

III B.Sc. Statistics

Subject name: Testing of statistical hypothesis

Subject code : CST61

Unit. : 01

Testing of hypothesis

Statistical hypothesis - Simple & composite

A statistical hypothesis is some statistic
are assertion about a population (or) equality
about the probability density characteristic
a population which we want to verify
on the basis of information available
from a sample.

If the statistical hypothesis is
specified the population completely then it
is termed as a simple statistic hypothesis
otherwise it is called a composite
hypothesis.

If x_1, x_2, \dots, x_n is a random sample
size of n from a normal population with
mean (μ) and variance (σ^2) then the
hypothesis

$$H_0: \mu = \mu_0, \sigma^2 = \sigma_0^2$$

is the "simple hypothesis" where as each
of following hypothesis is composite

hypothesis

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A

$$\mu < \mu_0, \sigma^2 < \sigma_0^2$$

$$\mu > \mu_0, \sigma^2 = \sigma_0^2$$

$$\mu = \mu_0, \sigma^2 < \sigma_0^2$$

$$\mu < \mu_0, \sigma^2 > \sigma_0^2$$

$$\mu = \mu_0, \sigma^2 > \sigma_0^2$$

Test of statistical hypothesis

The test of a statistical hypothesis is a two action decision problem after the experiment sample value have been observed that two action being the acceptance and the rejection of the hypothesis is under consideration.

Hypothesis

A defined statement about the population parameter is known as hypothesis.

Null hypothesis

A hypothesis which is usually of no difference is called null hypothesis. It is denoted by H_0 .

Null hypothesis which is test for possible rejection and under the assumption that is true

Eg: In case of single statistics H_0 will be that the sample statistic does not differ significant from the hypothetical parameter value

In case of statistic H_0 will be be that the sample statistics do not differ significant

Alternative hypothesis

Any hypothesis which is complementary to null hypothesis is called an alternative hypothesis. It is denoted H_1 .

Eg: If we want to test the null hypothesis that the population has a specified mean μ_0

$$H_0: \mu = \mu_0$$

then alternative hypothesis would

if $H_1: \mu \neq \mu_0$ [ie] $\mu > \mu_0$ (or) $\mu < \mu_0$ is known as two tailed]

ii) $H_1: \mu < \mu_0$ [left tailed alternative]

iii) $H_1: \mu > \mu_0$ [right tailed alternative]

This testing of alternative hypothesis is very important since it enables to decide whether to use single t-test (right or left) or two tailed test.

Types of Error:

The error of rejecting H_0 . When H_0 is true Type I error and the error of accepting H_0 . When H_0 is false is called Type - II error.

The probability of Type - I error and Type - II error denoted by α and β respectively.

Type - I = α = Probability of rejecting H_0 . When H_0 is true

Type - II = β = Probability of accepting H_0 . When H_0 is false

		Decision from theory	
		Reject H_0	Accept H_0
True statement	H_0 True	Reject H_0	(correct)
	H_0 False	(correct)	Type - II error (Wrong)

Level of significance

The probability of Type - I error (α) is known as the level of significance as test. It is also called as size of critical region.

Power of test $(1-\beta)$

Defined $(1-\beta)$

$$\int_{\omega} \mathbb{1}_1 dx = 1 - \int_{\omega} \mathbb{1}_0 dx = 1 - \beta$$

$$\beta = \int_{\omega} \mathbb{1}_0 dx$$

$$\int_{\omega} \mathbb{1}_1 dx = 1 - \beta$$

and

$$P(x \in \omega | H_1) = 1 - \beta$$

is called the power function of the test hypothesis H_0 against the alternative hypothesis H_1 .

The value of the power function at a parameter point is called "power of the test" at that point.

Most powerful test [MP-test]

The critical region ω is the most powerful critical region of the size (α) and the corresponding test has most powerful test level (α) for testing

$$H_0 : \theta = \theta_0 \text{ against } H_1 : \theta = \theta_1$$

$$P(x \in \omega | H_0) = \int_{\omega} f_0 dx = \alpha \rightarrow \textcircled{A}$$

and

$$P(x \in \omega | H_1) \geq P(x \in \omega | H_1) \rightarrow \textcircled{B}$$

For every other critical region ω' satisfying equation \textcircled{A} .

Neyman - Pearson - Lemma theorem

Statement

Let $K > 0$ be a constant & ω be a critical region of size α such that

$$W = \left\{ x \in \Omega : \frac{q(x, \theta_1)}{q(x, \theta_0)} > k \right\}$$

$$\bar{W} = \left\{ x \in \Omega : \frac{l_1}{l_0} \leq K \right\} \rightarrow \textcircled{1}$$

and

$$W' = \left\{ x \in \Omega : \frac{l_1}{l_0} > k \right\} \rightarrow \textcircled{2}$$

where l_0 & l_1 are the likelihood functions of the sample observation

$x = [x_1, x_2, x_3, \dots, x_n]$ under H_0 and H_1 respectively, then W is the most powerful critical region of the hypothesis $H_0 : \theta = \theta_0$ against the alternatives $H_1 : \theta = \theta_1$.

Proof

$$P(x \in W | H_0) = \int_W l_0 dx = \alpha \rightarrow \textcircled{3}$$

$$P(x \in W | H_1) = \int_W l_1 dx = 1 - \beta \rightarrow \textcircled{4}$$

To prove the lemma

We have the proof that as no critical region ω , $\epsilon \omega$ which is most powerful than ω .

Let w another critical decision size

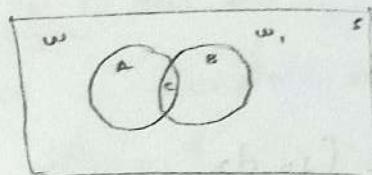
size $\alpha_1 \leq \alpha$

and the power $1 - P_1$,

$$P(x \in w_1 / H_0) = \int_{w_1} f_0 dx = \alpha_1 \rightarrow ⑤$$

$$P(x \in w_1 / H_1) = \int_{w_1} f_1 dx = 1 - P_1 \rightarrow ⑥$$

We have to remove $1 - P_0 \geq 1 - P_1$



$$w = A \cup C$$

$$w_1 = B \cup C$$

(C is empty, $w \neq w_1$, disjoint)

If $\alpha_1 \leq \alpha$

$$\int_{w_1} f_0 dx \leq \int_w f_0 dx$$

$$\underbrace{\int_B f_0 dx}_{B \cup C} \leq \underbrace{\int_A f_0 dx}_{\text{Ave}}$$

$$\int_B f_0 dx + \int_C f_0 dx \leq \int_A f_0 dx + \int_C f_0 dx$$

$$\underbrace{\int_B f_0 dx}_{B} \leq \underbrace{\int_A f_0 dx}_{A}$$

$$\underbrace{\int_A f_0 dx}_{A} \geq \underbrace{\int_B f_0 dx}_{B}$$

Once ACW in ①

$$\frac{f_1}{f_0} > k$$

$$\alpha_1 > k \cdot \alpha_0$$

Taking integration

$$\int_A f_1 dx > k \int_A f_0 dx \rightarrow ⑦$$

$$\int_A f_1 dx \geq k \int_B f_0 dx \rightarrow ⑧$$

Taking the equation ②

$$\frac{\int_1}{\int_0} \leq k$$

when $x \in \bar{w}$

$$k \int_B f_0 dx \leq \int_A f_1 dx$$

$$\int_{\bar{w}} f_1 dx \leq k \int_{\bar{w}} f_0 dx \rightarrow ⑨$$

Thus result also holds for any subset to \bar{w}

$$\bar{w} \cap w = B$$

$$\bar{w} - \bar{w} \cap w = B$$

$$\int_B f_1 dx \leq k \int_B f_0 dx$$

$$\int_B f_1 dx \leq \int_A f_1 dx$$

add $\int_C f_1 dx$ on both sides

$$\int_B f_1 dx + \int_C f_1 dx \leq \int_A f_1 dx + \int_C f_1 dx$$

$$\int_{B \cup C} f_1 dx \leq \int_{A \cup C} f_1 dx$$

$$\int_w f_1 dx \leq \int_w f_1 dx$$

In equation (4) and (6)

$$1-\beta_1 \leq 1-\beta$$

$$1-\beta \geq 1-\beta_1$$

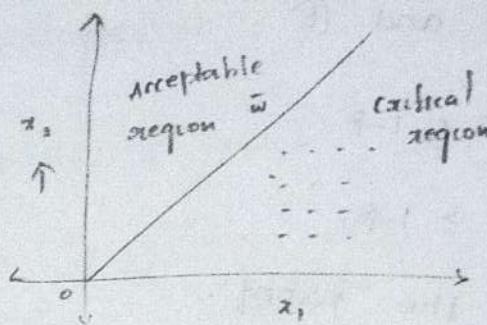
Hence the proof.

Critical Region

Let x_1, x_2, \dots, x_n be the sample observation denoted by 'o'. All the values of 'o' will be aggregate of a sample and they constitute a space called sample space & denoted by 'S'. Since the sample value x_1, x_2, \dots, x_n can be taken as a point in n-dimensional space and whether this point lie within this region (or) outside the region. We divide whole sample space into 2 disjoint point w and $S-w$ (or) \bar{w} (or) w' . The null hypothesis H_0 is rejected if the observed sample point fall in w and if it fall in w' and if we reject H_1 and accept H_0 .

The region of rejection of H_0 when H_0 is true is the region of outcome set where H_0 is rejected if the sample point falls in that region & is called critical region.

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Problem

- 14 Given the frequency function $f(x, \theta) = \frac{1}{\theta} e^{-x/\theta}$ that you are testing the null hypothesis $H_0: \theta = 1$ against $H_1: \theta = 2$. By mean of a sample observe value of x . What would be the size of type-II error. If you choose the interval i) $0.5 \leq x$, ii) $1 \leq x \leq 1.5$ as the critical regions. Also obtain the power function test.

$$f(x, \theta) = \frac{1}{\theta} e^{-x/\theta}, \theta \leq x \leq 0$$

$$H_0: \theta = 1, H_1: \theta = 2$$

If $0.5 \leq x$

$$\omega = \{x; 0.5 \leq x\}$$

$$\bar{\omega} = \{x; x \geq 0.5\}$$

$$\bar{\bar{\omega}} = \{x; x < 0.5\}$$

$$\alpha = P\{x \in \omega / H_0\}$$

$$= P\{x \geq 0.5 / \theta = 1\}$$

$$= P\{0.5 \leq x \leq \theta / \theta = 1\}$$

$$\alpha = P\{0.5 \leq x \leq 1\}$$

$$\alpha = \int_{0.5}^1 \frac{1}{\theta} e^{-x/\theta} dx$$

$$= \frac{1}{\theta} \int_{0.5}^1 e^{-x/\theta} dx$$

$$= \frac{1}{\theta} [-e^{-x/\theta}]_{0.5}^1$$

$$= 1 - e^{-0.5/\theta}$$

$$\alpha = 0.15$$

$$\beta = P(x \in \bar{W}/H_1)$$

$$= P(x \leq 0.5 / \theta = 2)$$

$$= \boxed{\int_{0.5}^1 P(0 \leq x \leq 0.5 / \theta = 2)}$$

$$\beta = \int_0^1 \frac{1}{\theta} e^{-x/\theta} dx$$

$$= \frac{1}{\theta} \int_0^{0.5} e^{-x/\theta} dx$$

$$= \frac{1}{2} [e^{-0.5/\theta}]_0^{0.5}$$

$$= \frac{1}{2} [0.5 - 0]$$

$$= \frac{0.15}{2}$$

$$= 0.25 \quad 1 - 0.25$$

The size of Type-I error $\alpha = 0.15$

The size of Type-II error $\beta = 0.25$

$$1 - \beta = 1 - 0.25$$

$$= 0.75$$

$$ii) 1 \leq x \leq 1.5$$

$$\omega = \{x : 1 \leq x \leq 1.5\}$$

$$\alpha = P\{x \in \omega | \theta = 1\}$$

$$= P\{1 \leq x \leq 1.5 | \theta = 1\}$$

$$= \int_1^{1.5} \frac{1}{\theta} dx$$

$$= \frac{1}{\theta} \int_1^{1.5} dx$$

$$= \frac{1}{1} [x]_1^{1.5}$$

$$= 1.5 - 1$$

$$\alpha = 0.5$$

$$\beta = P(x \in \bar{\omega} | H_1)$$

$$= 1 - P(x \in \omega | H_1)$$

$$= 1 - \boxed{P\{1 \leq x \leq 1.5 | \theta = 2\}}$$

$$= 1 - \int_1^{1.5} \frac{1}{\theta} dx$$

$$= 1 - \frac{1}{2} \int_1^{1.5} dx$$

$$= 1 - \frac{1}{2} [x]_1^{1.5}$$

$$= 1 - \frac{1}{2} [1.5 - 1]$$

$$= 1 - \frac{1}{2} [0.5]$$

$$\beta = \boxed{0.75}$$

The size of type-I error $\alpha = 0.5$

The size of type-II error $\beta = 0.75$

$$\text{power test} = \boxed{1 - \beta}$$

$$= \boxed{1 - 0.75}$$

$$= 0.25$$

Q) If $\theta = 2$ as a critical region for the testing $H_0 : \theta = 2$ against $H_1 : \theta < 2$ on the basis of sample observation from population $f(x, \theta) = \theta e^{-\theta x}$, observe obtain the value of Type I and Type II

$$W = \{x : x \geq 1\}$$

$$\bar{W} = \{x : x \leq 1\}$$

$$\alpha = P\{x \in W | H_0\}$$

$$\beta = P\{x \in \bar{W} | H_1\}$$

$$H_0 : \theta = 2, \quad H_1 : \theta < 2$$

$$f(x, \theta) = \theta e^{-\theta x}$$

$$\alpha = P(x \in W | H_0)$$

$$= \{x \geq 1 | \theta = 2\}$$

$$= \int_1^\infty 2e^{-2x} dx$$

$$= 2 \left[\frac{-e^{-2x}}{2} \right]_1^\infty$$

$$= 2 \left\{ \left(\frac{-e^{-2\infty}}{2} \right) - \left(\frac{-e^{-2}}{2} \right) \right\}$$

$$= 2 \left[\frac{e^{-2}}{2} \right]$$

$$= \frac{1}{e^2}$$

$$\beta = P(x \in \bar{W} | H_1)$$

$$= P(x \leq \frac{1}{e^2})$$

$$= \int_{-\infty}^{1/e^2} 2e^{-2x} dx$$

$$= \int_0^1 e^{-x} dx$$

$$= \int_0^1 e^{-x} dx$$

$$= \left[-\frac{e^{-x}}{1} \right]_0^1$$

$$= \left[-\frac{e^{-1}}{1} - \left(-\frac{e^0}{1} \right) \right]$$

$$= -e^{-1} + 1$$

$$= 1 - \frac{1}{e}$$

$$= \frac{e-1}{e}$$

The size of type-I error $\alpha = \frac{1}{e^2}$

The size of type-II error $\beta = 1 - \frac{1}{e}$

- 3) Let p be the probability that a will fall in a single toss. In order to test $H_0: p = \frac{1}{2}$ against $H_1: p = \frac{3}{4}$ the coin is tossed 5 times if H_0 is rejected. If more than 3 heads are obtained. Find the probability of the type-II error and power of the test.

Soln

$$H_0: p = \frac{1}{2}, H_1: p = \frac{3}{4}$$

To prove

$$H_0: p = \frac{1}{2}$$

$$q = 1-p$$

$$q = 1 - \frac{1}{2} = \frac{1}{2}$$

$$H_1: P = 2/4$$

$$q = 1 - 2/4$$

$$P = \frac{1}{2}$$

$$q = \frac{1}{4}$$

$$q = \frac{1}{2}$$

$$n = 5$$

$$\Omega = \{x : x \geq 3\}$$

$$\bar{\omega} = \{x : x \leq 3\}$$

$$\alpha = n c_x P^x q^{n-x}$$

$$= 5 C_4 P^4 q^{5-4} + 5 C_5 P^5 q^{5-5}$$

$$= 5 C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{4}\right) + 5 C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{4}\right)$$

$$= 5 (0.0625)(0.5) + 1 (0.03125) [1]$$

$$= 0.1875$$

$$\bar{\omega} = \{x : x \leq 3\}$$

$$\beta = n c_x P^x q^{n-x}$$

$$= 5 C_0 \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^5 + 5 C_1 \left(\frac{3}{4}\right)^1 \left(\frac{1}{4}\right)^4 \\ + 5 C_2 \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^3 + 5 C_3 \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^2$$

$$= \frac{1}{1024} + \frac{15}{1024} + \frac{90}{1024} + \frac{270}{1024}$$

$$= \frac{316}{1024} = \frac{47}{128}$$

The size of type-I error $\alpha = 0.1875$

The size of type-II error $\beta = 47/128$

$$\text{Power of test} = 1 - \beta$$

$$= 1 - 47/128$$

$$= \frac{81}{128}$$

4) Let x_1, x_2, \dots, x_{10} be unknown, To test $H_0: \mu = 1$ against $H_1: \mu > 1$ based on a sample of size 10 from the population we use the critical region

$$x_1 + 2x_2 + \dots + 10x_{10} \geq 0$$

what is its size? what is the power of the test?

Soln

$$\text{critical region } W = \{x : x_1 + 2x_2 + \dots + 10x_{10} \geq 0\}$$

$$\text{Let, } u = x_1 + 2x_2 + \dots + 10x_{10}$$

Since x_i 's identically independent distribution

$$(i.i.d) N(\mu, \sigma^2)$$

$$\mu \sim N \left[(1+2+\dots+10)\mu, (1^2+2^2+\dots+10^2)\sigma^2 \right]$$

$$= N \left[55\mu, 385\sigma^2 \right]$$

$$= N \left[55\mu, 385(4) \right]$$

$$= N \left[55\mu, 1540 \right]$$

The size α of the critical region is

$$\alpha = P(x \in W | H_0)$$

$$= P(u \geq 0 | H_0)$$

$$H_0: \mu = 1$$

$$\mu \sim N[-55, 1540]$$

$$Z = \frac{u - F(u)}{\sigma_u}$$

$$= \frac{u + 55}{\sqrt{1540}}$$

Under H_0 when $\mu = 0$

$$z = \frac{55}{\sqrt{1540}}$$
$$= \frac{55}{39.2458}$$

$$\approx 1.4015$$

$$\alpha = P(z \geq 1.4015)$$

$$\approx 0.5 - P(z \leq 1.4015)$$

$$\approx 0.5 - 0.4192$$

$$= 0.1808$$

Alternatively

$$\alpha = 1 - P(z \leq 1.4015)$$

$$= 1 - \Phi(1.4015)$$

Where $\Phi(\cdot)$ as the distribution function of standard normal variate

Power of the test = $1 - \alpha$

$$= P(z \in w(H_1))$$

$$= P(z \geq 0.4015)$$

Under $H_1 : \mu = 1$

$$u \sim N(55, 1540)$$

$$z = \frac{u - E(u)}{\sigma u}$$

$$= \frac{-55}{\sqrt{1540}}$$

$$= -1.40$$

$$\alpha = P(x \in \omega / H_0)$$

$$= P(x_1 + x_2 \geq 9.5 / H_0)$$

on sampling from the given exponential distribution

$$\frac{\sigma}{\theta} \sum x_i \sim \chi^2(2n) \quad n=2$$

$$u = \frac{\sigma}{\theta} (x_1 + x_2) \sim \chi^2(4)$$

$$\alpha = P \left[\frac{\sigma}{\theta} (x_1 + x_2) \geq \frac{\sigma}{\theta} \times 9.5 / H_0 \right]$$

$$= P \left[\chi_{(4)}^2 \geq 9.5 \right]$$

$$\alpha = 0.05$$

$$\text{power of test} = 1 - \beta$$

$$= P(x \in \omega / H_1)$$

$$= P(x_1 + x_2 \geq 9.5 / H_1)$$

$$= P \left[\frac{\sigma}{\theta} (x_1 + x_2) \geq \frac{\sigma}{\theta} \times 9.5 / H_1 \right]$$

$$= P \left[\chi_{(4)}^2 \geq 19 \right]$$