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UNIT - 4

RIEMANN INTEGRATION [COND]

Rolle's theorem:

Theorem:

Let f be continuous real valued fun on closed bounded interval $[a,b]$. If maximum value for f is attained at c where $a<c<b$ and if $f'(c)$ exists then $f'(c)=0$.

Proof:

Suppose the contrary i.e., $f'(c) \neq 0$.

If $f'(c)>0$ then $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c)>0$

and so $\frac{f(x)-f(c)}{x-c} > 0$ for $0<|x-c|<\delta_1$ where δ_1 is suitable possible no. $x \in [c, c+\delta_1]$ then $x-c>0$ and hence $f(x)-f(c)>0$

This contradicts the hypothesis, f attains maximum at c. If $f'(c)<0$

Then $\frac{f(x)-f(c)}{x-c} < 0$ for $0<|x-c|<\delta_2$

If $x \in [c-\delta_2, c]$ then $x-c<0$ and hence

$$f(x)-f(c)>0$$

which again contradiction

Hence, $f'(c)=0$.

Rolle's theorem:

Statement:

Let f be continuous real valued fun on closed bounded interval $[a,b]$ with $f(a)=f(b)=0$. If $f'(x)$ exist $\forall x$ in (a,b) then there is some point $c \in (a,b)$ where $f'(c)=0$

Proof:

If f is identically zero on $[a,b]$ then theorem is obvious.

Let $f(x) \neq 0$ if $f(x) > 0$ for some $x \in (a,b)$ the maximum value of f on $[a,b]$ will not be attained at a (or) b.

Since $f(a) = f(b) = 0$ hence f will attain its maximum value at some $c \in (a,b)$.

By theorem,

Let f be continuous real valued fun on closed bounded interval $[a,b]$ if the maximum value for f is attained at c when $a < c < b$ & if $f'(c)$ exist then $f'(c)=0$.

Hence $f'(c)=0$

If $f(x) < 0$ for some x in (a,b) the minimum value of f in $[a,b]$ will not be attained at a (or) b.

Since $f(a) = f(b) = 0$

Hence f will attained minimum value at some $c \in (a,b)$.

$\therefore f'(c) = 0$.

Darbau's property:

If f has derivative at every paint of $[a,b]$ then f' takes on every value b/w $f'(a)$ & $f'(b)$

Proof:

It is sufficient to consider case in which $f'(a) < f'(b)$. then if

$f'(a) < \gamma < f'(b)$ we must show that $\exists$ $c \in (a,b)$ such that $f'(c) = \gamma$.

Let $g(x) = f(x) - \gamma x \quad (a \leq x \leq b)$

$g'(x) = f'(x) - \gamma \quad (a \leq x \leq b)$

Then since $g'(x)$ exist $\forall$ $x \in [a,b]$ by theorem,

'If real valued fun f has derivative at paint $c \in R'$, then f is continuous at c'.

Show that g is continuous on $[a,b]$

Hence by theorem,

'If real valued fun f is continuous on closed bounded interval $[a,b]$ f attains maxi & mini value at point of $[a,b]$'.

g takes on mini value at some point $c \in [a,b]$.

But $g'(a) = f'(a) - \gamma < 0$ and so g cannot attains at mini value at a.

III^{ly} since $g'(b) = f'(b) - \gamma > 0$, g cannot attains its mini value at b.

Thus, $\exists$ $a < c < b$ such that g attains its mini value at c.

Hence, $g'(c) = 0$.

$$g'(c) = f'(c) - \gamma = 0$$

$$\Rightarrow f'(c) - \gamma = 0$$

$$\Rightarrow f'(c) = \gamma.$$

1. Find suitable point c of rolle's theorem for $f(x) = (x-a)(x-b)$, $a \leq x \leq b$.

Soln:

$$f(x) = (x-a)(x-b)$$

$$= x^2 - bx - ax + ab$$

$$f'(x) = 2x - (a+b)$$

By rolle's theorem, $f'(c) = 0$

$$2c - (a+b) = 0$$

$$2c = a+b$$

$$c = \frac{a+b}{2}$$

$\therefore$ The value clearly lies in $a \leq x \leq b$.

2. Thus the fun $f(x) = \sin x$, $0 \le x \le \pi$ satisfy rolle's theorem.

Soln:

$$f(x) = \sin x$$

$$\Rightarrow f(0) = \sin 0 = 0$$

$$\Rightarrow f(\pi) = \sin \pi = 0$$ satisfies rolle's theorem.

By theorem, $f'(c) = 0$

$$\cos c = 0$$

$$c = \cos^{-1}(0)$$

$$c = \pi/2$$

3. P.T. there is no value k such that $x^3 - 3x + k = 0$ has two distinct roots in $[0,1]$.

Soln:

Let us assume that $f(x) = x^3 - 3x + k$ becomes zero at $x = a$ & $x = b$ in $[0,1]$ with $a \neq b$ & for some k.

Hence by rolle's theorem $\exists$ $c \in [a,b]$ such that $f'(c) = 0$

$$f'(c) = 3c^2 - 3$$

$$3c^2 - 3 = 0$$

$$3c^2 = 3$$

$$c^2 = 1$$

$$c = \pm 1$$

This contradiction becoz $c \in [a,b]$ & $0 < a < b < 1$

$\therefore$ There is no k such that equ $x^3 - 3x + k = 0$ has two distinct roots in $[0,1]$.

4. If f is defined by $f(x) = x^2 \sin \frac{1}{x} (x \neq 0)$, $f(0) = 0$ then f has derivative at each pt of $(-\infty, \infty)$ & $f'(0) = 0$. but f' is continuous at 0.

Soln:

$$f(x) = x^2 \sin\left(\frac{1}{x}\right)$$

$$f'(x) = x^2 \left(\cos \frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + \sin\left(\frac{1}{x}\right) 2x$$

$$= -\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right)$$

$\therefore$ For $x \neq 0$, $f'(x)$ exist

To show that $f'(0)$ exist, we have for $x \neq 0$.

$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin\left(\frac{1}{x}\right)}{x}$$

$$= x \sin\left(\frac{1}{x}\right)$$

$$\therefore \left|\frac{f(x) - f(0)}{x - 0}\right| \leq |x|$$ & it follows

that $\lim_{x \to 0} \left[\frac{f(x) - f(0)}{x - 0}\right] = 0$

Thus $f'(0) = 0$.

$\therefore$ Thus we have shown that $f'(x)$ exist $\forall x$ but f' is continuous (not) at 0.

5. P.T. if $\frac{a_0}{n+1} + \frac{a_1}{n} + \dots \frac{a_{n-1}}{2} + a_n = 0$ then $a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x$ has one root b/w 0 & 1.

Soln:

Consider,

$$f(x) = \frac{a_0 x^{n-1}}{n+1} + \frac{a_1 x^n}{n} + \dots + \frac{a_{n-1}x^2}{2} + a_n x$$

Then $f(0)=0$ & $f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_{n-1}}{2} + a_n$

By hypothesis,

$f(x)$ is polynomial

Hence, it is continuous on $[0,1]$ & differentiate on (a,b).

Hence by theorem: $\exists$ $c \in (a,b)$ such that $f'(c) = 0$

Now, $f(x) = \frac{a_0 x^{n+1}}{n+1} + \frac{a_1 x^n}{n} + \dots + \frac{a_{n-1}x^2}{2} + a_n x$

$$f'(x) = \frac{a_0 (n-1) x^n}{n+1} + \frac{a_1 n x^{n-1}}{n} + \dots + \frac{a_{n-1} 2x}{2} + a_n$$

$f'(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0$ for $c \in [0,1]$.

Hence, $a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0$ has atleast one root b/w 0 & 1.

Law of mean:

Statement:

If f is continuous fun on closed bounded interval $[a,b]$ & if $f'(x)$ exist $\forall$ x in $[a,b]$, then $\exists$ c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Proof:

Let h be defined as $h(x) = f(x) - f(a)$

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b-a}(x-a) \quad (a \le x \le b)$$

Clearly $h(x)$ is continuous $a \le x \le b$ &

$h'(x)$ exist for $a \le x \le b$.

Then, $h(a) = 0 = h(b)$ & h obeys

hypothesis of Rolle's theorem.

Hence, $h'(c) = 0$ for some $c \in (a,b)$ but

$$h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$f'(c) - \frac{f(b)-f(a)}{b-a} = 0$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Theorem:

f is continuous real valued fun on interval J & if $f'(x) > 0$ $\forall x$ in J except possibly the end points of J (if there are any) then f is strictly increasing on J (& hence f is one-one).

Proof:

If $(a,b) \in J$ with $a < b$ we have by law of mean $f(b) - f(a) = b-a$ in $f'(c)$ for some c b/w a & b. but $f'(c) > 0$ $f(a) < f(b)$

By hypothesis & hence $f(b) - f(a) > 0$

i.e., if $a < b$ then $f(a) < f(b)$.

Hence proved.

Generalized law of mean:

Statement:

Let f & g be continuous function on the closed bounded interval [a,b] with $g(a) \neq g(b)$. If both f & g have derivative at each point of (a,b) & $f'(t)$ & $g'(t)$ are not both equal to zero for any $t \in (a,b)$ then $\exists$ point $c \in (a,b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Proof:

Let $h(x) = f(x) - f(a) - \dfrac{f(b)-f(a)}{g(b)-g(a)} [g(x) - g(a)]$

then $h(a) = 0$, $h(b) = 0$ & h satisfies the other hypothesis of rolle's theorem.

Hence, $h'(c) = 0$ for some $c \in (a,b)$

4) $$h'(x) = f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(x)$$

$$\Rightarrow h'(c) = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c)$$

$$f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0$$

$$f'(c) = \frac{f(b)-f(a)}{g(b)-g(a)} g'(c)$$

If $g'(c)$ were 0 then $f'(c)$ would be zero. contradicting cover hypothesis.

Hence, $g'(c) \neq 0$

$$f'(c) = \frac{f(b)-f(a)}{g(b)-g(a)} \, g'(c)$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Hence proved.

1. Calculate value of c for which $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ for each of following pairs of fun. $f(x) = x$, $g(x) = x^2$, $a = 0$, $b = 1$.

Soln:

$f(0) = 0$, $g(0) = 0$

$f(1) = 1$, $g(1) = 1$

$f'(x) = 1 \Rightarrow f(c) = 1$

$g'(x) = 2x \Rightarrow g'(c) = 2c$

$$\frac{1}{2c} = \frac{1-0}{1-0}$$

$$\frac{1}{2c} = 1$$

$$1 = 2c$$

$$c = 1/2$$

$f(x) = \sin x$, $g(x) = \cos x$, $[-\pi/2 \leq x \leq 0]$.

$$f(-\pi/2) = \sin(-\pi/2)$$

$$= -\sin \pi/2$$

$$= -1$$

$$f(0) = 0$$

$$g(-\pi/2) = \cos(-\pi/2) = 0$$

$$g(0) = \cos(0) = 1$$

$$f'(x) = \cos x \Rightarrow g'(x) = -\sin x$$

$$f'(c) = \cos c \qquad g'(c) = -\sin c$$

$$\frac{\cos c}{-\sin c} = \frac{0-(-1)}{1-0}$$

$$-\cot c = 1$$

$$c = \cot^{-1}(-1)$$

$$c = \pi/4. \qquad \therefore\ c = -\pi/4.$$

If $f'(x) = 0\ \forall\ x$ in (a,b) P.T. f is constant on (a,b)

Proof:

Let x be any point of interval (a,b).

we suppose that $f'(x) = 0$ in (a,b) since $f'(x)$ exist in (a,b)

The condition of mean value theorem are satisfied in (a,b) & in particular a, x

Hence applying mean value theorem in (a,x) we have $f'(c) = \frac{f(x)-f(a)}{x-a}$ where $a<c<x$ but $f'(c) = 0$

By hypothesis,

$$\frac{f(x)-f(a)}{x-a} = 0.$$

$$f(x) - f(a) = 0$$

$$f(x) = f(a)\ \forall\ x \text{ in } (a,b)$$

$\Rightarrow f(x)$ is constant.

Fundamental theorem of Calculus.

First fundamental theorem of calculus:

If f is continuous on the closed bounded interval $[a,b]$ and if $F(x) = \int_a^x f(t)dt$ $(a \le x \le b)$ then $F'(x) = f(x)$ $(a \le x \le b)$.

Proof:

Let $x \in [a,b]$ be fixed if $h \neq 0$ and $x+h \in [a,b]$. we have,

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,dt - \int_a^{x} f(t)\,dt$$

$$= \int_x^{x+h} f(t)\,dt \longrightarrow (1)$$

Since f is continuous on closed bounded interval $[x, x+h]$ then there pt of $[x, x+h]$ at which f attains maximum value M and minimum value m.

i.e., $m \le f(t) \le M$ $\quad (x \le t \le x+h)$ and $f(t_1) = m$, $f(t_2) = M$ for some $t_1 t_2$ in $[x, x+h]$

then we have

$$\int_x^{x+h} m\,dt \le \int_x^{x+h} f(t)\,dt \le \int_x^{x+h} M\,dt$$

$$\text{i.e., } mh \le \int_x^{x+h} f(t)\,dt \le Mh.$$

and finally $m \le \theta \le Mh$

where $\theta = \frac{1}{h} \int_x^{x+h} f(t)\,dt$

By known theorem,

that must be paint $c(h)$ in $[x, x+h]$ such that $f[c(h)] = \theta$.

Thus, we have shaw that if $h > 0$. Fi $c(h)$ in $[x, x+h]$ such that

$$f[c(h)] = \frac{1}{h} \int_x^{x+h} f(t)\,dt$$

From ①, we get

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)\,dt = f[c(h)] \rightarrow ②$$

But clearly $\lim_{h \to 0} c(h) = x$

Thus as $h \to 0$ the RHS of ② has limit $f(x)$.

Since, f is continuous at x.

Hence, the LHS of ② approaches $F(x)$ as

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

Theorem:

If $f \in R[a,b]$ if $F(x) = \int_a^x f(t)\,dt$, $(a \leq x \leq b)$ and if f is continuous at $x_0 \in [a,b]$ then $F'(x_0) = f(x_0)$.

Proof:

For $h > 0$, let I_h denote the interval $[x_0, x+h]$. Then write

$$\omega[f; I_h] = M[f, I_h] - m[f, I_h]$$

where, $M[f, I_h] = \underset{x \in I_h}{\text{l.u.b.}}\, f(x)$;

$$m[f, I_h] = \underset{x \in I_h}{\text{g.l.b.}}\, f(x)$$

we have,

$$|f(t) - f(x_0)| < \omega[f, I_h] \text{ where } t \in I_h$$

$$-\omega[f; I_h] \leq f(x) - f(x_0) \leq \omega[f; I_h]$$

$$\Rightarrow f(x_0) - \omega[f; I_h] \leq f(t) \leq f(x_0) + \omega[f; I_h] \quad (t \in I_h)$$

Hence, we have integrating b/w x_0 and $x_0 + h$.

$$h[f(x_0) - \omega[f; I_h]] \leq \int_{x_0}^{x_0+h} f(t)\,dt$$

$$\leq h[f(x_0) + \omega[f; I_h]]$$

i.e., $f(x_0) - \omega[f, I_n] \leq \int_{x_0}^{x_0+h} f(t)dt$

$\leq f(x_0) + \omega[f; I_n]$

But,

$$\frac{F(x_0+h) - F(x)}{h} = \frac{1}{h}\int_{x_0}^{x_0+h} f(t)dt$$

we get,

$$f(x_0) - \omega[f, I_n] \leq \frac{F(x_0+h) - F(x_0)}{h}$$

$$\leq f(x_0) + \omega[f; I_n] \rightarrow ①$$

But since by hypothesis, f is continuous at x_0, we have

$$\lim_{h\to 0} \omega[f; I_n] = 0 \rightarrow ②$$

Taking $\lim_{h\to 0}$ in ① and using ②

we get,

$$f(x_0) \leq F'(x_0) \leq f(x_0)$$

$$F'(x_0) = f(x_0).$$

Theorem:

If $f'(x)=0$ for every x in closed bounded interval $[a,b]$ then f is constant on $[a,b]$ i.e., $f(x)=c$ $(a \leq x \leq b)$ for some $c \in R$

Proof:

For any x with $a \leq x \leq b$

By theorem,

'If real valued function f has derivative at point $c \in R$ then f is continuous at c'.

f is continuous on interval $[a,b]$. By law of mean $\exists$ $c \in [a,b]$ such that

$$f'(c) = \frac{f(x) - f(a)}{x-a}$$

But by hypothesis $F'(c) = 0$

Thus $f(x) = f(a)$ $\forall$ x in $[a,b]$

and theorem is proved with $c = f(a)$.

Theorem:

If $f'(x) = g'(x)$ $\forall$ x in closed bounded interval $[a,b]$ then $f-g$ is constant i.e., $f(x) = g(x) + c$ $(a \leq x \leq b)$ some $c \in R$.

Proof:

W.K.T.

If f & g both have derivative then

$$(f-g)'(x) = f'(x) - g'(x) \quad \forall \; x \text{ in } [a,b]$$

Hence by hypothesis

$$(f-g)'(x) = 0 \quad \forall \; x \in [a,b]$$

Hence by above theorem,

$f-g$ is continuous.

i.e., $f(x) = g(x) + c \ (a \leq x \leq b)$ for some $c \in R$.

Second fundamental theorem of calculus:

If f is continuous function on the closed bounded interval $[a,b]$ and if $\phi'(x) = f(x) \ (a \leq x \leq b)$ then $\int_a^b f(x)dx = \phi(b) - \phi(a)$.

Proof:

we have $\phi'(x) = f(x) \ (a \leq x \leq b) \rightarrow (1)$

If we have

$F(x) = \int_a^x f(t)dt$ then by first fundamental theorem of calculus.

$$F'(x) = f(x) \ (a \leq x \leq b) \rightarrow (2)$$

From (1) & (2), we get

$\phi'(x) \ \forall \ x$ in $[a,b]$

$F'(x) = f(x) \ (a \leq x \leq b)$

Hence by above theorem

Integreting we get

$$F(x) = \phi(x) + c \quad (a \leq x \leq b) \text{ for some } c \in R$$

$$F(b) - F(a) = [\phi(b) + c] - [\phi(a) + c]$$

$$= \phi(b) - \phi(a)$$

But $F(a) = \int_a^a f(t)dt = 0$

Thus, $F(b) = \phi(b) - \phi(a)$

But $F(b) = \int_a^b f(t)\,dt$

$\therefore \int_a^b f(x)\,dx = \phi(b) - \phi(a).$

$\therefore$ Hence proved.

Change of variable:

Theorem:

Let ϕ be real valued function on closed bounded interval $[a,b]$ such that ϕ' is continuous on $[a,b]$. Let $A = \phi(a)$, $B = \phi(b)$ then if f is continuous on $\phi([a,b])$ we have $\int_A^B f(x)\,dx = \int_a^b f[\phi(u)]\,\phi'(u)\,du$.

Proof:

Since f is continuous function on closed bounded interval $\phi([a,b])$.

Then by first fundamental theorem of calculus there is function F such that $F'(x) = f(x)$, $x \in \phi([a,b])$.

Let $G(u) = F[\phi(u)]$ for $(a \leq u \leq b)$

then by chain rule,

$$G'(u) = F'[\phi(u)]\,\phi'(u)$$

$$= f[\phi(u)]\phi'(u) \quad (a \le u \le b)$$

using 2^{nd} fundamental theorem on calculus, we get,

$$\int_a^b f[\phi(u)]\phi'(u)\,du = \int_a^b G'(u)\,du$$

$$= G(b) - G(a)$$

$$= F[\phi(b)] - F[\phi(a)]$$

$$= F(B) - F(A)$$

$$= \int_A^B F'(x)\,dx$$

$$= \int_A^B F(x)\,dx$$

$$\therefore \int_A^B F(x)\,dx = \int_a^b F[\phi(u)]\phi'(u)\,du.$$

1) Evaluate $\int_1^2 x^3 dx$.

Soln:

Let $f(x) = x^3 \quad 1 \le x \le 2$

then f is continuous on $[a,b]$

Further if $F(x) = \frac{x^4}{4} \quad 1 \le x \le 2$

$$F'(x) = x^3 = f(x)$$

By 2^{nd} fundamental theorem,

$$\int_1^2 x^3\,dx = F(2) - F(1)$$

$$= \frac{2^4}{4} - \frac{1^4}{4}$$

$$= \frac{16}{4} - \frac{1}{4}$$

$$= \frac{15}{4}.$$

2. If f is continuous func on $[0,1]$ P.T. $\int_0^1 f(x)dx = \int_0^{\pi/2} f(\sin u)\cos u\,du$. If result then for the interval $[\pi, 9\pi/2]$ on RHS ? Explain.

Soln:

Let f be continuous func on $[0,1]$

Let $\phi(u) = \sin u$. we have,

$$\phi(0) = 0 \ , \ \phi(\pi/2) = 1$$

Since $\phi'(u) = \cos u$ is continuous on $[0, \pi/2]$ and since f is continuous on

$$\phi\left(\left[0, \frac{\pi}{2}\right]\right) = [0,1]$$

By change of variable

$$\int_A^B f(x)dx = \int_a^b f\,[\phi(u)]\,\phi'(u)\,du.$$

$$\int_0^1 f(x)dx = \int_0^{\pi/2} f(\sin u)\cos u\,du.$$

we also observe that $\phi(\pi) = 0$, $\phi(a\pi/2) = 1$ since ϕ' is continuous on

$[\pi, 9\pi/2]$ we would also have $\int_0^1 f(x)dx = \int_{\pi}^{9\pi/2} f(\sin u)\cos u$. provided that f is continuous on $[-1,1]$ which is image of $[\pi, \frac{9\pi}{2}]$ under ϕ.

Thus, $f(x) = \sqrt{x}$

$$\int_0^1 \sqrt{x}\, dx = \int_0^{\pi/2} \sqrt{\sin u}\, \cos u\, du \text{ is true.}$$

Taylor's theorem:

Drive the taylor series for function

Suppose that for all x in some interval J the function f may be expressed as

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \cdots + A_n(x-a)^n + \cdots \rightarrow ①$$

where $a \in J$ we say that equ ① is an expansion of f in power of $x-a$.

If we set $x=a$ in ① we get

$$f(a) = A_0$$

Differentiate both sides of equ①

w.r.t. 'x' we get,

$$f'(x) = 0 + A_1 + 2A_2(x-a) + \cdots + nA_n(x-a)^{n-1} + \cdots \rightarrow ②$$

so that $f'(a) = A_1$

Again diff equ② w.r.t. 'x' we get

$$f''(x) = 0 + 2A_2 + \dots + n(n-1)A_n(x-a)^{n-2} + \dots \longrightarrow ③$$

so that $f''(a) = A_2$.

In general for any $n = 0, 1, 2, \dots$

we have

$$f^{(n)}(x) = n!\,A_n + (n+1)n \dots 2(A_{n+1})(x-a) + (n-2)(n+1)(n) \dots 3A_{n+2}(x-a)^2 + \dots$$

If $x = a$ in above

$$f^{(n)}(a) = n!\,A_n$$

$$\Rightarrow A_n = \frac{f^n(a)}{n!} \quad n = 0, 1, \dots$$

$\therefore$ equ ① becomes

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \frac{f^{(n)}}{n!}(x-a)^n + \dots \longrightarrow ④$$

The series equ ④ is called taylor series about $x = a$ for func f.

Maclaurian series:

The taylor series about $x = a$ for function is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x).$$

where R_{n+1} is remainder.

If $a=0$, $f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$ is called the maclaurian series for f.

Lemma:

Let f be real valued function on interval $[a, a+h]$ such that $f^{(n+1)}(x)$ exist for every $x \in [a, a+h]$ and $f^{(n+1)}$ is continuo on $[a, a+h]$.

$$R_{k+1}(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)} t \, dt$$

$$(x \in [a, a+h]; k = 0, 1, \ldots n]$$

then $R_k(x) - R_{k+1}(x) = \frac{f^{(k)}(a)}{k!}(x-a)^k$ [$x \in [a, a+h]$; $k = 1, 2, \dots n$]. The same result holds if $h < 0$ and $[a, a+h]$ is replaced as $[a+h, a]$.

Proof:

$$R_{k+1}(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)\,dt$$

[$u = (x-t)^k$ $\qquad dv = f^{(k)}(t)$

$du = -k(x-t)^{k-1}dt \qquad v = f^{(k)}(t)$

$\int u\,dv = uv - \int v\,du$]

$$= \frac{1}{k!}\left[(x-t)^k f^{(k)}(t)\right]_a^x - \int_a^x f^{(k)}(t) - k(x-t)^{k-1}dt$$

$$= \frac{1}{k!}\left[(x-x)^k f^{(k)}(x) - (x-a)^k f^{(k)}(a)\right] + \frac{k}{k!}\int_a^x (x-t)^{k-1} f^{(k)}(t)\,dt.$$

$$= -\frac{1}{k!}(x-a)^k f^{(k)}(a) + \frac{k}{k(k-1)!}\int_a^x (x-t)^{k-1} f^{(k)}(t)\,dt$$

$$= \frac{-(x-a)^k}{k!} f^{(k)}(a) + \frac{1}{(k-1)!}\int_a^x (x-t)^{k-1} f^{(k)}(t)\,dt$$

$$R_{k+1}(x) = -\frac{(x-a)^k}{k!} f^{(k)}(a) + R_k(x)$$

$$R_k(x) - R_{k+1}(x) = \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Taylor's formula with the integral form of the reminder:

Statement:

Let f be real valued fun on $[a, h]$ such that $f^{(n+1)}$ exist for every $x \in [a, a+h]$. and $f^{(n+1)}$ is continuous on $[a, a+h]$ then

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n + R_{n+1}(x) \quad (x \in [a, a+h])$$

where

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dx.$$

The same result hold if $h < 0$ & $[a, a+h]$ is replaced by $[a+h, a]$.

Proof:

W.K.T.

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)dx \rightarrow ①$$

$$R_{k+1}(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)dt \rightarrow ②$$

$$R_k(x) = R_{k+1}(x) = \frac{f^{(n)}(0)}{k!}(x-a)^k \rightarrow ③$$

Put $k=0$ in equ ②

$$② \Rightarrow R_{k+1}(x) = \frac{1}{k!}\int_a^x (x-t)^k f^{(k+1)}(t)dt$$

$$R_1(x) = \int_a^x f'(t)dt$$

$$= [f(t)]_a^x$$

$$R(x) = f(x) - f(a) \rightarrow ④$$

Put $k = 1, 2, \dots n$ in equ ②

$$R_1(x) - R_2(x) = \frac{f'(a)}{1!}(x-a)$$

$$R_2(x) - R_3(x) = \frac{f''(a)}{2!}(x-a)^2$$

$\vdots$

$$R_n(x) - R_{n+1}(x) = \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Adding all the above equ^n.

$$R_1(x) - R_2(x) + R_2(x) - R_3(x) + \dots + R_n(x) - R_{n+1}(x)$$

$$= \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}a}{n!}(x-a)^n$$

$$R_1(x) - R_{n+1}(x) = \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

(2). $$R_1(x) = \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x)$$

(3) W.K.T.

$$R_1(x) = f(x) - f(a)$$

$$f(x) - f(a) = \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x)$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x).$$

i.e., the taylor series is converges to $f(x)$.

Theorem:

Let ϕ be continuous real valued fun on closed bounded interval $[a,b]$. Let g be continuous fun on $[a,b]$ such that $g(t) \geq 0$ $(a \leq t \leq b)$ then $\exists$ no. c with $a \leq c \leq b$ such that $\int_a^b \phi(t)\, g(t)\, dt = \phi(c) \int_a^b g(t)\, dt$.

Proof:

The continuous fun ϕ on compact interval $[a,b]$ attains maximum value M and a minimum value m.

Then, since $g(t) \geq 0 \ \forall t$

$$m \int_a^b g(t)\,dt \leq \int_a^b \phi(t) g(t)\,dt$$

$$\leq M \int_a^b g(t)\,dt \rightarrow ①$$

If g is identically zero then theorem is obvious.

Let us assume that $g(t) > 0$ for some t, so that $\int_a^b g(t)dt > 0$

From equ ①, we get

$$m \leq \theta \leq M \text{ where } \theta = \frac{\int_a^b \phi(t) g(t)\,dt}{\int_a^b g(t)\,dt} \rightarrow ②$$

Since M and m are in range of ϕ.

i.e., θ in the range of ϕ.

i.e., $\phi(c) = \theta$ for some $c \in [a,b]$

$$② \Rightarrow \phi(c) = \int_a^b \phi(t) g(t)\,dt \Big/ \int_a^b g(t)\,dt.$$

$$\int_a^b \phi(t)g(t)\,dt = \phi(c)\int_a^b g(t)\,dt.$$

Taylor's formula with legrange's form of remainder:

Let f be real valued function on $[a, a+h]$ such that $f^{(n+1)}(x)$ exist for every $x \in [a, a+h]$ and $f^{(n+1)}$ is continuous on $[a, a+h]$ then if $x \in [a, a+h]$ $\exists$ no. c with $a \le c \le x$ such that $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$

same result holds if $h < 0$ and $[a, a+h]$ is replaced by $[a+h, a]$.

Proof:

Applying above theorem in the interval $[a, x]$ we have

$$\int_a^x \phi(t)g(t)\,dt = \phi(c)\int_a^x g(t)\,dt$$

we take $\phi = f^{(n+1)}$ and $g(t) = \frac{(x-t)^n}{n!}$

so that $g(t) \ge 0$

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-1)^n R^{(n+1)}(t)\,dt$$

$$= \frac{f^{(n+1)}(c)}{n!}\int_a^x (x-t)^n\,dt$$

$$= \left[\frac{f^{(n+1)}(c)}{n!} \frac{(x-t)^{(n+1)}}{(n+1)(-1)} \right]_a^x$$

$$= \frac{f^{(n+1)}(c)}{n!} \left[-0 + \frac{(x-a)^{n+1}}{n+1} \right]$$

$$= \frac{f^{(n+1)}(c)}{n!} \frac{(x-a)^{n+1}}{n+1}$$

$$= \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}.$$

Thus,

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

$$\frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Taylor's formula with cauchy form of remainder.

Let f be real valued function on $[a, a+h]$ such that $f^{(n+1)}(x)$ exist for every $x \in [a, a+h]$ and $f^{(n+1)}$ is continuous on $[a, a+h]$ then if $x \in [a, a+h]$ $\exists$ no. c with $a \le c \le x$ such that $f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \dots + \frac{f^{(n)}(a)}{2!}$

$$(x-a)^n + \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a).$$

Proof:

The taylor's formula with lagrange

form of remainder

we have,

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n\,dt$$

Take $\phi(t) = f^{(n+1)}(t)(x-t)^n$ & $g(t) = 1$

By theorem,

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(c)(x-c)^n\,dt$$

$$= \frac{f^{(n+1)}(c)}{n!}(x-c)^n \int_a^x dt$$

$$= \frac{f^{(n+1)}(c)}{n!}(x-c)^n [t]_a^x$$

$$R_{n+1}(x) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a) \text{ for some } c \in I$$

$\therefore$ The taylor's formula with cauchy form of remainder.

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a).$$

1. Find the taylor series about $x=2$ for $f(x) = x^3 + 2x + 1$ $(-\infty < x < \infty)$ P.T. taylor series converges to $f(x)$ for every real x.

Soln:

$$f(x) = x^3 + 2x + 1$$

$f(x) = x^3 + 2x + 1 \quad \Rightarrow f(2) = 13$

$f'(x) = 3x^2 + 2 \quad \Rightarrow f'(2) = 14$

$f''(x) = 6x \quad \Rightarrow f''(2) = 12$

$f'''(x) = 6$

Taylor series is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$$= 13 + 14(x-2) + \frac{12}{3!}(x-2)^2 + \frac{6}{3!}(x-3)^3$$

$$= 13 + 14x - 28 + 6(x^2 + 4 - 4x) + x^3 - 3x^2(2) + 3(2)^2 x - 2^3$$

$$= 6x^2 + 9 - 8 + x^3 - 6x^2 + 12x + 14x - 24x$$

$$= x^3 + 2x + 1.$$

$\therefore$ The taylor series converges to $f(x)$.

2. Write taylor series formula with lagrange form of remainder $f(x) = \log(1+x)$ $(-1 < x < \infty)$. $a = 2$, $n = 4$.

Soln:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots +$$

$$\frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

$$f(x) = \log(1+x) \Rightarrow f(2) = \log 3$$

$$f'(x) = \frac{1}{1+x} \Rightarrow f'(2) = \frac{1}{3}$$

$$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(2) = -\frac{1}{9}$$

$$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(2) = \frac{2}{27}$$

$$f(x) = \log 3 + \frac{1}{3}(x-2) - \frac{1}{9}(x-2)^2 + \frac{2}{27}(x-2)^3 + \cdots$$