

III B.Sc. Statistics

Subject name: Testing of statistical hypothesis

Subject code : CST 61

Unit :2

Uniformly most powerful test [UMPT]

The region w is called uniformly most powerful critical region of size α [and the corresponding test has uniformly most powerful level α] for the testing H_0 as false

$$H_0: \theta = \theta_0 \text{ against } H_1: \theta \neq \theta_0$$

$$\text{If } P(x \in w | H_0) = \int_{w} f_{\theta_0}(x) dx = \alpha \rightarrow ①$$

$$P(x \in w | H_1) = P(x \in w | H_0) \vee \theta \neq \theta_0 \rightarrow ②$$

whatever the region w , satisfying eqn ① maybe

Likelihood ratio test

Likelihood A test as the critical region for testing H_0 [against H_1 as an interval $0 < x \leq \lambda_0$]

λ_0 is some number (≤ 1) determine by a distribution of λ and the probability of type I error i.e) λ_0 as given by the eqn $P(x < \lambda_0 | H_0)$

$P(x < \lambda_0 | H_0) = \alpha$ as a likelihood ratio to

for testing H_0

Eg: If $g(\cdot)$ is the probability density function of λ when λ_0 as determinate from the eqn

$$\int_0^{\lambda_0} g(\lambda | H_0) d\lambda = \alpha$$

Theorem

Statement

If λ is the likelihood ratio for testing a simple hypothesis H_0 and if $U = \phi(\lambda)$ is the monotonic increasing [decreasing] function of the λ , then the test based on U is equivalent to the likelihood ratio test based critical region for the test based on U is

$$\phi(0) < U < \phi(\lambda_0) \quad [\phi(\lambda_0) < U < \phi(0)]$$

Proof:

The critical region for the likelihood ratio test is given by $0 < \lambda \leq \lambda_0$.

Where,

$$\lambda_0 \text{ is determined by } \int_0^{\lambda_0} g(\lambda/H_0) d\lambda = \alpha \rightarrow ①$$

Let $U = \phi(\lambda)$ be a monotonically increasing function of λ then eqn ①

$$\alpha = \int_0^{\lambda_0} g(\lambda/H_0) d\lambda$$

$$= \int_{\phi(0)}^{\phi(\lambda_0)} h(U/H_0) dU$$

Where $h(U/H_0)$ is the pdf $h(u)$ when H_0 is true hence the critical region $0 < \lambda \leq \lambda_0$ transform to $\phi(0) < U < \phi(\lambda_0)$

However if $U = \phi(\lambda)$ is monotonically decreasing function that inequality is reversed and we get critical region has $\phi(\lambda_0) < U < \phi(0)$

Properties of likelihood ratio test

Likelihood ratio test is an asymptotic test. If we are testing a simple hypothesis H_0 against the simple alternative hypothesis H_1 , then the likelihood ratio principle leads to the same test as given by the Neymann-Pearson criterion. This suggests that likelihood ratio has some properties especially large sample properties.

In likelihood test the probability of type-I error is controlled suitably choosing cut off points λ_0 .

Asymptotic properties

- * Likelihood ratio test is generally UMPT if all exist we state below the asymptotic properties of likelihood ratio test
- * Under certain condition -2 loge has the asymptotic chi-square distribution
- * Under certain assumptions likelihood ratio test is consistent

Parameter space :-

Let us consider a random variable x with PDF $f(x, \theta)$ the functional form of the population distribution is assumed to be known except for the value of some unknown parameters which may take any value of set Θ .

It is expressed by writing the pdf
form $f(x, \theta)$, $\theta \in \Theta$.

The set Θ is the test set of all possible
values of θ is called parameter space

Example

If $X \sim N(\mu, \sigma^2)$ then the $f(x, \sigma)$ parameter space

$$\Theta = \begin{cases} (\mu, \sigma^2), & -\infty < \mu < \infty \\ \emptyset, & 0 < \sigma^2 < \infty \end{cases}$$

Testing for the mean of normal population

Let us take the problem of testing, if
the mean of a normal popl. has a specified value

let (x_1, x_2, \dots, x_n) be a random sample of
size n from the normal population with
mean μ and variance σ^2 , where μ and σ^2
are unknown

To test the (composite) null hypothesis

$$H_0: \mu = \mu_0 \text{ (specified)}, 0 < \sigma^2 < \infty$$

against the (composite) alternative hypothesis

$$H_1: \mu \neq \mu_0; 0 < \sigma^2 < \infty$$

In this case, the parameter space (ii)
is given by

$$(ii) = \{(\mu, \sigma^2) : -\infty < \mu < \infty, 0 < \sigma^2 < \infty\}$$

and the subspace \mathbb{H}_0 determined by
null hypothesis H_0 as given by

$$\mathbb{H}_0 = \{(\mu, \sigma^2) : \mu = \mu_0 ; 0 < \sigma^2 < \infty\}$$

The likelihood function of the sample
observation (x_1, x_2, \dots, x_n) as given by

$$L = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} \exp \left\{ \frac{-1}{2\sigma^2} \sum (x_i - \mu)^2 \right\}$$

The maximum likelihood estimates of
and σ^2 are given by

$$\hat{\mu} = \frac{1}{n} \sum x_i = \bar{x} \rightarrow ②$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum (x_i - \bar{x})^2 = s^2$$

Hence, substituting in eqn ①, the max
of L in the parameter space \mathbb{H}_0 as given,

$$L(\hat{\mu}) = \left(\frac{1}{2\pi s^2} \right)^{n/2} \exp \left(-\frac{n}{2} \right) \rightarrow ③$$

In \mathbb{H}_0 , the only variate parameter
and the maximum likelihood estimator
(MLE) of σ^2 for given $\mu = \mu_0$ is given by

$$\hat{\sigma}^2 = \frac{1}{n} \sum (x_i - \mu_0)^2 \rightarrow ④$$

$$= s_0^2 \text{ (say)}$$

$$= \frac{1}{n} \sum (x_i - \bar{x} + \bar{x} - \mu_0)^2$$

$$= \frac{1}{n} \left[\sum (x_i - \bar{x})^2 + \frac{1}{n} (\bar{x} - \mu_0)^2 \right]$$

Since, the product term vanishes

$$\sum (x_i - \bar{x})(\bar{x} - \mu_0) = (\bar{x} - \mu_0) \sum (x_i - \bar{x}) = 0$$

$$\hat{\sigma}^2 = s^2 + (\bar{x} - \mu_0)^2 = s_0^2 \quad \text{--- (4)}$$

Here, sub in eqn ①

$$L(\hat{\mu}_0) = \left(\frac{1}{2\pi s^2} \right)^{n/2} \exp \left\{ -\frac{n}{2s^2} \right\} \rightarrow ⑤$$

The ratio of eqn ④ and ⑤ gives the likelihood ratio criterion

$$\begin{aligned} \lambda &= \frac{L(\hat{\mu}_0)}{L(\bar{x})} \\ &= \left(\frac{s^2}{s_0^2} \right)^{n/2} \rightarrow ⑥ \\ &= \left[\frac{s^2}{s_0^2 + (\bar{x} - \mu_0)^2} \right]^{n/2} \\ &= \left(\frac{1}{\frac{s^2}{s_0^2} + \frac{(\bar{x} - \mu_0)^2}{s_0^2}} \right)^{n/2} \end{aligned}$$

Under H_0 is the test statistic, $\frac{\bar{x} - \mu_0}{s/\sqrt{n}}$

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

where

$$s^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$$

$$= \frac{ns^2}{n-1}$$

Thus, follows student t distribution

$(n-1)$ degrees of freedom

Thus,

$$t = \frac{\bar{x} - H_0}{s/\sqrt{n}}$$

$$= \frac{\bar{x} - H_0}{s/\sqrt{n-1}} \sim t_{n-1}$$

$$t = \frac{\bar{x} - H_0}{s/\sqrt{n-1}}$$

Squaring on b.s

$$\hat{t}^2 = \frac{(\bar{x} - H_0)^2}{s^2/(n-1)}$$

$$\hat{t}^2 = \frac{(\bar{x} - H_0)^2}{s^2} (n-1)$$

$$\frac{\hat{t}^2}{n-1} = \frac{(\bar{x} - H_0)^2}{s^2} \rightarrow ⑨$$

$$\lambda = \left(1 + \frac{\hat{t}^2}{n-1} \right)^{-n/2}$$

$$= \phi(\hat{t}^2) \text{ (say)} \rightarrow ⑩$$

The LR test for testing H_0 against H_1

consisting in finding the critical region
of the type $\Omega \subset \lambda < \lambda_0$ where λ_0 is given

$$\int_{-\infty}^{\lambda_0} g(\lambda/H_0) d\lambda = \alpha$$

Which requires the distribution of λ under H_0

In this case it is not necessary to obtain the distribution of λ

Since $\lambda = \Phi(t)$ monotonic function of t^2 ,
the test be carried on with t^2 as an criterion with λ

$$t^2 = 0 \Leftrightarrow \lambda = 1$$

$$t^2 = \infty \Leftrightarrow \lambda = 0$$

The critical region of the LR test $\text{OLR} < \lambda_0$
is using in eqn ⑩ is equivalent to

$$\left(1 + \frac{t^2}{n-1} \right)^{-n/2} \leq \lambda_0$$

$$\left(1 + \frac{t^2}{n-1} \right)^{n/2} \geq \lambda_0^{-1}$$

Raising power $n/2$ on b.s

$$\left(1 + \frac{t^2}{n-1} \right)^{-2/n} \geq \lambda_0$$

$$\frac{t^2}{n-1} \geq \lambda_0^{-1} - 1$$

$$t^2 \geq (n-1) [\lambda_0^{-2/n} - 1]$$

$$t^2 \geq A^2 \text{ (say)}$$

Thus the critical region we will be

$$|t| \geq \frac{(\bar{x} - \mu_0) \sqrt{n}}{\sigma} \geq A \rightarrow (11)$$

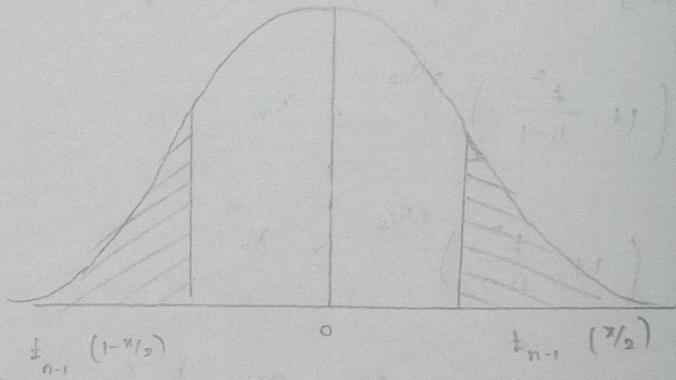
the constant A as determined such that

$$P[|t| \geq A/\mu_0] = \alpha \rightarrow (12)$$

Since under statistics 't' follows student distribution with $(n-1)$ dof

$$P[t > t_{n-1}(\alpha)] = \int_{t_{n-1}(\alpha)}^{\infty} f(t) dt = \alpha \rightarrow (13)$$

The critical region is shown the following diagram



Thus for testing $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$ (σ as unknown) we have a 2 tailed test as

$$|t| = \frac{\sqrt{n}(\bar{x} - \mu_0)}{\sigma} > t_{n-1} (\alpha/2)$$

H_0 is rejected

and if $|t| < t_{n-1} (\alpha/2)$ H_0 may be accepted

Test for the variance of a normal population

Let us consider the problem of testing if the variance of a normal population has a specified value σ_0^2 on the basis of the random sample x_1, x_2, \dots, x_n of size n from normal population $N(\mu, \sigma^2)$

To test the hypothesis

$$H_0: \sigma^2 = \sigma_0^2$$

against alternative hypothesis

$$H_1: \sigma^2 \neq \sigma_0^2$$

We have

$$\Theta = [(\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 > 0]$$

$$\Theta_0 = [(\mu, \sigma^2) : -\infty < \mu < \infty, \sigma^2 = \sigma_0^2]$$

The likelihood function of the sample observation is given by

$$L = \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \right\} \rightarrow ①$$

$$L(\hat{\theta}) = \left(\frac{1}{2\pi\hat{\sigma}^2} \right)^{n/2} \exp \left(-\frac{n}{2\hat{\sigma}^2} \right)$$

In Θ_0 , we have only one variable parameter viz μ and $L(\theta)$

$$L(\hat{\theta}_1) = \left(\frac{1}{2\pi\sigma_1^2} \right)^{n/2} \exp \left\{ -\frac{1}{2\sigma_1^2} \sum_{i=1}^n (x_i - \mu)^2 \right\}$$

The MLE for μ is given by

$$\frac{\partial}{\partial \mu} \log L = 0$$

$$\Rightarrow \hat{\mu} = \bar{x}$$

$$\therefore L(\hat{\theta}_1) = \left(\frac{1}{2\pi\sigma_1^2} \right)^{n/2} \exp \left\{ -\frac{1}{2\sigma_1^2} \sum_{i=1}^n (x_i - \bar{x})^2 \right\}$$

$$\cdot \left(\frac{1}{2\pi\sigma_1^2} \right)^{n/2} \exp \left\{ -\frac{n\bar{x}^2}{2\sigma_1^2} \right\}$$

The likelihood ratio criterion is given by

$$\lambda = \frac{L(\hat{\theta}_1)}{L(\hat{\theta}_0)}$$

$$\lambda = \left(\frac{\sigma^2}{\sigma_1^2} \right)^{n/2} \exp \left\{ -\frac{1}{2} \left(\frac{n\bar{x}^2}{\sigma_1^2} - n \right) \right\}$$

We know that under H_0 , the statistics follow

$$\chi^2 = \frac{n\bar{x}^2}{\sigma_1^2} \sim (n-1) \text{ d.o.f}$$

Follows Chi-square distribution with $(n-1)$ d.o.f. In term of χ^2 , we have

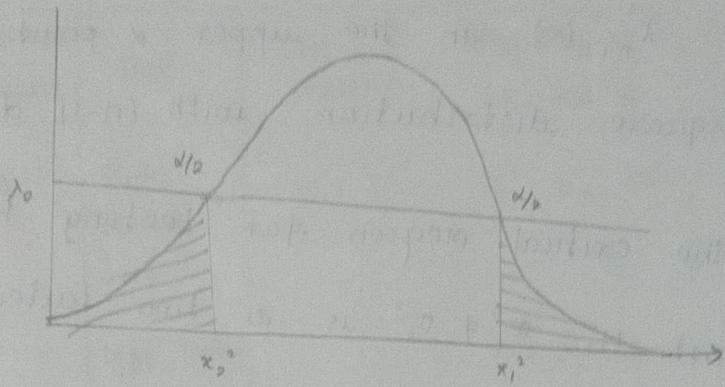
$$\lambda = \left[\frac{\chi^2}{n} \right]^{n/2} \exp \left\{ -\frac{1}{2} (\chi^2 - n) \right\}$$

Since λ is monotonic function of χ^2 , the test may be done using χ^2 as a criterion.

The critical region in $0 < \chi^2 < \lambda_0$ is now equivalent

$$\left(\frac{\chi^2}{n}\right)^{n/2} \exp\left\{-\frac{1}{2}(\chi^2 - n)\right\} < \lambda_0$$

$$\exp\left(-\frac{1}{2}\chi^2\right) (\chi^2)^{n/2} < \lambda_0 (ne^{-1})^{n/2} = B \quad (\text{say})$$



Since χ^2 has a chi-square distribution

with $(n-1)$ d.o.f the critical region $\text{ogn } \textcircled{2}$ is determined by a pair of intervals $x_1^2 < \chi^2 < x_2^2$ and $x_1^2 < \chi^2 < \lambda_0$. Where x_1^2 and x_2^2 are to be determined such that the ordinate are equal

$$(\chi^2)^{n/2} \exp\left\{-\frac{1}{2}\chi^2\right\} = (x_0^2)^{n/2} \exp\left\{-\frac{1}{2}x_0^2\right\}$$

critical region is shown in shaded region

in the above diagram.

← x_1^2 & x_2^2 are defined by the equation.

$$P(\chi^2 > x_1^2) = \frac{\alpha}{2} \quad \text{and}$$

$$P(\chi^2 > x_2^2) = 1 - \frac{\alpha}{2}$$

In other words

$$\chi^2 > \chi^2_{n-1} (\alpha/2) \text{ and}$$

$$\chi^2 < \chi^2_{n-1} (1 - \alpha/2)$$

where $\chi^2_{n-1} (\alpha)$ is the upper α point of chi-square distribution with $(n-1)$ d.f.

The critical region for testing $H_0: \sigma^2 = \sigma_0^2$ against $H_1: \sigma^2 \neq \sigma_0^2$ is a two tailed region given by

$$\chi^2 > \chi^2_{n-1} (\alpha/2) \text{ and}$$

$$\chi^2 < \chi^2_{(n-1)} (1 - \alpha/2)$$

Thus in this case we have a 2-tailed test

Remark

i) To test $H_0: \sigma^2 = \sigma_0^2$ against the alternative hypothesis $H_1: \sigma^2 \neq \sigma_0^2$ we get 1-tailed test [left tailed test] with critical region

$$\chi^2 < \chi^2_{(n-1)} (\alpha)$$

while for testing H_0 against $H_1: \sigma^2 > \sigma_0^2$ we have right tailed test with critical region

$$\chi^2 > \chi^2_{n-1} (\alpha)$$

The test statistic, the test criterion & the confidence interval for the parameter σ^2 for testing $H_0: \sigma^2 = \sigma_0^2$ (μ unknown) against the variance alternative hypothesis

Alternative Hypothesis	Test	Test statistic	Reject H_0 at ' α ' level if	Confidence $(1-\alpha)$ interval for σ^2
$\sigma^2 > \sigma_0^2$	Right tailed test	$\chi^2 = \frac{ns^2}{\sigma_0^2}$	$\chi^2 > \chi^2_{n-1}(\alpha)$	$\sigma^2 \geq \frac{ns^2}{\chi^2_{n-1}(\alpha)}$
	Left tailed test		$\chi^2 < \chi^2_{n-1}(1-\alpha)$	$\sigma^2 \leq \frac{ns^2}{\chi^2_{n-1}(1-\alpha)}$
	Two tailed test		$\chi^2 > \chi^2_{n-1}(\alpha/2)$ $\chi^2 < \chi^2_{n-1}(1-\alpha/2)$	$\frac{ns^2}{\chi^2_{n-1}(\alpha/2)} \leq \sigma^2 \leq \frac{ns^2}{\chi^2_{n-1}(1-\alpha/2)}$