SOLUTION OF DIFFERENCE EQUATIONS THROUGH DISCRETE CONVOLUTION AND ITS INVERSION

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ABSTRACT

In this paper, we present the solution to the difference equation problem using deconvolution method which provides numerical values to linear non homogeneous difference equation. Convolution is a fundamental mathematical operation which takes two functions and produces a third function that represents the amount of overlap between one of the functions and a reversed and translated version of the other function .Similarly, the deconvolution techniques have a great importance in solving different kinds of equations. .Deconvolution is the inversion of convolution equation. This method can be implemented in many applications. Here we used discrete convolution and deconvolution to obtain solution to the difference equations.

Key Words: Convolution, Deconvolution, Difference equations

1. INTRODUCTION

Convolution is a fundamental mathematical operation which takes two functions and produces a third function that represents the amount of overlap between one of the functions and a reversed and translated version of the other function [9]. Similarly, the deconvolution techniques have a great importance in solving different kinds of equations. [2]. Deconvolution is the inversion of convolution equation. This method can be implemented in many applications. Here we used discrete convolution and deconvolution to obtain solution to the difference equations.

Solving the difference equation using these concepts makes great interest. In section I we discussed on the notation of discrete convolution and deconvolution of finite and infinite sequences. In section II we presented our main concepts which are the unique solution the solution to the non homogeneous linear difference equation.

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Further we used these concepts to solve the numerical solution of initial value problem and boundary value problem with suitable examples.

2. LINEAR DISCRETE CONVOLUTION AND DECONVOLUTION

Let $a = (a_0, a_1, a_2, \dots \dots\ a_n)$ and b= $(b_{0,} b_1, b_2, \dots \dots\ b_n)$ are the finite sequence with same number of elements. The discrete convolution of these sequences is given by

$$c = a * b = (c_0, c_1, c_2, \dots \dots\ c_n) \quad (1)$$

Where c is a finite sequence and defined as below

$$c_0 = a_0 b_0$$

$$c_1 = a_1 b_0 + a_0 b_1$$

$$\dots\dots.. \quad (2)$$

$$c_k = \sum_{i=o}^{n} a_{n-i} b_i$$

Here a and c are known finite sequences with $a_0 \neq 0$, we can determine the finite sequence b provided the condition (i) to be satisfied. This is known as deconvolution of the sequence c by the sequence a.This was denoted by $b = c/a$ (3)

The relations are defined as follows

$$b_0 = c_0/a_0$$

$$b_1 = \frac{1}{a_0}(c_1 - a_1 b_0)$$

$$\dots\dots\dots\dots \quad (4)$$

$$b_k = \frac{1}{a_0}(c_k - \sum_{i=0}^{n-1} a_{n-i} b_i)$$

Inverse of the finite sequence of a is given by $a^{-I} = \delta/a$, such that

$$\frac{c}{a} = c * a^{-I} \quad (5)$$

We consider the infinite case with same definition, where n is arbitrary

Therefore

$$(a_0, a_1, a_2, \dots \dots\ a_n \dots.) * (b_{0,} b_1, b_2, \dots \dots\ b_n \dots.)$$
$$= (((a_0, a_1, a_2, \dots \dots\ a_n \dots.) * (b_{0,} b_1, b_2, \dots \dots\ b_n \dots.)) : n = 0, 1, 2 \dots.$$

2.1 LINEAR DIFFERNCE EQUATION WITH CONSTANT COEFFICIENT

Consider the following non homogeneous linear difference equation

$$\sum_{i=0}^{k} a_{n-i} u_{i+n} = b_n , \quad n = 1,2, \ldots . \tag{6}$$

With the coefficients $a_0 \neq 0$ and we denote

$$a = (a_0, a_1, a_2, \ldots \ldots\ a_k, 0,0 \ldots) = a = (a_0, a_1, a_2, \ldots \ldots\ a_n, \ldots .) \tag{7}$$

such that $k = 0$ if $n > k$

and b= $(b_0, b_1, b_2, \ldots \ldots\ b_n, \ldots ..)$ (8)

2.1.1 Theorem

The unique solution $u = (u_0, u_1, u_2, \ldots \ldots\ u_n, \ldots)$ of the equation (6) with the initial values $u_0, u_1, \ldots \ldots u_{k-1}$ is given by

$$u = ((a_0, a_1, a_2, \ldots \ldots\ a_{k-1}) * ((u_0, u_1, u_2, \ldots \ldots\ u_{k-1}), b) * a^{-1} \tag{9}$$

Proof

Let us denote $c = (c_0, c_1, c_2, \ldots \ldots\ c_n, \ldots .) = a * u$ (10)

Using convolution product, we get

$$c_0 = a_0 u_0$$

$$c_1 = a_1 u_0 + a_0 u_1$$

$$\ldots \ldots . \tag{11}$$

$$c_{k-1} = \sum_{i=o}^{k-1} a_{k-1-i} u_i$$

Therefore

$$c = (c_0, c_1, c_2, \ldots \ldots\ c_{k-1}) = (a_0, a_1, a_2, \ldots \ldots\ a_{k-1}) * (u_0, u_1, u_2, \ldots \ldots\ u_{k-1}) \tag{12}$$

Changing the index $i = j + k$, and take $a_{k+n} = 0, n = 1,2, \ldots .$ which implies

$$c_{k+n} = \sum_{i=o}^{k+n} a_{k+n-i} u_i$$

$$= \sum_{j=-k}^{n} a_{n-j} u_{j+k}$$

$$= \sum_{j=0}^{n} a_{n-j} u_{j+k}$$

$$= b_k \qquad n = 1,2, \ldots . \tag{13}$$

Hence , the equations (12) and (13) gives

$$u = ((a_0, a_1, a_2, \ldots \ldots\ a_{k-1}) * ((u_0, u_1, u_2, \ldots \ldots\ u_{k-1}), b)$$

$$= (a_0 u_0 , a_1 u_0 + a_0 u_1 , \ldots \ldots , \sum_{i=o}^{k-1} a_{k-1-i} u_i , b_0 , b_1 , b_2, \ldots \ldots\ b_n, \ldots. \quad (14)$$

From (11) and (14) gives the solution to the non homogeneous linear difference equation (6)

The converse is also true

2.2 NUMERICAL PROBLEMS

EXAMPLE 2.2.1

The linear difference equation given by $u_{n+2} - 2u_{n-1} - 3u_n = n, n = 0,1,2, \ldots.$ has the initial conditions $u_0 = u_1 = 1$ has $k = 3$ with $a = (1, -2, -3,0,0, \ldots)$ and $b = (0,1,2, \ldots)$

Solution

We know that

$$u = ((a_0, a_1) * (u_0, u_1), b)$$

$$= ((1, -2) * (1,1), b)$$

$$= (1, -1,0,1,2,3, \ldots)$$

This results the sequence

$$u = \frac{c}{a} = (1,15,14,45, \ldots)$$

Hence the solution

EXAMPLE 2.2.2

The boundary value problem formed by the difference equation

$u_{n+2} + u_n = b_n, n = 0,1,2, \ldots$ with $b = (2,0, -2,0,2,0, -2,0, \ldots)$ has $a = (1,0,1,0,1,0,1,0 \ldots)$ and $a^{-1} = (1,0, -1,0,1,0, -1,0 \ldots)$

Solution

Consider

$$u = (0,0,1 * a^{-1}) + u_0 . (1,0,0 \ldots) * a^{-1} + u_1 . (0,1,0,0 \ldots) * a^{-1}$$

$$= (0,0,2,0 - 4,0,6,0, \ldots) + u_0 . (1,0, -1,0,1,0, -1,0, \ldots) + u_1 . (0,1,0 - 1,0,1,0, -1 \ldots)$$

Case 1: If $u_3 = 0, u_4 = -3,$

Which gives $u_1 = -u_3 = 0$ and $u_0 = 1$

Hence the boundary value problem has the unique solution

$$u = (1,0,1,0,-3,0,5,0, \dots)$$

Case 2: If $u_2 = 1, u_4 = -3,$

From $u_2 = 2 - u_0 = 1$ and $u_4 = -4 + u_3 = -3$

We get $u_0 = 1$

Hence the boundary value problem has infinity of solution by the ralation

$$u = (1,0,1,0,-3,0,5,0, \dots) + u_1 \,.\, (0,1,0-1,0,1,0,-1 \dots)$$

Where u_1 is arbitrary

Case 2: If $u_2 \neq 1, u_4 = -3,$

Which gives both $u_0 = 1$ and $u_0 \neq 1$

This leads to contradiction

Therefore it has no solution

CONCLUSION

The deconvolution method is applied for the initial value problems and for their generalizations. We also arrived that how the discrete convolution and deconvolution, be used to compute the numerical values of the solutions of the initial value problems for linear non-homogeneous difference and with constant coefficients, through some solved examples.

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