{maximum } Z = 2x_1 + 2x_2 + 5x_3 - 2x_4 + 0x_5 + A_1 - Mx_1 - Mx_2 - Mx_3$$

Subject to constraints

$$x_1 + 2x_2 + 5x_3 + 0x_4 + 0x_5 + A_1 = 15$$

$$2x_1 + x_2 + 5x_3 - x_4 + 0x_5 + A_2 = 20$$

$$x_1 + 2x_2 + x_3 + x_4 + 0x_5 - x_2 + A_3 = 10$$

$$x_1, x_2, x_3, \overset{x_4}{S_1}, S_2, A_1, A_2, A_3 \geq 0$$

Initial Iteration

$C_B$	$C_j$	1	2	3	-1	0	0	-M	-M	-M	Ratio
$y_B$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$S_1$	$S_2$	$A_1$	$A_2$	$A_3$	
-M	$A_1$	1	2	5	0	0	0	1	0	0	15/5=3
-M	$A_2$	2	1	5	0	-1	0	0	1	0	20/5=4
-M	$A_3$	1	0	1	1	0	-1	0	0	1	10/1=10
	$Z_j$	-1M	-5M	-9M	-M	M	M	-M	-M	-M	
	$Z_j^*$	-4M-1	-5M-2	-9M-3	-M+1	M	M	0	0	0	
	$C_j$										

Since  $Z_j^* - C_j < 0$

1	0	-1	0	0	0
1	0	-1	0	0	0
0	0	0	1	0	0
0	0	0	0	1	0

	$c_j$	1	2	3	-1	0	0	$m$	$-m$	
CB	$\bar{Y}_B$	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$A_1$	$A_3$	Ratio
-M	$A_1$	$-1/5$	$\boxed{1/5}$	0	0	$3/5$	0	1	0	$15/7$
3	$x_3$	$2/5$	$1/5$	1	0	$-1/5$	0	0	0	0
-M	$A_3$	$3/5$	$9/5$	0	1	$1/5$	-1	0	1	$30/9$
	$Z_j$	$\frac{-2m+6}{5}$	$\frac{-16m+3}{5}$	3	$-m$	$\frac{-4m+3}{5}$	$m$	$-m$	$-m$	

$$Z_j - \frac{-2m+11}{5} - \frac{16m-7}{5} 0 -m+1 - \frac{4m+3}{5} m 0 0$$

$c_j$

	$c_j$	1	2	3	-1	0	0	$m$	
CB	$\bar{Y}_B$	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$A_3$	
2	$x_1$	$-1/4$	1	0	0	$3/4$	0	0	
3	$x_3$	$3/4$	0	1	0	$-2/4$	0	0	
-M	$A_3$	$6/4$	0	0	1	$-4/4$	-1	1	

$$Z_j - \frac{-6m+7}{4} 2 3 -m \frac{4m}{4} m -m$$

$$Z_j - \frac{-6m}{4} 0 0 -m+1 \frac{4m}{4} m 0$$

$c_j$

$Z_j - c_j < 0$ , the current basic

feasible solution is not optimal

$x_4$  → Enter

$A_3$  → Leave

$c_j$	1	2	3	-1	0	0		
$C_B$	$y_B$	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$	$x_5$
2	$x_2$	$-1/1$	1	0	0	$3/1$	0	
3	$x_3$	$3/1$	0	1	0	$-2/1$	0	
1	$x_4$	$6/1$	0	0	1	$-4/1$	1	
	$z_j^*$	$1/1$	2	3	4	$4/1$	1	
	$z_j^* - c_j$	$-6/1$	0	0	0	$4/1$	1	
	$c_j$							

	1	2	3	-1	0	0	
$C_B$	$y_B$	$x_1$	$x_2$	$x_3$	$x_4$	$s_1$	$s_2$
2	$x_2$	0	1	0	$1/6$	$1/3$	$-1/6$
3	$x_3$	0	0	1	$-1/2$	0	$1/2$
1	$x_1$	1	0	0	$7/6$	$-2/3$	$-1/6$
	$z_j^*$	1	2	3	0	0	0
	$z_j^* - c_j$	0	0	0	1	0	0
	$c_j$						

since all  $z_j^* - c_j \geq 0$ , the current basic feasible solution is optimal

∴ the optimal solution is

maximum  $Z = 15$

$$x_1 = 5/2, x_3 = 5/2$$

$$x_2 = 5/2, x_4 = 0$$

7/2

## Duality of LPP

### Formulation of dual problem

These are two important forms of primal-dual pairs, namely symmetric form and unsymmetric form

#### Symmetric form

Consider the following LPP

$$\text{maximum } Z = c_1x_1 + c_2x_2 + c_3x_3 + \dots + c_nx_n$$

Subject to constraints

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2$$

.....

.....

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m$$

$$\text{and } x_1, x_2, \dots, x_n \geq 0$$

i.e., maximum  $\rightarrow Z = cx$

subject to

$$Ax \leq b$$

$$x \geq 0$$

where  $c = c_1, c_2, \dots, c_n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Problem

- 1) Find the dual of the following LPP

$$\text{maximum } z = x_1 + 2x_2 + x_3$$

Subject to

$$2x_1 + x_2 - x_3 \leq 2$$

$$-2x_1 + x_2 - 5x_3 \geq -6$$

$$4x_1 + x_2 + x_3 \leq 16$$

$$x_1, x_2, x_3 \geq 0$$

Solution

The given optimal LPP is

$$\text{maximum } z = x_1 + 2x_2 + x_3$$

Subject to

$$2x_1 + x_2 - x_3 \leq 2$$

$$-2x_1 + x_2 - 5x_3 \geq -6$$

$$4x_1 + x_2 + x_3 \leq 16$$

$$x_1, x_2, x_3 \geq 0$$

That is

$$\max z = x_1 + 2x_2 + x_3$$

subject to

$$2x_1 + x_2 - 2x_3 \leq 2$$

$$2x_1 - x_2 + 5x_3 \leq 6$$

$$4x_1 + x_2 + x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0$$

Primal  $\longrightarrow$

$$x_1 \quad x_2 \quad x_3 \leq \text{value}$$

$$\begin{array}{ccccc} \text{Dual } y_1 & 2 & 1 & -1 & 2 \\ \downarrow & y_2 & 2 & -1 & 5 \\ y_3 & 4 & 1 & 1 & 6 \end{array}$$

$$\begin{array}{ccccc} \geq & & & & \\ & 1 & & 0 & 1 \\ \text{value} & & & & \end{array}$$

Since the primal problem is maximum type with constraints 3 and variables 3, the dual problem is minimization type (2) with 3 constraints and 3 variables  $y_1, y_2, y_3$ .

∴ The dual problem is

$$\min z' = 2y_1 + 6y_2 + 6y_3$$

subject to

$$2y_1 + 2y_2 + 4y_3 \geq 1$$

$$y_1 - y_2 + y_3 \geq 2$$

$$-y_1 + 5y_2 + y_3 \geq 1$$

and  $y_1, y_2, y_3 \geq 0$

2) Exhibit the dual of LPP

$$\text{minimum } z = 3x_1 - 2x_2 + 4x_3$$

Subject to

$$3x_1 + 5x_2 + 4x_3 \geq 7$$

$$6x_1 + x_2 + 3x_3 \geq 4$$

$$7x_1 - 2x_2 - x_3 \leq 10$$

$$x_1 - 2x_2 + 5x_3 \geq 3$$

$$4x_1 + 7x_2 - 2x_3 \geq 2$$

The given primal LPP is

$$\text{minimum } Z = 3x_1 - 2x_2 + 4x_3$$

Subject to

$$3x_1 + 5x_2 + 4x_3 \geq 7$$

$$6x_1 + x_2 + 3x_3 \geq 4$$

$$7x_1 - 2x_2 - x_3 \leq 10$$

$$x_1 - 2x_2 + 5x_3 \geq 3$$

$$4x_1 + 7x_2 - 2x_3 \geq 2$$

That is

$$\text{minimum } z = 3x_1 - 2x_2 + 4x_3$$

subject to

$$3x_1 + 6x_2 + 4x_3 \geq 7$$

$$6x_1 + x_2 + 3x_3 \geq 4$$

$$-7x_1 + 2x_2 + x_3 \geq 10$$

$$x_1 - 2x_2 + 5x_3 \geq 3$$

$$4x_2 + 7x_3 - 2x_1 \geq 2$$

Primal  $\longrightarrow$

$$x_1 \quad x_2 \quad x_3 \quad \geq \text{value}$$

Dual

	$y_1$	3	5	4	7
$\downarrow$	$y_2$	6	1	3	4
	$y_3$	-7	2	1	10
	$y_4$	1	-2	5	3
	$y_5$	4	7	-2	2
$\leq$		3	-2	4	
	value				

$\therefore$  The dual problem is

$$\max z' = 7y_1 + 4y_2 + 10y_3 + 3y_4 + 2y_5$$

subject to

$$3y_1 + 6y_2 - 7y_3 + 4y_4 + 4y_5 \leq 3$$

$$5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2$$

$$4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4$$

b) Write the dual of LPP

$$\text{maximum } z = 3x_1 + 10x_2 + 2x_3$$

s.t.

$$2x_1 + 3x_2 + 2x_3 \leq 7$$

$$3x_1 - 2x_2 + 4x_3 = 3$$

$$\text{and } x_1, x_2, x_3 \geq 0$$

Solution

Given primal LPP is

$$\text{maximum } z = 3x_1 + 10x_2 + 2x_3$$

s.t.

$$2x_1 + 3x_2 + 2x_3 \leq 7$$

$$3x_1 - 2x_2 + 4x_3 = 3$$

$$x_1, x_2, x_3 \geq 0$$

Since the primal problem contains 2 constraints and 3 variable,  
the dual problem will contain 3 constraints and 2 dual variable.

Since the 2<sup>nd</sup> constraint in the primal problem is an equality,  
the corresponding 2<sup>nd</sup> dual variable  $y_2$  is unrestricted in sign.

Therefore the dual problem

B

optimal  $\rightarrow$

$$x_1 \quad x_2 \quad x_3 \leq \text{value}$$

$$\text{Dual } y_1 \quad 2 \quad 9 \quad 2 \quad 7$$

$$\downarrow \quad y_2 \quad 3 \quad -2 \quad 4 \quad 5$$

$$\geq \quad 3 \quad 10 \quad 2$$

value

$$\min z' = 7y_1 + 3y_2$$

Subject to

$$2y_1 + 3y_2 \geq 3$$

$$8y_1 - 2y_2 \geq 10$$

$$5y_1 + 4y_2 \geq 2$$

$$y_1 \geq 0$$

$y_2$  is unrestricted

(Q) Write down the dual of the following LPP and solve it (or)

Hence write down the solution of the primal

$$\text{maximum } z = 4x_1 + 2x_2$$

Subject to constraints

$$-x_1 - x_2 \leq -3$$

$$-x_1 + x_2 \geq -2$$

$$x_1, x_2 \geq 0$$

### Solution

Given the problem of the primal LPP is

$$\text{maximum } z = 4x_1 + 2x_2$$

Subject to

$$-x_1 - x_2 \leq -3$$

$$x_1 - x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

The dual problem is

$$\min z' = -8y_1 + 2y_2$$

$$-y_1 + y_2 \geq 4$$

$$-y_1 - y_2 \geq 2$$

$$y_1, y_2 \geq 0$$

$$\begin{array}{l} \text{max } z^1 = 3y_1 - 2y_2 \\ \hline -y_1 + y_2 \boxed{1} \\ -y_1 - y_2 \boxed{2} \end{array}$$

add multiple  
variables

$$\text{max } z = 3y_1 - 2y_2 + 0S_1 + 0S_2 - MA_1 - MA_2$$

$$-y_1 + y_2 - 1S_1 + 0S_2 + A_1 = 4$$

$$-y_1 - y_2 + 0S_1 - 1S_2 + A_2 = 2$$

$\rightarrow$   $y_1 = 2(x)$   
Artificial

$$y_1, y_2, A_1, A_2 \geq 0$$

Initial

$c_j^0$	3	-2	0	0	-M	-M
$C_B$	$y_B$	$y_1$	$y_2$	$S_1$	$S_2$	$A_1$
-M	$A_1$	-1	1	-1	0	1
-M	$A_2$	-1	-1	0	-1	0

$$z_j^0 = 2m - 3$$

$$z_j^0 = 2m - 3 - 2 = m - M$$

$$c_j$$

since all  $z_j^0 - c_j \geq 0$  and the artificial variable  $A_1$  and  $A_2$  appear in the basis at non-zero level.

$\therefore$  There exist no finite optimum solution to the given primal LPP

Apply the principle of duality to  
solve the LPP

$$\text{maximum } z = 3x_1 + 2x_2$$

Subject to

$$x_1 + x_2 \geq 1$$

$$x_1 + x_2 \leq 7$$

$$x_1 + 2x_2 \leq 10$$

$$x_2 \leq 3$$

and  $x_1, x_2 \geq 0$

Solution

Given the problem of the  
primal LPP is

$$\text{maximum } z = 3x_1 + 2x_2,$$

s.t.

$$x_1 + x_2 \leq 1$$

$$x_1 + x_2 \leq 7$$

$$x_1 + 2x_2 \leq 10$$

$$x_2 \leq 3$$

and  $x_1, x_2 \geq 0$

The dual problem is

$$\min z' = -y_1 + -y_2 + 10y_3 + 3y_4$$

$$-y_1 + y_2 + y_3 \geq 3$$

$$-y_1 + y_2 + 2y_3 + y_4 \geq 2$$

$$\text{maxi } z' = y_1 - y_2 - 10y_3 - 3y_4$$

$$y_1 - y_2 - 10y_3 - 3y_4$$

$$-y_1 + y_2 + y_3 \geq 3$$

$$-y_1 + y_2 + 2y_3 + y_4 \geq 2$$

$$\max z : y_1 + 7y_2 - 10y_3 - 3y_4 + 0s_1 + 0s_2 \\ - MA_1 - MA_2$$

$$-y_1 + y_2 + y_3 - s_1 + 0s_2 + A_1 \leq 3$$

$$-y_1 + y_2 + 2y_3 + y_4 + 0s_1 - 1s_2 + A_2 \leq 2$$

$c_j$	1	-1	-10	-3	0	0	-M	-M			
$C_B$	$y_B$	$y_1$	$y_2$	$y_3$	$y_H$	$s_1$	$s_2$	$A_1$	$A_2$	$X_B$	$\bar{X}_{A_1}$
-M	$A_1$	-1	1	1	0	-1	0	1	0	3	
-M	$A_2$	-1	1	2	1	0	-1	0	1	2	
$Z_j$	2m	-2m	-3m	-M	-M	M	-m	-M			
$Z_j^-$	2m-1	-2m+1	-3m+10	-M+3	-M	M	0	0	0		
$c_j$											

Entering variable =  $y_3$

Exit variable =  $A_2$

$C_B$	$y_B$	1	-1	-10	-3	0	0	-M	
$-M$	$A_1$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$\frac{1}{2}$	1	2
$-10$	$y_3$	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	1
$Z_j$	$\frac{M/2+5}{2}$	$-\frac{M/2-5}{2}$	$-\frac{10}{2}$	$\frac{M/2-5}{2}$	m	$-\frac{M/2+5}{2}$	-M		
$Z_j^-$	$\frac{M+8}{2}$	$-\frac{M+4}{2}$	0	$\frac{M-4}{2}$	M	$-\frac{M+5}{2}$	0		
$c_j$	$\frac{M+8}{2}$	$-\frac{M+4}{2}$	0	$\frac{M-4}{2}$	M	$-\frac{M+5}{2}$	0		

			1	-1	-10	-3	0	0	M
CB	YB	Y <sub>1</sub>	Y <sub>2</sub>	Y <sub>3</sub>	Y <sub>4</sub>	Y <sub>11</sub>	S <sub>1</sub>	S <sub>2</sub>	A <sub>1</sub> X <sub>B</sub>
-M	A <sub>1</sub>	0	0	-1	-1	-1	1	1	M+1
-1	Y <sub>2</sub>	-1	1	2	1	0	-1	0	2

$$z_j^* - c_j \geq 0 \quad M+1 \leq z_j^* \leq M$$

$$z_j^* - c_j \geq 0 \quad M+1 \leq z_j^* \leq M$$

			1	-1	-10	-3	0	0	
CB	YB	Y <sub>1</sub>	Y <sub>2</sub>	Y <sub>3</sub>	Y <sub>4</sub>	Y <sub>11</sub>	S <sub>1</sub>	S <sub>2</sub>	G
0	S <sub>2</sub>	0	0	0	-1	-1	-1	1	1

$$z_j^* - c_j \geq 0 \quad -1 \leq z_j^* \leq 0 \quad 3$$

$$z_j^* - c_j \geq 0 \quad -1 \leq z_j^* \leq 0$$

$$z_j^* - c_j \geq 0 \quad 0 \leq z_j^* \leq 3$$

c<sub>j</sub>

$$z_j^* - c_j \geq 0$$

maximum z = -21

min z = 21