MARUDHAR KESARI JAIN COLLEGE FOR WOMEN, VANIYAMBADI

PG & RESEARCH DEPARTMENT OF MATHEMATICS

CLASS : I M. C. A

SUBJECT CODE : 23PCA11

SUBJECT NAME : DISCRETE MATHEMATICS

SYLLABUS

UNIT 1

RELATIONS

Binary relations-Operations on relations- properties of binary relations in a set Equivalence relations Representation of a relation by a matrix -Representation of a relation by a digraph Functions-Definition and examples-Classification of functions-Composition of functions-Inverse function.

Syllabus

UNIT I : RELATIONS - Binary Relations - operations on Relation - Properties of binary Relation in a Set - Equalance Relation - Representation of Relation by a Matrix - Representation of Relation by a digraph - Functions - Definition and examples - Classification of functions - Composition of function - Inverse function.

UNIT 2 :

Mathematical logic - logical Connectiveness - Well formed formula - Truth table of Well formed formula - Algebra of Preposition - Quine's Method - Normal forms of well formed formula - Disjunctive Normal form - Principle disjunctive Normal form - Conjunctive Normal form - Principle Conjunctive Normal form - Rules of inference for Prepositional Calculus - Quantifiers - universal quantifiers - Existential quantifiers.

UNIT 3 :

Recurrence Relation - Formulation - Solving recurrence relation by iteration - Solving recurrence relation - Solving linear homogeneous recurrence relation of order two - Solving linear non homogeneous recurrence relation -

Permutations - Cyclic Permutation - Permutations with repetation of Permutations of Set With indistinguishable Objects - Combinations - Combination with repetation.

UNIT 4:

Matrices - Special types of matrices - Determinents - Inverse of a Square Matrix - Cramer's rule for Solving linear equations - Elementary Operations - Rank of a matrix - Solving a System of linear equations - Characteristic roots and Characteristic Vectors - Cayley Hamilton theorem - Problems.

UNIT 5:

Graphs - Connected graphs - Euler graph - Euler line - Hamiltonian Circuit and Paths - Planar graph - Complete graph - Bipartite graph - Hyper Cube graph - Matrix representation of graph.

Textbook :-

M. Chandrasekaran & M. umaparvathi, discrete Mathematics, learning Private limited, New delhi, 2005.

Relation :

1. Let A and B be any two sets. A relation R from A to B is a subset of $A \times B$.

If R is a subset of $A \times B$ and A, B $(a,b) \in R$. We say that A is related to b (i.e) $a\,R\,b$

Eg:-

$A = \{1,2\}$

$B = \{a,b,c\}$

$R = \{(1,a)\ (1,b),\ (2,a),\ (2,c)\}$

2. Domain and Range of a Relation :

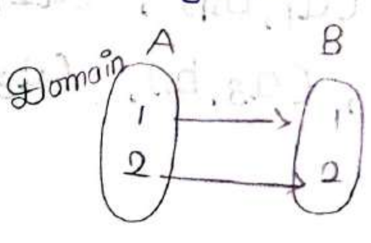
Let R be a relation from A to B. Let R be a subset of $A \times B$. The domain D of a relation is the set of all first element of ordered pairs which belongs to R.

domain

$D = \{a / a \in A$ and $(a,b) \in R$ for some $b \in B\}$

Eg:- $A = \{1,2\}$ $B = \{1,2\}$

$R = \{(1,1),\ (2,2)\}$



The range E of the relation is the set of all second elements of the ordered pair in R.

$E = \{ b \mid b \in B \text{ and } (a,b) \in R \text{ for some } a \in A \}$

Let $A = \{1,2,3,4\}$ & $B = \{r,s,t\}$
$R = \{(1,r), (2,s), (3,r)\}$ find domain & range ?

Domain = $\{1,2,3\}$

Range = $\{r,s\}$

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3. **Matrix of a Relation :**

If $A = \{a_1, a_2, \ldots . a_m\}$ and $B = \{b_1, b_2 \ldots b_n\}$ are finite set containing m & n elements respectively & R is a relation from A to B.

We can represent R by $m \times n$ matrix.

$M_R = [m_{ij}]$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Eg:-

$A = \{a_1, a_2, a_3\}$

$B = \{b_1, b_2, b_3, b_4\}$

$R = \{(a_1,b_1), (a_1,b_4), (a_2,b_2), (a_2,b_3), (a_3,b_1), (a_3,b_3)\}$

$$\therefore M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array}\begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

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4. **Digraph of a Relation :-**

2^{m} A Small Circle is drawn for each element of A & marked with the Corresponding element. These Circles are Called Vertices.

2^{m} An arc is drawn from the vertex a_i to vertex a_j if and only if $a_i R a_j$. This is Called edge.

This Pictorial representation of R is Called a directed graph or digraph of R.

In a digraph of R, the indegree of a vertex is the number of edges terminating at the Vertex.

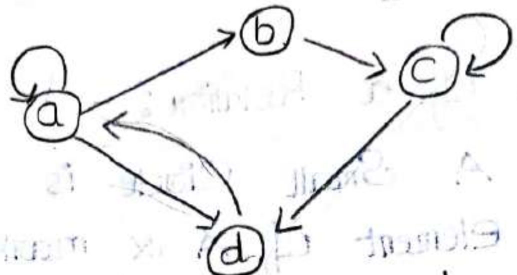
The Outdegree of a vertex is the number of edges leaving the vertex.

Eg:- Let $A = \{a, b, c, d\}$ & R has the relation on A,

$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Construct a digraph of R & list the indegree and Outdegree of the vertices.

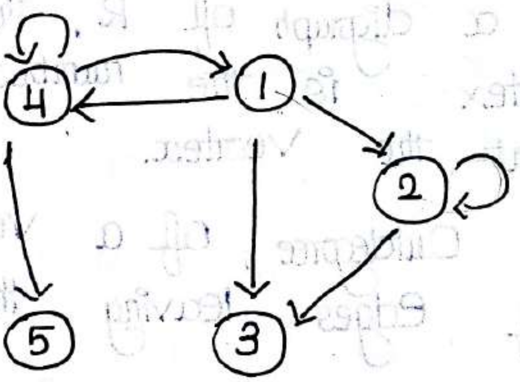
$R = \{(a,a), (a,b), (a,d), (b,c), (c,c), (c,d), (d,a)\}$



	a	b	c	d
Indegree	2	1	2	2
Outdegree	2	1	1	

HW:

① Find the relation determined by the following digraph.



Solution:-

$R = \{(1,2), (1,3) (1,4), (2,2), (2,3), (4,1), (4,4), (4,5)\}$

Operations on relations

5. Let R & S be relations from A to B

Then (i) $\overline{R} = \{(a,b) \in A \times B \mid (a,b) \notin R\}$ is the Complement of the relation R.

(ii) $R \cap S = \{(a,b) \in A \times B \mid (a,b) \in R$ & $(a,b) \in S\}$

is the Intersection of the relations R & S.

(iii) $R \cup S = \{(a,b) \in A \times B \mid (a,b) \in R$ (or) $(a,b) \in S\}$ is the union of the relation R & S.

Eg:- $A = \{1,2,3\}$ $B = \{a,b\}$

$R = \{(1,a), (2,a)\}$ $S = \{(1,a), (3,b)\}$

$R \cup S = \{ \ldots \}$

$R \cap S = \{(1,a)\}$

6. Inverse Relation:

Every Relation R from a Set A into a Set B has an inverse relation R^{-1} from B to A which is defined by

$$R^{-1} = \{(b,a) \mid (a,b) \in R\}$$

Eg:-

$A = \{a,b,c\}$, $B = \{1,2\}$

$R = \{(a,1), (a,2), (c,1)\}$

$R^{-1} = \{(1,a), (2,a), (1,c)\}$

Composition of a Relation :

Let A, B, C are non empty sets
Let R as a relation from A to B
Let S is a relation from B to C.
then Composition of R & S is represented as $S o R$ is a relation from A to C.

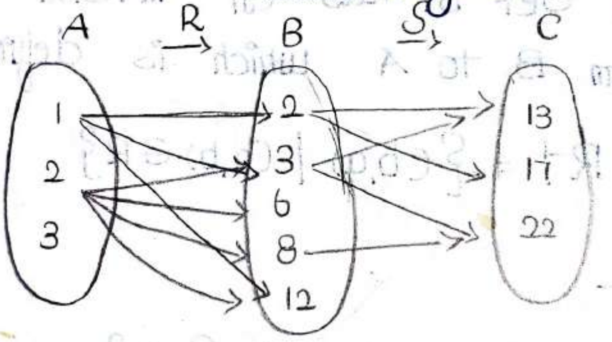
Eg:-

Suppose $A = \{1,2,3\}$ $B = \{2,3,6,8,12\}$
$C = \{13,17,22\}$ and $R = \{(1,2), (1,3), (1,12), (2,3), (2,6), (2,8), (2,12)\}$

$S = \{(2,13), (2,17), (3,13), (3,22), (8,22)\}$

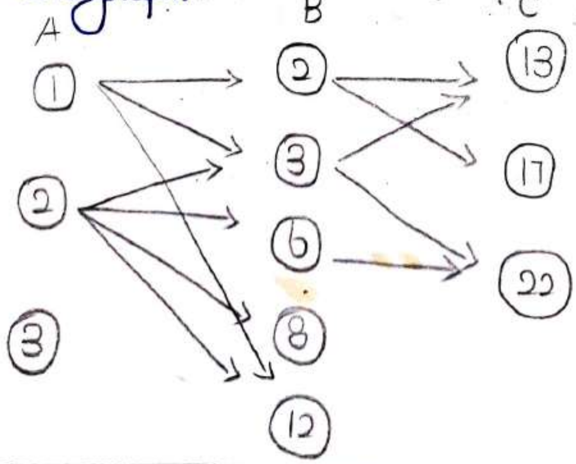
Find $S o R$.

Arrow diagram



$S o R = \{(1,13), (2,13), (1,17), (2,22), (1,22)\}$

Diagraph:



Set Representation :

$(1,2) \in R$ & $(2,13) \in S \Rightarrow (1,13) \in S_oR$

$(1,2) \in R$ & $(2,17) \in S \Rightarrow (1,17) \in S_oR$

$(1,3) \in R$ & $(3,13) \in S \Rightarrow (1,13) \in S_oR$

$(1,3) \in R$ & $(3,22) \in S \Rightarrow (1,22) \in S_oR$

$(2,8) \in R$ & $(8,22) \in S \Rightarrow (2,22) \in S_oR$

$(2,3) \in R$ & $(2,13) \in S \Rightarrow (2,13) \in S_oR$

$(2,3) \in R$ & $(3,22) \in S \Rightarrow (2,22) \in S_oR$

$S_oR = \{(1,13), (1,17), (1,22), (2,13), (2,22)\}$

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8. __Theorem :__

Associative law for Composition of Relation.

Statement:

Let A, B, C & D be Sets. Suppose R is a relation from A to B, S is a relation from B to C & T is a relation from C to D Then

$(T_oS)_oR = T_o(S_oR)$

Proof :

Let $R: A \to B \quad \forall (a,b) \in R$

$S: B \to C \quad \forall (b,c) \in S$

$T: C \to D \quad \forall (c,d) \in T$

Let $(a,d) \in (T \circ S) \circ R$

$\Rightarrow (a,b) \in R$ and $(b,d) \in T \circ S$

$(c,d) \in T$ & $(b,c) \in S$

As $(a,b) \in R$ & $(b,c) \in S \Rightarrow (a,c) \in S \circ R$

Now $(a,c) \in S \circ R$ & $(c,d) \in T \Rightarrow (a,d) \in T \circ (S \circ R)$

Hence $T \circ (S \circ R) \in (T \circ S) \circ R \rightarrow (1)$

$(T \circ S) \circ R \overset{\text{Subset}}{\subset} T \circ (S \circ R)$

Similarly we have

$T \circ (S \circ R) \supset (T \circ S) \circ R \rightarrow (2)$

From (1) & (2)

$T \circ (S \circ R) = (T \circ S) \circ R$

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9. Given $A = \{1,2,3,4\}$ and
$R = \{(1,2), (1,1), (1,3), (2,4), (3,2)\}$ and
$S = \{(1,4), (1,3), (2,3), (3,1), (4,1)\}$
are relations on A. Find $S \circ R$.
(by matrix method)

$$M_R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 \end{array}$$

$$M_S = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \end{array}$$

$M_{SoR} = M_R \circ M_S$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In matrix representation take only 0 & 1

$$= \begin{bmatrix} 0+0+1+0 & 0+0+0+0 & 1+1+0+0 & 1+0+0+0 \\ 0+0+0+1 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+1+0+0 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+0 \end{bmatrix}$$

$$M_{SoR} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$S o R = \{(1,1)\ (1,3)\ (1,4)\ (2,1)\ (3,3)\}$

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10 Let $A = \{a, b\}$

Let $R = \{(a,b), (b,a), (b,b)\}$ &

$S = \{(a,a), (b,a), (b,b)\}$ be relations on A.

Find SoR and RoS Comment on your result.

$$M_R = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \quad M_S = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$M_{SoR} = M_R \circ M_S$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1 & 0+1 \\ 1+1 & 0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$S o R = \{(a,a)\ (a,b), (b,a), (b,b)\}$

$$M_{R\circ S} = M_S \circ M_R$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & 1+0 \\ 0+1 & 1+1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$R\circ S = \{(a,b), (b,a), (b,b)\}$

Therefore,

$S\circ R = \{(a,a), (a,b), (b,a), (b,b)\}$

$R\circ S = \{(a,b), (b,a), (b,b)\}$

$$\boxed{S\circ R \neq R\circ S}$$

W

11. Let $A = \{1,2,3,4\}$.

Let $R = \{(1,1), (1,2), (2,3), (2,4), (3,4), (4,1), (4,2)\}$ and

$S = \{(3,1), (4,4), (2,3), (2,4), (1,1), (1,4)\}$

be two relations on A.

(a) Is $(1,3) \in R\circ R$?

(b) Is $(4,3) \in S\circ R$?

(c) Is $(1,1) \in R\circ S$?

(d) Compute $S\circ R$, $R\circ S$, $R\circ R$ and $S\circ S$.

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array} \quad M_S = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$M_{SoR} = M_R \circ M_S$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0+0 & 0+0+0+0 & 0+1+0+0 & 1+1+0+0 \\ 0+0+1+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+1 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+1 \\ 1+0+0+0 & 0+0+0+0 & 0+1+0+0 & 1+1+0+0 \end{bmatrix}$$

$$M_{SoR} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$SoR = \{(1,1), (1,3)(1,4), (2,1), (2,4), (3,4), (4,1),$
$(4,3), (4,4)\}$

$M_{RoS} = M_S \circ M_R$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0+1 & 1+0+0+1 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+1 & 0+0+0+1 & 0+0+0+0 & 0+0+1+0 \\ 1+0+0+0 & 1+0+0+0 & 0+0+0+0 & 0+0+0+0 \\ 0+0+0+1 & 0+0+0+1 & 0+0+0+0 & 0+0+0+0 \end{bmatrix}$$

$$M_{RoS} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$RoS = \{(1,1), (1,2), (2,1), (2,2), (2,4), (3,1)$
$(3,2), (4,1), (4,2)\}$

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$M_{RoR} = M_R \circ M_R$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

12. If R is given by the matrix $M_R = \begin{bmatrix} 1 & 0 & 1 \\ \ldots \end{bmatrix}$

Find the matrices of RoR & $RoRoR$.

$$= \begin{bmatrix} 1+0+0+0 & 1+0+0+0 & 0+1+0+0 & 0+1+0+0 \\ 0+0+0+1 & 0+0+0+1 & 0+0+0+0 & 0+1+0+1+0 \\ 0+0+0+1 & 0+0+0+1 & 0+0+0+0 & 0+0+0+0 \\ 1+0+0+0 & 1+0+0+0 & 0+1+0+0 & 0+1+0+0 \end{bmatrix}$$

$$M_{RoR} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$RoR = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,4), (3,1), (3,2), (4,1), (4,2), (4,3), (4,4)\}$

$$M_{SoS} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0+0 & 0+0+0+0 & 0+0+0+0 & 1+0+0+1 \\ 0+0+1+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+1 \\ 1+0+0+0 & 0+0+0+0 & 0+0+0+0 & 1+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & 0+0+0+0 & 0+0+0+1 \end{bmatrix}$$

$$M_{SoS} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$SoS = \{(1,1), (1,4), (2,1), (2,4), (3,1), (3,4), (4,4)\}$

(a) Yes., $(1,3) \in R \circ R$

(b) Yes, $(4,3) \in S \circ R$

(c) Yes, $(1,1) \in R \circ S$

(d) $S \circ R = \{(1,1), (1,3), (1,4), (2,1), (2,4), (3,4), (4,1), (4,3), (4,4)\}$

$R \circ S = \{(1,1), (1,2), (2,1), (2,2), (2,4), (3,1), (3,2), (4,1), (4,2)\}$

$R \circ R = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,4), (3,1), (3,2), (4,1), (4,2), (4,3), (4,4)\}$

$S \circ S = \{(1,1), (1,4), (2,1), (2,4), (3,1), (3,4)$
$(4,4)\}$

ASS 13. Find the inverse of the relations
$R = \{(2,4), (1,2), (3,4), (5,6)\}$

ASS 12a. If R is given by the matrix $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Find the matrices of $R \circ R$ & $R \circ R \circ R$.

$$M_{R \circ R} = M_R \circ M_R$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+1 & 1+0+1 \\ 1+1+0 & 0+1+0 & 1+0+0 \\ 0+1+0 & 0+1+1 & 0+0+1 \end{bmatrix}$$

$$M_{R \circ R} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$R \circ R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$$M_{RoRoR} = M_{RoR} \circ M_R$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+0 & 0+1+1 & 1+0+1 \\ 1+1+0 & 0+1+1 & 1+0+1 \\ 1+1+0 & 0+1+1 & 1+0+1 \end{bmatrix}$$

$$M_{RoRoR} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$RoRoR = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

Ass

13. Find the inverse of the relation

$R = \{(2,4), (1,2), (3,4), (5,6)\}$

$R^{-1} = \{(4,2), (2,1), (4,3), (6,5)\}$

Ass

14. Let R be the relation from $A = \{1,3,5,7,9\}$ to $B = \{2,4,6,8\}$ which is defined as aRb iff & only iff $a > b$. List the elements of R & find its domain & range & also find R^{-1}.

$R = \{(3,2), (5,2), (5,4), (7,2), (7,4), (7,6), (9,2), (9,4), (9,6), (9,8)\}$

Domain = $\{3,5,7,9\}$

Range = $\{2,4,6,8\}$

$R^{-1} = \{(2,3), (2,5), (4,5), (2,7), (4,7), (6,7),$
$(2,9), (4,9), (6,9), (8,9)\}$

15. Let $A = \{1,2,3,4\}$ & $B = \{1,2,3,4\}$ In Each of the following, find all the Pairs of $A \times B$ that belongs to R.

(a) $R = \{(x,y) \mid x \geq y\}$
(b) $R = \{(x,y) \mid x > y\}$
(c) $R = \{(x,y) \mid x \leq y\}$
(d) $R = \{(x,y) \mid x = y^2\}$

$A \times B = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3)$
$(2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3)$
$(4,4)\}$

(a) $R = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1),$
$(4,2), (4,3), (4,4)\}$

(b) $R = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\}$

(c) $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3),$
$(2,4), (3,4), (3,3), (4,4)\}$

(d) $R = \{(1,1), (4,2)\}$

16. Identity Relation:

If A is a Set then $\{(a,a) \mid a \in A\}$ is also a relation We denoted it by A & it is Called identity relation A.

$$A = \{(a,a) \mid a \in A\}$$

17. Reflexive:

A relation R on a Set A is Said to be reflexive if

$$(a,a) \in R, \forall x \in A$$

a?

18. Irreflexive:

A relation R on a Set A is Said to be irreflexive if $(a,a) \notin R, \forall a \in A$

19. Symmetric:

A relation R on Set A is Said to be Symmetric if $(a,b) \in R \Rightarrow (b,a) \in R$

20. Anti Symmetric:

A relation R on Set A is Said to be anti Symmetric if $(a \neq b$ and $(a,b) \in R \Rightarrow (b,a) \notin R$

21. Transitive:

A relation R on Set A is Said to be transitive if

$$(a,b) \in R \text{ \& } (b,c) \in R \Rightarrow (a,c) \in R$$

22. Assymetric :

A relation R Set on A Said to be assymetric if $(a,b) \in R \Rightarrow (b,a) \notin R$

Note :

A relation R on the Set A is

(i) reflexive if $\Delta \subset R$

(ii) Irreflexive if $R \cap \Delta$

(iii) Symmetric if $R^{-1} = R$

(iv) AntiSymmetric if $R \cap R^{-1} \subset \Delta$

(v) Transitive if $R \circ R \leq R$

$\Delta \subset R$

$R \cap \Delta$

$R^{-1} = R$

23. Equivalence Relation :

A relation R on a Set A is Said to be an equivalence relation if is Satisfies the following Condition.

(i) Reflexive $(a,a) \in R \ \forall a \in R$

(ii) Symmetric $(a,b) \in R \Rightarrow (b,a) \in R$

(iii) Transitive $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$

24. Partial Order Relation :

A relation R on a Set A is Said to be PoR if it Satisfies

(i) Reflexive $(a,a) \in R$

(ii) AntiSymmetric $a \neq b$ & $(a,b) \in R \Rightarrow (b,a) \notin R$

(iii) Transitive $(a,b) \in R$, $(b,c) \in R \rightarrow$ $(a,c) \in R$

Theorem :

Let R is a relation on a Set A then Prove that

(i) if R is reflexive then R^{-1} is also reflexive

(ii) If R is transitive then R^{-1} is also transitive.

(iii) If R is an Equivalence relation then R^{-1} is also an Equivalence Relation.

25 Prove that the relation " Congruence $(\equiv)$ Modulo m" over the Set of Positive integer is an Equivalence Relation.

Proof:

$$x \equiv y \pmod{m}$$

$$x - y = k\underset{\text{Constant}}{m}$$

(i) Reflexive :

$$x - x = 0$$

$$x - x = 0.m$$

$$x \equiv x \bmod m$$

$\therefore$ Reflexive is Satisfied.

(ii) Symmetric :

Let $x \equiv y \bmod m$

$$x - y = km$$

Multiply (-) on both Side

$$y - x = -km$$

$$y - x = km$$

$$y \equiv x \pmod{m}$$

$\therefore$ Symmetric is Satisfied.

(iii) Transitive:

Let $x \equiv y \bmod m$ & $y \equiv z \bmod m$

$$x - y = km \rightarrow ① \quad y - z = lm \rightarrow ②$$

① + ②

$$x - \cancel{y} + \cancel{y} - z = km + lm$$

$$x - z = m(k + l)$$

$$x - z = cm$$

$$x = z \bmod m$$

$\therefore$ Transitive is Satisfied

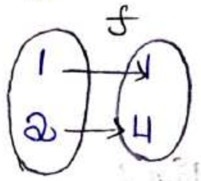
$\therefore$ All 3 Conditions are Satisfied then the given Congruence Modulo m is a Equivalence Relation.

FUNCTIONS : / Mappings / Transformation.

A Relation f from A to B is Said to be a function from A to B if $\forall\, a \in A$ there is Exactly one $b \in B$ Such that $(a,b) \in f$.

(i.e) $b = f(a)$

$f(x) = x^2$



Eg:-

$A = \{1, 2\}$

$B = \{1, 4\}$

Note:

1. Domain:

The Set of all Elements in A is Called domain of f.

2. Co-domain:

The Set of all Elements of B is Called Co-domain of f.

3. Image:

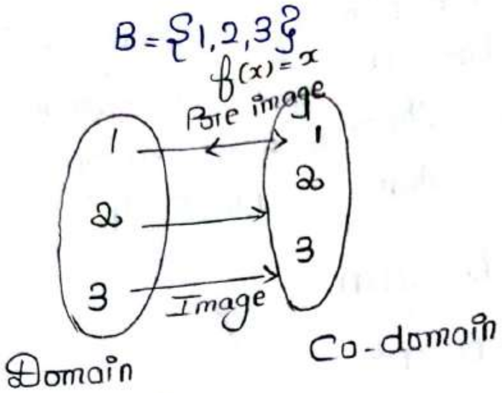
The element b is Said to be f image.

4. Pre-image:

The element a is Said to be f Pre-image.

$A = \{1,2,3\}$

$B = \{1,2,3\}$



$f(1) = 1$

$f(2) = 2$

---

29. **Operator / Transformation :** (Identity Function)

If the domain & Co-domain of a function f are both the same set says $f : A \to A$ then f is called operator on A.

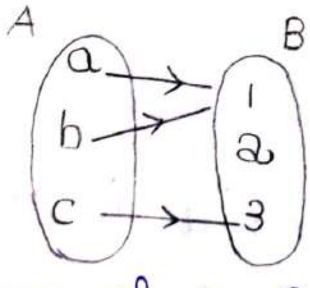
---

30. **Range :**

Let $f : A \to B$. The range of f consists of exactly those elements in B which appears as the image of atleast one element of A.

(i.e) $R(f) = f(A) = \{ f(a) \in B \mid a \in A \}$

Eg :-



Range of $A = \{1, 3\}$

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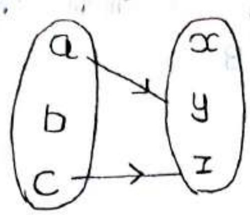
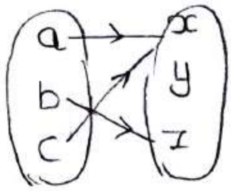
31. State whether the following diagram are functions of $A = \{a, b, c\}$ into $B = \{x, y, z\}$.

(i)



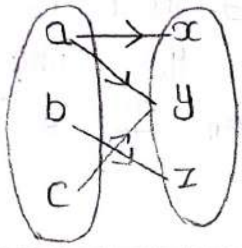
$\therefore$ It is not a function.

(ii)



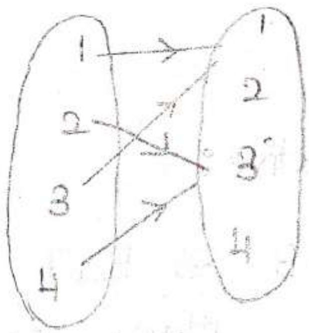
$\therefore$ It is a function

(iii)



$\therefore$ It is not a function.

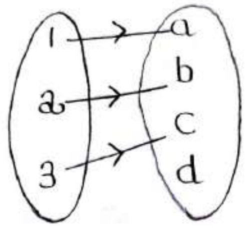
32. Find the range



$R(f) = \{1, 3\}$

33. Functions :

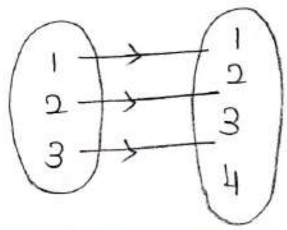
A function $f: A \to B$ is a rule which associates every element of A with a unique element of B.



34. Into / Injective : / one to one

If $f: A \to B \ni$ there is atleast one element in B is not the image of any element in A then f is into function.

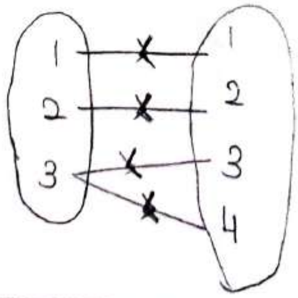
$f(A) \subset B$



35. Onto function / Surjective :

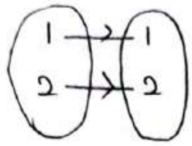
If $f: A \to B \ni$ Each element in B is the image of atleast one element in A then f is onto.

$f(A) = B$



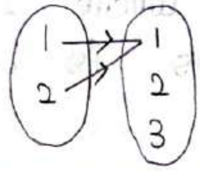
36. One to one function :

The function $f\ A \to B$ is $1=1$. if $x_1, x_2 \in A$ $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$



37. Many to one :

If $f: A \to B$ is many to one if distinct element of A has the Same element image in B



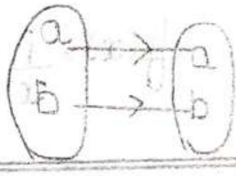
38. 1-1 Correspondence / Bijective :

If $f: A \to B$ is 1-1 & onto then f is Called 1-1 Correspondence between A & B.

39. Identity Mapping :

$f: A \to B$ defined by $f(x) = x\ \forall\ x \in A$

is Called an IM.

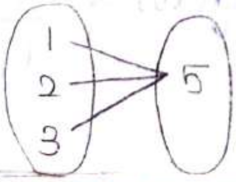


$f(a) = a$
$f(b) = b$

40. Constant function :

$f: A \to B$ defined by $f(x) = c\ \forall\ x \in A$

is Called CF.



41. **Projection :**

$f: S \times T \to S$ defined by $f(a,b) = a$ $\forall\, a \in S,\ b \in T$ is Called Projection of $S \times T$ onto S.

---

42. **Equal Functions :**

Two functions $f: A \to B$ & $g: A \to B$ are Said to be Equal if $f(x) = g(x)$ $\forall\, x \in A$.

---

43. Prove that the mapping $f\ Z^+ \to Z^+$ defined by $f(x) = x^2$ where Z is the Set of Positive integers is 1-1 is onto

5

$Z^+ = \{1, 2, 3 \dots\}$

$\because f(x) = x^2 \Rightarrow f(x) = \{1, 4, 9, \dots\}$

$\therefore x_1, x_2 \in Z^+$

$f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

$x_1 = x_2$

$f(x') = f(y')$

$x' = y'$

$\therefore f$ is 1-1 vs onto

To Prove : Onto

$y = f(x)$

$y = x^2$

$\Rightarrow x = \sqrt{y}$

$\forall\, x \in Z^+\ \exists\ \sqrt{y} \in Z^+$

---

44. Prove that $f: X \to X$ where $X = \{x \in R$ is $x \neq 0\}$ $x = R - \{0\}$ defined by $f(x) = \frac{1}{x}$ is 1-1 is onto.

one to one :

$X = R - \{0\}$, $f: X \to X$ defined by

$$f(x) = \frac{1}{x}$$

$\therefore$ let $f(x_1) = f(x_2) \Rightarrow \dfrac{1}{x_1} = \dfrac{1}{x_2} \Rightarrow x_1 = x_2$

onto : $y = f(x)$

Let $x \in X$ then $y = \frac{1}{x}$

$$f\left(\frac{1}{x}\right) = \frac{1}{\left(\frac{1}{x}\right)} = x \quad x = \frac{1}{y} \quad \forall x \in R - \{0\} \ \exists \ \frac{1}{y} \in R - \{0\}$$

$\therefore$ hence f is 1-1 & onto

45. Inverse function :

Let $f : A \to B$ be 1-1 & onto. Then the inverse function $f^{-1} : B \to A$ is $y = f(x)$ $\forall x \in A, y \in B$.

46. Prove that $f : R \to R$ defined by $f(x) = ax + b$ where $a, b, x \in R, a \neq 0$ is invertible.

one to one :

$$f : R \to R \text{ by } f(x) = ax + b$$

$x_1, x_2 \in R$

$$f(x_1) = f(x_2)$$

$$ax_1 + b = ax_2 + b$$

$$ax_1 = ax_2$$

$$x_1 = x_2$$

$y = f(x)$ $f(x) = y$.

onto :

If $y \in R$ then $y = f(x)$.

$$y = ax + b \Rightarrow x = \frac{y - b}{a}$$

Now $x \in R$ $\exists$ $\frac{y-b}{a} = x \in R$ $\ni$

$$f(x) = f\left(\frac{y-b}{a}\right) \quad ax + b$$

$$= a\left(\frac{y-b}{a}\right) + b$$

$$f(x) = y$$

$\therefore$ Hence f^{-1} exists defined by

$$f^{-1}(y) = \frac{y-b}{a}$$

---

47. Composition :

Let $f: A \rightarrow B$ & $g: B \rightarrow C$ be two functions. The Composition of f & g is the function $g \circ f: A \rightarrow C$ defined by

$$(g \circ f)\, x = g\,[f(x)] \quad \forall\, x \in A$$

---

48. If x = $\{$ $f: R \rightarrow R$ & $g: R \rightarrow R$ are defined by $f(x) = x^2$ & $g(x) = \sin x$ & find $f \circ g$ & $g \circ f$.

$$(f \circ g)\, x = f\,[g(x)]$$
$$= f\,[\sin x]$$
$$= \sin x^2$$

$$(g \circ f)\, x = g\,[f(x)]$$
$$= g\,[x^2]$$
$$= \sin x^2$$

---

49. If $f(x) = x+2$, $g(x) = x-2$ & $h(x) = 3x^2$ for $x \in R$ find $f \circ g \circ h$ & $f \circ h \circ g$.

$$f \circ g \circ h = f\,[g(h(x))]$$
$$= f\,[g(3x^2)]$$
$$= f\,[3x^2 - 2]$$
$$= 3x^2 - 2 + 2$$

$= 3x^2$

$f \circ h \circ g = f[h(g(x))]$

$= f[h(x-2)]$

$= f[3(x-2)^2]$

$= 3(x-2)^2 + 2$ $= 3x^2 - 12x + 14$

---

50. $f(x) = 3x+4$, $g(x) = x^2+2$

$f \circ g = f[g(x)]$

$= f[x^2+2]$

$= 3(x^2+2)+4$

$= 3x^2+6+4$

$= 3x^2+10$

$g \circ f = g[f(x)]$

$= g[3x+4]$

$= (3x+4)^2+2$

$= (3x)^2 + 16 + 2(3x)(4) + 2$

$= 9x^2 + 16 + 24x + 2$

$= 9x^2 + 24x + 18$

---

51. $f(x) = 3x^2+2$, $g(x) = x-4$, $h(x) = \sin x$

$f \circ g \circ h = f[g(h(x))]$

$= f[g(\sin x)]$

$= f[\sin x - 4]$

$= 3(\sin x - 4)^2 + 2$

$= 3(\sin^2 x + 16 - 8\sin x) + 2$

$= 3\sin^2 x - 24\sin x + 48 + 2$

$= 3\sin^2 x - 24\sin x + 50$

$f o h o g = f[h(g(x))]$

$= f[h(x-4)]$

$= f[\sin(x-4)]$

$= 3[\sin(x-4)]^2 + 2$

Theorem:

Associative law of Composition of function.

Let $f: A \to B$, $g: B \to C$ & $h: C \to D$

then $(hog) of = ho(gof)$

Proof:

LHS $[(hog)of](x) = (hog) f(x)$

$= h[g(f(x))]$

R.H.S $[ho(gof)](x) = h[(gof)(x)]$

$= h[g(f(x))]$

$\therefore$ LHS = RHS

Theorem:

Let $f: A \to B$ & $g: B \to C$ be functions. Then

(i) If both f & g are one to one, gof is also one one.

(ii) If both f & g are onto, gof is also onto.

(i) Proof:

Given f & g are one to one $\rightarrow f(x_1) = f(x_2)$

To Prove

$g \circ f$ is also one to one

$(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow x_1 = x_2$

$g[f(x_1)] = g[f(x_2)]$ $\quad \because f$ is 1-1

$g(x_1) = g(x_2)$ $\quad \therefore g$ is 1-1

$\boxed{x_1 = x_2}$

$\therefore g \circ f$ is 1-1

(ii) Proof:

$f(x) = y$
$x \in A$
$y \in B.$

Given f & g are onto

To Prove

$g \circ f$ is onto

f is onto $\exists\, a \in A \rightarrow f(a) = b$

g is onto $\exists\, b \in B \rightarrow g(b) = c$

$(g \circ f)(a) = g[f(a)] = g(b) = c$

$\Rightarrow (g \circ f)(a) = c$

$\therefore a \in c$

Hence $g \circ f$ is onto

Let $x = \{1,2,3,4\}$ is a mapping $f : x \to x$ be given by $f = \{(1,2), (2,3), (3,4), (4,1)\}$ Form the Composite functions f^2, f^3, f^4.

$f = \{(1,2), (2,3), (3,4), (4,1)\}$

$f(1) = 2$

$f(2) = 3$

$f(3) = 4$

$f(4) = 1$

$f^2(x) = f(f(x))$

$f^2(1) = f[f(1)] = f(2) = 3$

$f^2(2) = f[f(2)] = f(3) = 4$

$f^2(3) = f[f(3)] = f(4) = 1$

$f^2(4) = f[f(4)] = f(1) = 2$

$\therefore f^2 = \{(1,3), (2,4), (3,1), (4,2)\}$

$f^3(x) = f(f^2(x))$

$f^3(1) = f[f^2(1)] = f(3) = 4$

$f^3(2) = f[f^2(2)] = f(4) = 1$

$f^3(3) = f[f^2(3)] = f(1) = 2$

$f^3(4) = f[f^2(4)] = f(2) = 3$

$\therefore f^3 = \{(1,4), (2,1), (3,2), (4,3)\}$

$f^4(x) = f(f^3(x))$

$f^4(1) = f[f^3(1)] = f(4) = 1$

$f^4(2) = f[f^3(2)] = f(1) = 2$

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$f^4(3) = f[f^3(3)] = f(2) = 3$

$f^4(4) = f[f^3(4)] = f(3) = 4$

$\therefore f^4 = \{(1,1), (2,2), (3,3), (4,4)\}$

Let $X = \{1,2,3\}$ & f, g, h and S be functions from X to X given by $f = \{(1,2), (2,3), (3,1)\}$, $g = \{(1,2), (2,1), (3,3)\}$, $h = \{(1,1), (2,2), (3,1)\}$ and $S = \{(1,1), (2,2), (3,3)\}$ Find $f\circ g$, $g\circ f$, $f\circ h\circ g$, $g\circ S$, $S\circ S$ and $f\circ S$

Given:

$f(1) = 2$	$g(1) = 2$	$h(1) = 1$	$S(1) = 1$
$f(2) = 3$	$g(2) = 1$	$h(2) = 2$	$S(2) = 2$
$f(3) = 1$	$g(3) = 3$	$h(3) = 1$	$S(3) = 3$

$\underline{f\circ g(x)}$

$f(g(x)) \Rightarrow f(g(1)) = f(2) = 3$

$f(g(2)) = f(1) = 2$

$f(g(3)) = f(3) = 1$

$f\circ g = \{(1,3), (2,2), (3,1)\}$

$\underline{g\circ f(x)}$

$g[f(x)] \Rightarrow g[f(1)] = g(2) = 1$

$g[f(2)] = g(3) = 3$

$g[f(3)] = g(1) = 2$

$\therefore g\circ f = \{(1,1), (2,3), (3,2)\}$

$\underline{f o h o g(x)} \Rightarrow f[h(g(x))]$

$f[h(g(1))] = f[h(2)]$
$= f(2) = 3$

$f[h(g(2))] = f[h(1)]$
$= f(1) = 2$

$f[h(g(3))] = f[h(3)]$
$= f(1) = 2$

$\therefore f o h o g = \{(1,3), (2,2), (3,2)\}$

$\underline{g o s(x)}$

$g[s(x)] \Rightarrow g[s(1)] = g(1) = 2$
$g[s(2)] = g(2) = 1$
$g[s(3)] = g(3) = 3$

$\therefore g o s = \{(1,2), (2,1), (3,3)\}$

$\underline{s o s(x)}$

$s[s(x)] \Rightarrow s[s(1)] = s(1) = 1$
$s[s(2)] = s(2) = 2$
$s[s(3)] = s(3) = 3$

$\therefore s o s = \{(1,1), (2,2), (3,3)\}$

$\underline{f o s(x)}$

$f[s(x)] = f[s(1)] = f(1) = 2$
$f[s(2)] = f(2) = 3$
$f[s(3)] = f(3) = 1$

$\therefore f o s = \{(1,2), (2,3), (3,1)\}$

Theorem:

Let $f: A \to B$ & $g: B \to C$ be both one to one & onto functions then Prove that $(gof)^{-1}(c) = f^{-1} o g^{-1}$

Proof :-

$$\text{Let } (gof)^{-1}(c) = a \to ①$$

$$c = (gof)(a)$$

$$= g[f(a)]$$

$$g^{-1}(c) = f(a)$$

$$f^{-1}[g^{-1}(c)] = a$$

$$(f^{-1} o g^{-1})(c) = a \to ②$$

$$1 = 2$$

$$\therefore (gof)^{-1} = f^{-1} o g^{-1}$$

Hence Proved