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PG & RESEARCH DEPARTMENT OF MATHEMATICS

CLASS : I M. C. A

SUBJECT CODE : 23PCA11

SUBJECT NAME : DISCRETE MATHEMATICS

SYLLABUS

UNIT 3

RECURRENCE RELATIONS

Formulation -solving recurrence Relation by Iteration- solving Recurrence Relations- Solving Linear Homogeneous Recurrence Relations of Order Two- Solving Linear Non homogeneous Recurrence Relations. Permutations-Cyclic permutation- Permutations with repetitions- permutations of sets with indistinguishable objects Combinations- Combinations with repetition.

Recurrence Relation :

A Recurrence is a way of giving information or instruction of prior knowledge.

Eg:

If Somebody requires the root of your house, you can say the instruction to reach at home is based on some prior knowledge or information. This process is called recursion.

Eg:-

1. The fibonacci numbers can be defined as follows.

$$F_0 = 1, \quad F_1 = 1$$

$$F_n = F_{n-1} + F_{n-2}$$

2. nC_r can be defined as follows.

$$nC_0 = 1$$

$$nC_n = 1$$

$$nC_r = (n-1)C_r + (n-1)C_{r-1}, \quad n > r > 0$$

3. Example :

Ackermann's function can be defined

$$A(0, y) = y+1$$

$$A(x+1, 0) = A(x, 1)$$

$$A(x+1, y+1) = A(x, A(x+1, y))$$

Recursion Iteration & Induction :

→ basic forward (Iteration) Recursion ↓ prior knowledge (backward)

While we calculate function values forth recursion, we start with the required expression in terms of earlier values.

We move backward until we reach the basis. but in iteration we start with basic and works forward using relation and stop when the required value is known.

Example :-

Calculate F_4 of the fibonacci S number using (i) Recursion (ii) Iteration.

(i) Recursion :

$$F_n = F_{n-1} + F_{n-2}$$

$$F_4 = F_{4-1} + F_{4-2}$$

$$= F_3 + F_2$$

$$= (F_{3-1} + F_{3-2}) + (F_{2-1} + F_{2-2})$$

$$= (F_2 + F_1) + (F_1 + F_0)$$

$$= F_2 + F_1 + F_1 + F_0$$

$$= (F_{2-1} + F_{2-2}) + F_1 + F_1 + F_0$$

$$= F_1 + F_0 + F_1 + F_1 + F_0$$

$\because F_0 = 1$

$\because F_1 = 1$

$$= 1 + 1 + 1 + 1 + 1$$

$$\therefore F_4 = 5$$

(ii) Iteration :

$F_0 = 1$

$F_1 = 1$

$F_2 = F_{2-1} + F_{2-2} = F_1 + F_0 = 1+1 = 2$

$F_3 = F_{3-1} + F_{3-2} = F_2 + F_1 = 2+1 = 3$

$F_4 = F_{4-1} + F_{4-2} = F_3 + F_2 = 3+2 = 5$

Recursion - Iteration (Difference)

- Usually Iterated Computation are faster than recursive Computation.
- But recursive definition give more insight into the interpretation of the given function.
- A Recursive Program is more difficult to a Corresponding iterative Program.
- In general, there are many Problems involving Recursion for which iterative Solution either do not exist or do not Easily found.

Recursion & Induction :

Induction is used for Proving Properties express as function of natural numbers.

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Property of induction:

An inductive definition of a Property or Set P is given as follows:

1. Given a finite Set A where elements have the Property P.

2. The elements of a Set B, all of which are Constructed from A have the Property P.

3. The elements Constructed as in (1) and (2) are the only elements Satisfying Property P.

So, in inductive definition, we use recursive definition in the forward direction.

Also note that Proof by induction Can be used whenever recursion is used.

Polynomial and their Calculation

Recursive Definition of Polynomial:

The Set S[x] of all Polynomials whose Coefficients are elements of S is defined as follows:

1. any element of S is a Polynomial of degree zero.

2. $P(x)x + a$ is a Polynomial of degree n when $P(x)$ is a Polynomial of degree $n-1$ and $a \in S$.

3. Only those expression obtained by using (1) and (2) finite number of times are Polynomials.

Example:-

Consider $F(x) = 5x^3 + 4x^2 + 3x + 2$. This can be defined using recursive definition as follows.

Solution:-

Given:

$$F(x) = 5x^3 + 4x^2 + 3x + 2$$

$$(((5)x + 4)x + 3)x + 2$$

A Polynomial define recursively is said to be a in telescopic form is the method of writing a Polynomial in recursive form (telescoping form) called Horner's Method.

Consider $P(x) = x^5 + 3x^4 + 5x^3 + x - 10$ in telescopic form.

Solution:-

$$P(x) = ((((1)x + 3)x + 5)x + 0)x + 1)x - 10$$

Use the Horner's method to write $P(x) = x^4 + 2x^3 + 3x^2 + 4x$ in telescoping form also mention the number of multiplication and additions / Subtraction involved in telescopic form Compare it with usual definition.

$$P(x) = ((((1)x + 2)x + 3)x + 4)x + 0$$

We require four multiplications & 4 additions

In the usual form we require 7 multiplication and 4 additions.

[for example

$2 \times 2 = 1$

$$P(2) = 2^4 + 2(2)^3 + 3(2)^2 + 4(2) + 0$$

2×2

$4(2^1) = 1$

$2^2 = 1$

$3(2^2) = 1$

$2^3 = 2(2^2) = 1$

$2(2^3) = 1$

$2^4 = 2(2^3) = 1$

$1(2^4) = 1$.

Thus, we required 7 multiplications

Thus by writing a Polynomial in telescopic form the number of multiplication is reduced from 7 to 4.

Sequence :

A Sequence of integer (discrete function) is a function from $N \times I$. Where N is a Set of all natural numbers & I is a Set of all integers.

Eg :-

Usually a Sequence is written as a list if S is a Sequence then it is usually written as $S_1, S_2, \ldots S_n \cdots$ When $S_n = S(n)$.

Eg :-

The fibonacci number F ~~f~~ $F_0 = 1$, $F_1 = 1$, $F_2 = 2 \ldots$

Reccurence Relation :

Let S be a Sequence of integer. A Recciurience Relation on S is a formula that relates all but a finite number of terms of S to Previous terms of S.

(i.e) There exists k_0 in the domain of S such that $S(k)$ for $k > k_0$ is expressed in terms of Sum of the terms of the Sequence Preceding S_k. The terms not defined by the formula are Said to form the initial Condition (basic or boundary Condition) of the Sequence.

Eg :-

The fibonacci Sequence divide fined by the relation $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2} \cdots$ are the initial Conditions are the basis.

Find the recurrence relation and basis for the Sequence. $(1, 3, 3^2, \dots)$

Take $\{0,1,2,\dots\}$ as the domain of the Sequence.

Then

$a_0 = 1$, $a_1 = 3$,

$a_2 = 3^2$

Hence the recurrence relation.

$a_n = 3a_{n-1}$ $\qquad a_1 = 3(a)^{1-1}$

$= 3(1) = 3$

$a_0 = 1$ is the basis. $\qquad a_2 = 3(a)^{2-1}$

$= 3(a)^1$

$= 3(3) = 9 = 3^2$

Consider D defined by $D(k) = 5.2^k$, $k \geq 0$ find the recurrence relation on D.

Sol:-

For $k \geq 0$

$D(k) = 5.2^k$ and $D(k-1) = 5.2^{k-1}$

So, $\dfrac{D(k)}{D(k-1)} = \dfrac{\cancel{5}.2^k}{\cancel{5}.2^{k-1}}$ $\qquad 2^k \cdot 2^{-k+1}$

$D(k) / D(k-1) = 2$

$D(k) = 2D(k-1)$

$D(k) - 2D(k-1) = 0 \qquad k \geq 1$

The initial Condition is $D(0) = 5$ $\qquad 2^0 = 1$

$\hookrightarrow 5.2^k$

5.2^0

$= 5.1$

$= 5$

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Definition :-

The recurrence relation on a sequence S is of order k if expressed as a function of $T(n-1) \ldots T(n-k)$ and $T(n-k)$ appears function.

Eg:

The relation $T(n) = 2(T(n-1)^2 - nT(n-3))$ is a recurrence relation of order 3. End

Definition:

A recurrence relation on a sequence S is a linear recurrence relation with constant coefficients if it is of the form

$$S(k) + C_1 S(k-1) + \ldots + C_n S(k-n) = f(k), k \geq n$$

where $C_1, C_2, C_3 \ldots C_n$ are numbers and f is a function defined for $k \geq n$, $C_n \neq 0$ then the relation is said to be of order n.

Definition:

For a recurrence relation $S(k) + C_1 S(k-1) + \ldots + C_n S(k-n) = f(k)$ then associated homogeneous relation is $S(k) + C_1 S(k-1) + \ldots + S(k-n) = \underline{0}$

Eg:-

Consider the recurrence relation

i) $D(k) - 3D(k-1) = 0$

ii) $C(k) - 5C(k-1) + 6C(k-2) = 2k-7$

iii) $S(k) - 4S(k-1) \div 17S(k-2) + 30S(k-3) = 4^k$

iv) $T(k) = T(k/2) + 5$, $k \geq 0$ where $k/2$ is the integral part of $k/2$.

Sol :-

i) Is a homogeneous of order 1.

ii) is a linear relation of order 2 but not homogeneous.

iii) is a linear non homogeneous relation on of order 3.

iv) is a recurrence relation of infinite order.

$$T(k) = T(k-n) + 5$$

Take $k = 2n$ in $T(k) = T\left(\frac{k}{2}\right) + 5$

$$k(2n) = T\left(\frac{2n}{2}\right) + 5$$

$$= T(n+5)$$

$$= T(2n-n) + 5$$

So it is not Possible to find a fixed +ve integer n Such that a relation of the type.

$$T(k) + C_1 T(k-1) + \dots + C_n T(k-n) = f(k)$$

$$k \geq n.$$

Find the recurrence relation satisfying $y_n = A(3)^n + B(-4)^n$

Note
∴ It is enough to find $n-1, n-2$

Sol:-

$$y_n = A(3)^n + B(-4)^n \rightarrow (1)$$

Take $n = n-1$

$$y_{n-1} = A(3)^{n-1} + B(-4)^{n-1}$$

3^{n-1+1}

$\times^{ly}$ 3

$$3y_{n-1} = A\,3^{n-\cancel{1}}.\cancel{3}^1 + B(-4)^{n-1}.3$$

$$3y_{n-1} = A(3)^n + B(-4)^{n-1}.3 \rightarrow (2)$$

(1) - (2)

$$y_n = \cancel{A(3)^n} + B(-4)^n$$

$$\underset{(-)}{3y_{n-1}} = \underset{(-)}{\cancel{A(3)^n}} \underset{(-)}{+} B(-4)^{n-1}.3$$

$$y_n - 3y_{n-1} = B(-4)^n + B(-4)^{n-1}3$$

$\mp \rightarrow$ doubt

$$= B(-4)^n \left(\frac{4}{4}\right) \mp B(-4)^{n-1}.3$$

$$= B(-4)^n\, 4 \cdot 4^{-1} \mp B(-4)^{n-1}.3$$

$$= B(-4)^{n-1}.4 - B(-4)^{n-1}.3$$

$$= B(-4)^{n-1}[4+3]$$

$$= B(-4)^{n-1}(7)$$

$$y_n - 3y_{n-1} = B(-4)^{n-1}(7) \rightarrow (3)$$

From (3) Take $n = n-1$

$$y_{n-1} - 3y_{n-2} = B(-4)^{n-2}(7) \rightarrow (4)$$

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$$④ \Rightarrow y_{n-1} - 3y_{n-2} = B(-4)^{n-2}(7).$$

$$= B(-4)^{n-1}(-4)^{-1}(7)$$

$$y_{n-1} - 3y_{n-2} = B(-4)^{n-1}\left(\frac{7}{-4}\right)$$

$$(-4)(y_{n-1} - 3y_{n-2}) = B(-4)^{n-1}(7)$$

$$(-4)(y_{n-1} - 3y_{n-2}) = y_n - 3y_{n-1} \quad \text{From (3)}$$

$$-4y_{n-1} + 12y_{n-2} = y_n - 3y_{n-1}$$

$$-4y_{n-1} + 12y_{n-2} - y_n + 3y_{n-1} = 0$$

$$-y_n - y_{n-1} + 12y_{n-2} = 0$$

$\times^{l}y\ (-)$

$y_n + y_{n-1} - 12y_{n-2} = 0$ which is the required recurrence relation.

Find the recurrence relation satisfying $y_n = (A + Bn)4^n$

Sol:-

$$y_n = A4^n + Bn.4^n \rightarrow ①$$

Put $n = n-1$

$$y_{n-1} = A4^{n-1} + B(n-1)4^{n-1} \qquad A4^{n-1}.4 + B(n-1)4^{n-1}.4$$

$$= A4^n4^{-1} + B(n-1)4^n4^{-1} \qquad A4^n + B(n-1)4^n$$

$$= \frac{1}{4}\left[A4^n + B(n-1)4^n\right]$$

$$4y_{n-1} = A4^n + B(n-1)4^n \rightarrow ②$$

① - ②

$$y_n = \cancel{A4^n} + Bn.4^n$$

$$4y_{n-1} = \cancel{A4^n} + B(n-1)4^n$$

$$(-) \quad (-) \quad (-)$$

$$y_n - 4y_{n-1} = Bn.4^n - B(n-1)4^n$$

$$= B\cancel{n}.4^n - B\cancel{n}.4^n + B4^n$$

$$y_n - 4y_{n-1} = B4^n \rightarrow ③$$

Put $n = n-1$ in ③

$$y_{n-1} - 4y_{n-2} = B.4^{n-1}$$

$$y_{n-1} - 4y_{n-2} = B4^n.4^{-1}$$

$$y_{n-1} - 4y_{n-2} = (B4^n)\frac{1}{4}$$

$$4(y_{n-1} - 4y_{n-2}) = y_n - 4y_{n-1}$$

$$4y_{n-1} - 16y_{n-2} - y_n + 4y_{n-1} = 0$$

$$-y_n + 8y_{n-1} - 16y_{n-2} = 0$$

$\times$ by $(-)$

$$y_n - 8y_{n-1} + 16y_{n-2} = 0$$

This is the required recurrence relation.

Find the recurrence relation for the fibonacci Sequence.

Sol:-

W.K.T $F_n = F_{n-1} + F_{n-2}$

Hence the recurrence relation for the fibonacci Sequence is

$$F_n - F_{n-1} - F_{n-2} = 0$$

For the Sequence defined by $A(k) = k^2 - k$, $k \geq 0$ obtain recurrence relation if A is a Sequence of integers.

Sol:-

$$A(k) = k^2 - k \rightarrow (1)$$

put $k = k-1$

$$A(k-1) = (k-1)^2 - (k-1) \rightarrow (2)$$

(1) - (2)

$k^2 + 1 -$

$$A(k) = k^2 - k$$
$$A(k-1) = (k-1)^2 - (k-1)$$
$$(-) \quad (-) \quad (+)$$

$$A(k) - A(k-1) = (k^2 - (k-1)^2)(-\cancel{k} + (\cancel{k}-1))$$

doubt

$$= k^2 - (k^2 + 1 - 2k) - 1$$

$$= \cancel{k^2} - \cancel{k^2} - 1 + 2k - 1$$

$$A(k) - A(k-1) = 2k - 2 \rightarrow (3)$$

put $k = k-1$ in (3)

$$A(k-1) - A(k-2) = 2(k-1) - 2$$

$A(k-1) - A(k-2) = 2k - 2 - 2$

$A(k-1) - A(k-2) = A(k) - A(k-1) - 2$

$A(k-1) - A(k-2) - A(k) + A(k-1) + 2 = 0$

$-A(k) + 2A(k-1) - A(k-2) + 2 = 0$

$\times$ by $(-)$

$A(k) - 2A(k-1) + A(k-2) - 2 = 0$. which is the required recurrence relation.

H.W

1. Find the recurrence relation for the sequence $S(k) = 2k + 9$.

$S(k) = 2k + 9 \rightarrow ①$

Put $k = k-1$

$S(k-1) = 2(k-1) + 9$

$= 2k - 2 + 9$

$S(k-1) = 2k + 7 \rightarrow ②$

① - ②

$S(k) = \cancel{2k} + 9$

$S(k-1) = \cancel{2k} + 7$

$(-) \quad (-) \quad (-)$

$S(k) - S(k-1) = 9 - 7$

$S(k) - S(k-1) = 2 \rightarrow ③$

~~From~~ put $k = k-1$ in ③

$S(k-1) - S(k-2) = 2$

$S(k-1) - S(k-2) = S(k) - S(k-1)$

$S(k-1) - S(k-2) - S(k) + S(k-1) = 0$

$-S(k) + 2S(k-1) - S(k-2) = 0$

2. Find the recurrence relation for the Sequence.

$$B(k) = 2k^2 + 1$$

Sol:-

$$B(k) = 2k^2 + 1 \rightarrow (1)$$

Put $k = k-1$

$$B(k-1) = 2(k-1)^2 + 1 \rightarrow (2)$$

(1) - (2)

$$B(k) = 2k^2 + \cancel{1}$$

$$B(k-1) = 2(k-1)^2 \cancel{+1}$$

$$(-) \quad (-) \quad (-)$$

$$B(k) - B(k-1) = 2k^2 - 2(k-1)^2$$

$$= 2k^2 - 2(k^2 + 1 - 2k)$$

$$= \cancel{2k^2} - \cancel{2k^2} - 2 + 4k$$

$$B(k) - B(k-1) = 4k - 2 \rightarrow (3)$$

Put $k = k-1$

$$B(k-1) - B(k-2) = 4(k-1) - 2$$

$$B(k-1) - B(k-2) = 4k - 4 - 2$$

$$= 4k - 2 - 4$$

$$= B(k) - B(k-1) - 4$$

$$B(k-1) - B(k-2) - B(k) + B(k-1) + 4 = 0$$

$$-B(k) + 2B(k-1) - B(k-2) + 4 = 0$$

$\times$ by $-$

$$B(k) - 2B(k-1) + B(k-2) - 4 = 0$$

$$B(k) - 2B(k-1) + B(k-2) = 4.$$

Solution of finite Order Homogeneous linear Relations:

Suppose we want to find $S(k)$

$\therefore S(k) - 7S(k-1) + 12S(k-2) = 0$

$S(0) = S(1) = 4$

To find $S(2) \Rightarrow S(2) - 7S(2-1) + 12S(2-2) = 0$

$$S(2) - 7S(1) + 12S(0) = 0$$

$$S(2) - 7(4) + 12(4) = 0$$

$$S(2) - 28 + 48 = 0$$

$$S(2) = -48 + 28$$

$$S(2) = -20$$

This Process can be repeated for finding $S(k)$. $k > 2$. But this Process is tedious and time consuming for large values.

If $S(k)$ is got as a fun" of k, then $S(k)$ can be directly evaluated. Such a fun is called a closed form expression for a sequence S.

Find closed form expression for the recurrence relation $D(k) - 2D(k-1) = 0$, $D(0) = 5$

$D(k) - 2D(k-1) = 0$

$D(k) = 2D(k-1) \rightarrow (1)$

As $D(0) = 5$

$(1) \Rightarrow D(1) = 2D(1-1)$

$= 2D(0)$

$= 2(5)$

$D(2) = 2D(2-1)$
$= 2D(1)$
$= 2(2(5))$
$= 2^2(5)$
$= 5.2^2$

We Can Prove by induction that
$D(k) = 5.2^k \quad \forall \; k \geq 0$

Hence $D(k) = 5.2^k$ is Closed form Expression for D.

Definition:

The Process of finding a closed form expression for the terms of a Sequence from its recurrence relation is Called Solving the relation.

Eg:-

We Solved the relation $D(k) = 2D(k-1)$
$D(0) = 5$

It is not Possible to Solve all recurrence relation. Also there is no Single algorithm to Solve the relations that are Solvable.

Definition:

The Characteristic Equation of the homogeneous relation of order n. $S(k) + C_1 S(k-1) + \ldots + C_n S(k-n) = 0$ is the n^{th} degree equation
$a^n + C_1 a^{n-1} + \ldots + C_{n-1} a + C_n = 0$

The L.H.S of this equation is Called Characteristic Polynomial.

Eg.

Find the Characteristic Equation of

$J(k) - 4J(k-1) + 4J(k-2) = 0$

Sol :-

Given

$$J(k) - 4J(k-1) + 4J(k-2) = 0 \quad \rightarrow (1)$$

The given Equation is of the form

$$S(k) + C_1 S(k-1) + C_2 S(k-2) = 0$$

$\therefore$ The n^{th} degree Equation

$$a^n + C_1 a^{n-1} + C_2 a^{n-2} = 0$$

The 2nd degree Equation

$$a^2 + C_1 a + C_2 = 0 \quad \rightarrow (2)$$

$\therefore$ The Characteristic Equation is

$$a^2 - 4a + 4 = 0$$

Algorithm for Solving n^{th} order homogeneous Recurrence Relation:

Step 1 : Write the Characteristic equation of the given homogeneous relation.

Step 2 : Find all the roots of the Characteristic equation. (They are called Characteristic roots)

Step 3 : (i) If the roots $a_1, a_2, \dots a_n$ are distinct then the general Solution of the recurrence relation is

$$S(k) = b_1 a_1^k + b_2 a_2^k + \dots b_n a_n^k \longrightarrow ①$$

(ii) If the root a_j is repeated P times, $b_j a_j^k$ is replaced by

$$(C_0 + C_1 k + \dots C_{P-1} k^{P-1}) a_j^k$$

(In Particular, if a_j is a double root then $b_j a_j^k$ is replaced by $(C_0 + C_1 k) a_j^k$).

Step 4 : If n initial Conditions are given, Obtain n linear equations in a unknowns $b_1, b_2, \dots b_n$ (got in (1)) by replacing L.H.S of (1) by the given values. If Possible, Solve these equations.

Note: We have a general

---

1. Solve the following recurrence relation

$$S(k) - 10S(k-1) + 9S(k-2) = 0 \quad , \quad S(0) = 3, \ S(1) = 11.$$

Sol:-

$$S(k) - 10S(k-1) + 9S(k-2) = 0$$

The characteristic equation is

$$a^2 - 10a + 9 = 0$$

$$a = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1 \quad b = -10 \quad c = 9$$

$$a = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(9)}}{2(1)}$$

$$= \frac{10 \pm \sqrt{100-36}}{2}$$

$$= \frac{10 \pm \sqrt{64}}{2}$$

$$= \frac{10 \pm 8}{2}$$

$$= \frac{10+8}{2}, \frac{10-8}{2}$$

$$= \frac{18}{2}, \frac{2}{2}$$

$$= 9, 1$$

The roots are distinct 9, 1

$$S(k) = b_1 \cdot 1^k + b_2 9^k$$

$$= b_1 + b_2 9^k$$

$$S(k) = b_1 + b_2 9^k$$

Put $k=0$ $\quad S(0) = b_1 + b_2 9^0 = 3$

$$= b_1 + b_2 = 3 \rightarrow (1)$$

$k=1$ $\quad S(1) = b_1 + b_2 9^1 = 11$

$$b_1 + 9b_2 = 11 \rightarrow (2)$$

Solve (1) & (2)

$$\cancel{b_1} + b_2 = 3$$

$$\cancel{b_1} + 9b_2 = 11$$

$$\cancel{(-)} \quad (-) \quad (-)$$

$$-8b_2 = -8$$

$\boxed{b_2 = 1}$ Sub in 1

$b_1 + 1 = 3$

$b_1 = 3 - 1$

$\boxed{b_1 = 2}$

$\therefore \; b_1 = 2, \; b_2 = 1$

Hence $S(k) = 2 + 9^k$

$\boxed{b_2 = 1}$ Sub in ①

$b_1 + 1 = 3$

$b_1 = 3 - 1$

$\boxed{b_1 = 2}$

$\therefore b_1 = 2, \; b_2 = 1$

Hence $S(k) = 2 + 9^k$

Solve $D(k) - 8D(k-1) + 16D(k-2) = 0$ Where $D(2) = 16$, $D(3) = 80$.

Sol:-

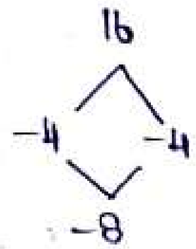
$$D(k) - 8D(k-1) + 16D(k-2) = 0$$

The Characteristic equation is

$$a^2 - 8a + 16 = 0$$

$$(a-4)(a-4) = 0$$

$$a = 4, 4$$



The roots are Same $4, 4$

$$D(k) = (C_0 + C_1 k)\, a_j^k$$

put $k = 2$ $= (C_0 + C_1 k)\, 4^k \longrightarrow$ Ⓐ

$$D(2) = (C_0 + C_1 2)\, 4^2 = 16$$

$$(C_0 + C_1 2)\, 16 = 16$$

$$C_0 + 2C_1 = 16/16$$

$$C_0 + 2C_1 = 1 \longrightarrow ①$$

Put $k=3$

$$D(3) = (C_0 + C_1(3))4^3 = 80$$

$$(C_0 + C_1(3))\,64 = 80$$

$$C_0 + 3C_1 = \frac{\overset{5}{\cancel{80}}}{\cancel{64}\,4}$$

$$C_0 + 3C_1 = \frac{5}{4} \longrightarrow (2)$$

Solve the equ (1) & (2)

$$\cancel{C_0} + 2C_1 = 1$$
$$\cancel{C_0} + 3C_1 = 5/4$$
$$(-) \quad (-) \quad (-)$$

$$2C_1 - 3C_1 = 1 - \frac{5}{4}$$

$$-C_1 = \frac{4-5}{4}$$

$$-C_1 = \frac{-1}{4}$$

$$\boxed{C_1 = 1/4}$$ Sub in 1

$$C_0 + 2\left(\frac{1}{4}\right) = 1$$

$$C_0 + \frac{1}{2} = 1$$

$$C_0 = 1 - \frac{1}{2}$$

$$= \frac{2-1}{2}$$

$$\boxed{C_0 = \frac{1}{2}}$$

Hence (A)

$$D(k) = \left(\frac{1}{2} + \frac{1}{4} \cdot k\right) 4^k$$

H.W

Find $f(n)$ if $f(n) = 7f(n-1) - 10f(n-2)$ given that $f(0) = 4$, $f(1) = 17$

Ans:-
$f(n) = 1(2^n) + 3(5^n)$
or
$2^n + 3(5^n)$

Sol:-

$f(n) - 7f(n-1) + 10f(n-2) = 0$

The Characteristic Equation is

$a^2 - 7a + 10 = 0$

10

$(a-5)(a-2) = 0$ $\quad -5 \quad -2$

$a = 5, 2$

-7

The roots are distinct $5, 2$

$$f(n) = b_1 \cdot 5^n + b_2 2^n \rightarrow (A)$$

Put $n = 0$

$$f(0) = b_1 \cdot 5^0 + b_2 2^0 = 4$$

$$b_1 + b_2 = 4 \rightarrow (1)$$

Put $n = 1$

$$f(1) = b_1 \cdot 5^1 + b_2 2^1 = 17$$

$$5 \cdot b_1 + 2b_2 = 17 \rightarrow (2)$$

Solve ① & ②

① × by 5 ⟹ $\cancel{5b_1} + 5b_2 = 20$

$\cancel{5b_1} + 2b_2 = 17$

$(-) \quad (-) \quad (-)$

$3b_2 = 3$

$b_2 = 3/3$

$\boxed{b_2 = 1}$

$b_2 = 1$ Sub in equ 1

$b_1 + b_2 = 4$

$b_1 + 1 = 4$

$b_1 = 4 - 1$

$\boxed{b_1 = 3}$

Hence, A

$f(n) = 3.5^n + 1.2^n$

$= 3(5)^n + 2^n$

Solve the recurrence relation

$S(k) - 4S(k-1) - 11S(k-2) + 30S(k-3) = 0$

$S(0) = 0, \quad S(1) = -35, \quad S(2) = -85$

Sol:-

The Characteristic equation is

$$a^3 - 4a^2 - 11a + 30 = 0$$

$$\begin{array}{c|cccc} & 1 & -4 & -11 & 30 \\ 2 & 0 & 2 & -4 & -30 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$$(a-2)(a^2 - 2a - 15) = 0$$

$a = 2$

$(a+3)(a-5) = 0$

$a = 2, -3, 5$

$$\begin{array}{ccc} & -15 & \\ +3 & & -5 \\ & -2 & \end{array}$$

$$\therefore S(k) = b_1(2)^k + b_2(-3)^k + b_3(5)^k \rightarrow (A)$$

put $k = 0$

$$S(0) = b_1(2)^0 + b_2(-3)^0 + b_3(5)^0 = 0$$

$$b_1 + b_2 + b_3 = 0 \rightarrow (1)$$

put $k = 1$

$$S(1) = b_1(2)^1 + b_2(-3)^1 + b_3(5)^1 = -35$$

$$2b_1 - 3b_2 + 5b_3 = -35 \rightarrow (2)$$

put $k = 2$

$$S(2) = b_1(2)^2 + b_2(-3)^2 + b_3(5)^2 = -85$$

$$4b_1 + 9b_2 + 25b_3 = -85 \rightarrow (3)$$

Solve ① & ②

$$① \times 2 \Rightarrow \cancel{2b_1} + 2b_2 + 2b_3 = 0$$

$$\cancel{2b_1} - 3b_2 + 5b_3 = -35$$

$$(-) \quad (-) \quad (-) \quad (-)$$

$$5b_2 - 3b_3 = 35 \rightarrow ④$$

Solve ① & ③

$$① \times \text{by } 4 \Rightarrow \cancel{4b_1} + 4b_2 + 4b_3 = 0$$

$$\cancel{4b_1} + 9b_2 + 25b_3 = -85$$

$$(-) \quad (-) \quad (-) \quad (-)$$

$$-5b_2 - 21b_3 = 85 \rightarrow ⑤$$

Solve ④ & ⑤

$$\cancel{5b_2} - 3b_3 = 35$$

$$\cancel{-5b_2} - 21b_3 = 85$$

$$-24b_3 = 120$$

$$b_3 = \frac{\cancel{120}^{5}}{-\cancel{24}}$$

$$\boxed{b_3 = -5}$$

$$\frac{\overset{2}{24 \times 5}}{120}$$

$b_3 = -5$ Sub in equ ④

$$5b_2 - 3(-5) = 35$$

$$5b_2 + 15 = 35$$

$$5b_2 = 35 - 15$$

$$b_2 = \frac{20}{5}$$

$$\boxed{b_2 = 4}$$

$b_3 = -5, \; b_2 = 4$ Sub in equ ①

$$b_1 + b_2 + b_3 = 0$$

$$b_1 + 4 - 5 = 0$$

$$b_1 = 5 - 4$$

$$\boxed{b_1 = 1}$$

Hence ; A

$$S(k) = 1 \cdot 2^k + 4(-3)^k + (-5)(5)^k$$

$$= 2^k + 4(-3)^k - 5(5)^k$$

Write the recurrence relation for Fibonacci ~~Sequence~~ number & Solve it.

Sol:-

The recurrence relation is

$$F(n) - F(n-1) - F(n-2) = 0$$

The characteristic equation is

$$a^2 - a - 1 = 0$$

$$a = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1, \; b = -1, \; c = -1$$

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

$$a = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

Hence

$$F(n) = b_1\left(\frac{1+\sqrt{5}}{2}\right)^n + b_2\left(\frac{1-\sqrt{5}}{2}\right)^n \rightarrow (A)$$

put $n=0$

$$F(0) = b_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + b_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = 1$$

$$b_1 + b_2 = 1 \rightarrow (1)$$

$$F(1) = b_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + b_2\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$$

$$b_1\left(\frac{1+\sqrt{5}}{2}\right) + b_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \rightarrow (2)$$

From equ (1) $b_2 = 1-b_1$

$b_2 = 1-b_1$ Sub in equ (2)

$$b_1\left(\frac{1+\sqrt{5}}{2}\right) + (1-b_1)\left(\frac{1-\sqrt{5}}{2}\right) = 1$$

$$\frac{1}{\cancel{2}}\cancel{b_1} + \frac{\sqrt{5}}{2}b_1 + \frac{1}{2} - \frac{\sqrt{5}}{2} - \frac{1}{\cancel{2}}\cancel{b_1} + \frac{\sqrt{5}}{2}b_1 = 1$$

$$\frac{\cancel{2}\sqrt{5}}{\cancel{2}}b_1 = 1 - \frac{1}{2} + \frac{\sqrt{5}}{2}$$

$$\sqrt{5}b_1 = \frac{2-1}{2} + \frac{\sqrt{5}}{2}$$

$$= \frac{1}{2} + \frac{\sqrt{5}}{2}$$

$$\sqrt{5}b_1 = \frac{1+\sqrt{5}}{2}$$

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$$b_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$b_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$ Sub in equ ①

$$\frac{1+\sqrt{5}}{2\sqrt{5}} + b_2 = 1$$

$$b_2 = 1 - \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$= \frac{2\sqrt{5} - 1 + \sqrt{5}}{2\sqrt{5}}$$

$$b_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}$$

Hence, A

$$F(n) = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$= \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Solution of Non-Homogeneous Relations:

In the case of non-homogeneous recurrence relations the general solution is the sum of

i) Solution for the corresponding homogeneous relation.

ii) Particular solution depending on the R.H.S of the given recurrence relation.

(i) Can be found in the previous section.

For finding the particular solution (ii) we adopt the following procedure.

Procedure for finding the particular Solution :

Step 1 : a) If the R.H.S of the recurrence relation is $a_0+a_1k+\dots a_mk^m$, Substitute $d_0+d_1k+\dots d_mk^m$ in place of $T(k)$, $d_0+d_1(k-1)+\dots a_m(k-1)^m$ in place of $T(k-1)$ etc, in the given recurrence relation.

b) If the R.H.S is ca^k, Substitute d_0a^k in place of $T(k)$, d_0a^{k-1} in place of $T(k-1)$ etc in the given relation.

Step 2 : At the end of Step 1 we get a Polynomial in k with Coefficients $d_0, d_1, \dots$ on L.H.S which is Equal to the R.H.S of the given recurrence relation. Equate the Coefficient of powers of k on both Sides to get values for $d_0, d_1, \dots$

Step 3 : The general Solution is the Sum of the Solution for the homogeneous relation and the Particular Solution got in Step 2. use initial Conditions for getting the values of unknowns (b_1, b_2, etc).

Note. (We discuss Some particular Cases now).

1. If the R.H.S of the given recurrence relation is a Constant a_0, then replace $T(k), T(k-1), \dots$ by d_0.

2. If the R.H.S is a_0+a_1k, replace $T(k)$ by d_0+d_1k, $T(k-1)$ by $d_0+d_1(k-1)$ etc.

3. When the R.H.S is ca^k and a Coincides with a Characteristic root, the above method fails. When a is a Simple root of the Characteristic Equation, take d_0ka^k. When a is a double root of the Characteristic equation take $d_0k^2a^k$.

1. Solve $T(k) - 7T(k-1) + 10T(k-2) = 6+8k$

with $T(0)=1$, $T(1)=2$

<u>Sol:-</u>

a) Homogeneous Solution

The characteristic equation is

$$a^2 - 7a + 10 = 0$$

10

-5 -2

-7

$$(a-2)(a-5)=0$$

$$a = 2, 5$$

$\therefore$ The roots are 2, 5

Hence the homogeneous Solution is

$$b_1 2^k + b_2 5^k.$$

b) Particular Solution :

The R.H.S of the given relation is

$$6+8k = d_0 + d_1 k$$

The Particular Solution $= d_0 + d_1 k$

Replace $T(k) = d_0 + d_1 k$

$$T(k-1) = d_0 + d_1(k-1)$$

$$T(k-2) = d_0 + d_1(k-2)$$

$$\therefore d_0 + d_1 k - 7(d_0 + d_1(k-1)) + 10(d_0 + d_1(k-2)) = 6+8k$$

$$d_0 + d_1 k - 7d_0 - 7d_1 k + 7d_1 + 10d_0 + 10d_1 k - 20d_1 = 6+8k$$

$$4d_0 - 13d_1 + 4d_1 k = 6+8k$$

Equating the Coefficient

$4d_0 - 13d_1 = 6$

$4d_1 = 8$

$d_1 = 8/4$

$\boxed{d_1 = 2}$

$4d_0 - 13(2) = 6$

$4d_0 - 26 = 6$

$d_0 = \frac{6+26}{4}$

$= 32/4$

$\boxed{d_0 = 8}$

Hence the Particular Solution is $8+2k$

The general Solution is

$T(k) = b_1 2^k + b_2 5^k + 8 + 2k$

Now,

Take $k=0$

$T(0) = b_1 2^0 + b_2 5^0 + 8 + 2(0) = 1$

$b_1 + b_2 + 8 = 1$

$b_1 + b_2 = 1 - 8$

$b_1 + b_2 = -7 \rightarrow (1)$

$k=1$,

$T(1) = b_1 2^1 + b_2 5^1 + 8 + 2(1) = 2$

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$$2b_1 + 5b_2 + 8 + 2 = 2$$

$$2b_1 + 5b_2 = 2 - 10$$

$$2b_1 + 5b_2 = -8 \rightarrow (2)$$

Solve equ (1) & (2)

(1) × by 2

$$\cancel{2b_1} + 2b_2 = -14$$

$$\cancel{2b_1} + 5b_2 = -8$$

$$(-) \quad (-) \quad (-)$$

$$-3b_2 = -6$$

$$b_2 = \frac{-6}{-3}$$

$$\boxed{b_2 = 2}$$

$b_2 = 2$ Sub in equ (1)

$$b_1 + 2 = -7$$

$$b_1 = -7 - 2$$

$$\boxed{b_1 = -9}$$

$$\therefore T(k) = -9 \cdot 2^k + 2 5^k + 8 + 2k$$

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2. Solve $S(k) - S(k-1) - 6S(k-2) = -30$

Where $S(0) = 20$, $S(1) = -5$.

Solu :-

a) Homogeneous Solution :

The characteristic equation is

$$a^2 - a - 6 = 0$$

$$(a+2)(a-3) = 0$$

$$a = -2, 3$$

-6

$2 \quad -3$

-1

The roots are $-2, 3$

Hence the homogeneous Solution is

$$b_1 \cdot (-2)^k + b_2 \cdot 3^k$$

b) Particular Solution.

The R.H.S of the recurrence relation is Constant

$\therefore$ Particular Solution $= d$

Replace $S(k) = d$

$$S(k-1) = d$$

$$S(k-2) = d$$

$$\therefore \cancel{d} - \cancel{d} - 6d = -30$$

$$-6d = -30$$

$$\boxed{d = 5}$$

Hence the Particular Solution = d = 5

The general Solution is

$$S(k) = b_1 \cdot (-2)^k + b_2 \cdot 3^k + 5$$

Take

$k = 0$

$$S(0) = b_1 (-2)^0 + b_2 \cdot 3^0 + 5 = 20$$

$$b_1 + b_2 + 5 = 20$$

$$b_1 + b_2 = 20 - 5$$

$$b_1 + b_2 = 15 \rightarrow (1)$$

$k = 1$

$$S(1) = b_1 (-2)^1 + b_2 (3)^1 + 5 = -5$$

$$-2b_1 + 3b_2 = -5 - 5$$

$$-2b_1 + 3b_2 = -10 \rightarrow (2)$$

Solve equ (1) & (2)

(1) × by 2

$$\cancel{2b_1} + 2b_2 = 30$$

$$\cancel{-2b_1} + 3b_2 = -10$$

$$5b_2 = 20$$

$$b_2 = \frac{20}{5}$$

$$\boxed{b_2 = 4}$$

$b_2 = 4$ Sub in equ (1)

$$b_1 + 4 = 15$$

$$b_1 = 15 - 4$$

$$\boxed{b_1 = 11}$$

Hence the Solution is

$$S(k) = 11.(-2)^k + 4(3)^k + 5$$

---

3. Solve $S(k) - 3S(k-1) - 4S(k-2) = 4^k$.

Solution :-

a) Homogeneous Solution

The characteristic equation is

$$a^2 - 3a - 4 = 0$$

$$(a+1)(a-4) = 0$$

$$a = -1, 4$$

The roots are $-1, 4$

Hence the homogeneous Solution is

$$b_1(-1)^k + b_2(4)^k$$

b) Particular Solution

The R.H.S of recurrence relation is 4^k.

4 is a root of characteristic equation.

Take Particular Solution $= dk4^k$

(Suppose 4 is not a root of equation it is enough to take $d4^k$)

Replace $S(k) = dk4^k$

$$S(k-1) = d(k-1)4^{k-1}$$

$$S(k-2) = d(k-2)4^{k-2}$$

$$\therefore dk4^k - 3d(k-1)4^{k-1} - 4d(k-2)4^{k-2} = 4^k$$

$$dk4^{k-2+2} - 3d(k-1)4^{k-1-1+1} - 4d(k-2)4^{k-2} = 4^{k-2+2}$$

$$dk4^{k-2+2} - 3d(k-1)4^{k-2+1} - 4d(k-2)4^{k-2} = 4^{k-2+2}$$

$$\cancel{4^{k-2}}\left[dk(4)^2 - 3d(k-1)4 - 4d(k-2)\right] = \cancel{4^{k-2}} \cdot 4^2$$

$$16dk - 12d(k-1) - 4d(k-2) = 16$$

$$16\cancel{dk} - 12\cancel{dk} + 12d - 4\cancel{dk} + 8d = 16$$

$$20d = 16$$

$$d = \frac{16}{20}$$

$$\boxed{d = 0.8}$$

Hence the Particular Solution $= dk4^k$

$$= (0.8)k4^k$$

$\therefore$ The general Solution is

$$S(k) = b_1(-1)^k + b_2(4)^k + 0.8k4^k$$

4. $S(k) - 4S(k-1) + 4S(k-2) = 3k + 2^k$

$S(0) = 1, \quad S(1) = 1$

<u>Sol</u> :-

a) Homogeneous Solution

The Characteristic equation is

$a^2 - 4a + 4 = 0$

$(a-2)(a-2) = 0$

$a = 2, 2$

4

-2 -2

-4

The roots are 2,2.

Hence the homogeneous Equation is

$$(C_0 + C_1 k) 2^k$$

b) Particular Solution for 3k

Particular Solution $= d_0 + d_1 k$

$$3k + 2^k = d_0 + d_1 k$$

Replace $S(k) = d_0 + d_1 k$

$$S(k-1) = d_0 + d_1 (k-1)$$

$$S(k-2) = d_0 + d_1 (k-2)$$

$$\therefore d_0 + d_1 k - 4[d_0 + d_1 (k-1)] + 4[d_0 + d_1 (k-2)] = 3k$$

$d_0 + d_1 k - \cancel{4d_0} - \cancel{4d_1 k} + 4d_1 + \cancel{4d_0} + \cancel{4d_1 k} - 8d_1 = 3k$

$d_0 - 4d_1 + d_1 k = 3k$

Equating the Coefficient

$d_1 k = 3k$

$d_1 = \frac{3\cancel{k}}{\cancel{k}}$

$\boxed{d_1 = 3}$

$d_0 - 4d_1 = 0$

$d_0 - 4(3) = 0$

$d_0 - 12 = 0$

$\boxed{d_0 = 12}$

$\therefore$ The Particular Solution for $d_0 + d_1 k$ is

$12 + 3k$

c) Particular Solution for 2^k.

Particular Solution $= dk^2 2^k$.

(Since the base of the R.H.S equation 2 is a double root of the characteristic equation)

Replace $S(k) = dk^2 2^k$

$$S(k-1) = d(k-1)^2 2^{k-1}$$

$$S(k-2) = d(k-2)^2 2^{k-2}$$

$$\therefore dk^2 2^k - 4[d(k-1)^2 2^{k-1}] + 4[d(k-2)^2 2^{k-2}] = 2^k$$

$$\cancel{2^k}[dk^2 - 4[d(k-1)^2 2^{-1}] + 4[d(k-2)^2 2^{-2}] = \cancel{2^k}$$

$$dk^2 - \frac{\overset{2}{\cancel{4}}}{\cancel{2}} d(k-1)^2 + \frac{\overset{1}{\cancel{4}}}{\cancel{2^2}} d(k-2)^2 = 1$$

$$dk^2 - 2d(k-1)^2 + d(k-2)^2 = 1$$

$$dk^2 - 2d(k^2 + 1 - 2k) + d(k^2 + 4 - 4k) = 1$$

$$\cancel{dk^2} - 2\cancel{dk^2} - 2d + 4\cancel{dk} + dk^2 + 4d - 4\cancel{dk} = 1$$

$$-2d + 4d = 1$$

$$2d = 1$$

$$\boxed{d = 1/2}$$

$\therefore$ The Particular Solution is $\left(\frac{1}{2}\right) k^2 2^k$

$\therefore$ The general Solution is

$$S(k) = (C_0 + C_1 k) 2^k + 12 + 3k + \left(\frac{1}{2}\right) k^2 2^k$$

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put $S(0) = (C_0+C_1(0))2^0 + 12+3(0)+\left(\frac{1}{2}\right)0^2 2^0 = 1$

$k=0$

$(C_0+0)+12+0=1$

$C_0 = 1-12$

$\boxed{C_0 = +1}$

$\boxed{C_0 = -11}$

put $k=1$

$S(1) = (C_0+C_1(1))2^1 + 12+3(1)+\frac{1}{2}(1)^2 2^1 = 1$

$(C_0+C_1)2+15+\frac{1}{\cancel{2}}(\cancel{2})=1$

$(C_0+C_1)2+16=1$

$(-11+C_1)2+16=1$

$-22+2C_1+16=1$

$2C_1-6=1$

$2C_1=1+6$

$\boxed{C_1 = 7/2}$

$\therefore S(k) = \left((-11)+\frac{7}{2}k\right)2^k + 12+3k+\left(\frac{1}{2}\right)k^2 2^k$

$= 12+3k+\left[-11+\frac{7}{2}k+\frac{1}{2}k^2\right]2^k$

$= 12+3k+\left(\frac{-22+7k+k^2}{2}\right)2^k$

$= 12+3k\,(-22+7k+k^2)\,2^{k-1}$

Permutation :

Combinatorial Analysis involves determining the number of Possibilities of Some event without enumerating all the Possibilites.

In order to develop the general Procedure for obtaining Possibilities we have to introduce the Concepts Called Permutations & Combinations.

$$nP_r = \frac{n!}{(n-r)!}$$

(or)

$$\frac{\lfloor n}{\lfloor n-r}$$

Permutation when Some of the things are alike taken all time.

All letters are replaced by distinct letters the number of arrangements of n things is $\lfloor n$ (or) $n!$.

$$\therefore \lfloor n = x \lfloor p \lfloor q \lfloor r \text{ (or)}$$

$$n! = x\, p!\, q!\, r!$$

i.e) $$x = \frac{n!}{p!\,q!\,r!}$$

Permutation when each thing may be repeated

$$\therefore n \times n \times n = n^r \text{ ways}$$

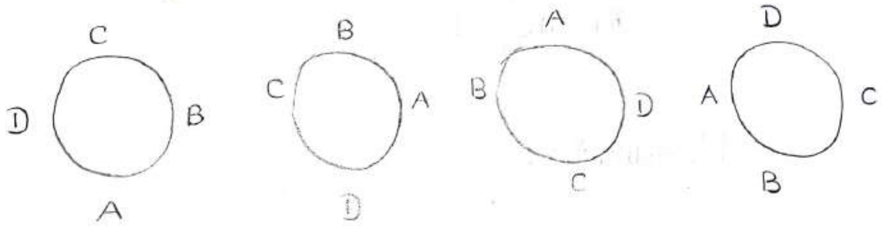
Circular permutation (or) Cyclic permutation.

We have seen permutation of n things in a row. Now we consider the permutations of n things along a circle.

Eg:-

In general n distinct things can be arranged along a circle.

$$\frac{\lfloor n}{n} = \lfloor n-1$$



Combination :

In permutation of n things taken r at a time we have considered the number of different arrangements.

Here we pay due regard of the order in which the different things occur.

On the other hand if we do not give importance to the order but only consider the selections of the or things out of n things we call it Combination.

$$nC_r = \frac{n!}{r!\,(n-r)!}$$

(or)

$$= \frac{\lfloor n}{\lfloor r \,\lfloor n-r}$$

Note:

i) $nC_0 = 1$

ii) $nC_1 = 1$

Permutation:

Any arrangement of a set of n objects is called Permutation. It is denoted by nP_r.

$$nP_r = \frac{n!}{(n-r)!}$$

$\because nC_n = 1$

Ex

Find the value of $^7P_2$ & $^{10}P_3$

Sol:-

$$^7P_2 = \frac{7!}{(7-2)!}$$

$$= \frac{7!}{(5)!}$$

$$= \frac{7\times6\times\cancel{5!}}{\cancel{5!}}$$

$$= 42$$

$$^{10}P_3 = \frac{10!}{(10-3)!}$$

$$= \frac{10\times9\times8\times\cancel{7!}}{\cancel{7!}}$$

$$= 720$$

Prove that $^nP_r = {}^{(n-1)}P_r + r\,{}^{(n-1)}P_{r-1}$

$$^nP_r = \frac{n!}{(n-r)!} \rightarrow \text{L.HS}$$

$$^{(n-1)}P_r = \frac{(n-1)!}{(n-1-r)!}$$

$$^{n-1}P_{r-1} = \frac{(n-1)!}{(n-1-(r-1))!}$$

$$= \frac{(n-1)!}{(n-\cancel{1}-r+\cancel{1})!}$$

$$= \frac{(n-1)!}{(n-r)!}$$

$$(n-1)P_r + r(n-1)P_{r-1} = \frac{(n-1)!}{(n-1-r)!} + \frac{(n-1)!}{(n-r)!}$$

Multiply & ÷ by $(n-r)$ on R.H.S

$$= \frac{(n-1)!\,(n-r)}{(n-1-r)!\,(n-r)} + r\,\frac{(n-1)!\,\cancel{(n-r)}}{(n-r)!\,\cancel{(n-r)}}$$

$$= \frac{(n-1)!\,(n-r)}{(n-r)\,((n-r)-1)!} + r\,\frac{(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!\,(n-r)}{(n-r)!} + \frac{r\,(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!\,[n-\cancel{r}+\cancel{r}]}{(n-r)!}$$ $\because n(n-1) = n!$

$$= \frac{n\,(n-1)!}{(n-r)!}$$

$$= \frac{n!}{n-r!} \longrightarrow \text{R.H.S}$$

$$= nP_r$$

A Committee of 3 to be choosen out of 5 english man, 4 french man & 3 indians. The Committee to contain one of the each nationality.

(i) In how many ways can done by this

one English man can be choosen from 5 ways.
one french man can be choosen from 4 ways
one indian can be choosen from 3 ways.

$\therefore$ Total no of ways $= 5 \times 4 \times 3$
$= 60$ ways

(ii) In how many arrangement will a Particular indian be included.

Sol:-
If a specific indian in the Committee there is only one way of choosing indian

$\therefore$ Total no of ways $= 5 \times 4 \times 1$
$= 20$ ways.

There are 5 trains to M to D & back to M. In how many way can be person go from M to D & return into a different train.

Sol:-
There are 5 ways to choosing a train from M to D.

There are 4 ways to choosing a train from D to M.

∵ He cannot choose the same train for return.

∴ Total no of ways = 5×4

$= 20$ ways.

How many no of 4 digits can be formed out of the digits 1, 2 .... 9 if repetition of digits is i) not allowed ii) allowed.

Sol :-

(i) Not allowed

1000	100	10	1	
$\square$	$\square$	$\square$	$\square$	←

The unit place can be filled in 9 ways.

The 10^{th} place can be filled in 8 ways.

The 100^{th} place can be filled in 7 ways

The 1000^{th} place can be filled in 6 ways.

∴ Total no of ways $= 9 \times 8 \times 7 \times 6$

$= 3024$ ways.

(ii) Allowed

If repetition is allowed all the 4 places can be filled in 9 ways.

∴ Total no of ways $= 9 \times 9 \times 9 \times 9 = 6591$ ways.

How many no of 4 digit can be formed out of the 4 digits 0, 1, 2 .... 9 if repetition of the digits

i) not allowed ii) allowed

Solution :—

1000^{th} 100^{th} 10^{th} 1^{th}

_ _ _ _

i) 0 cannot be filled in 1000^{th} place

$\therefore$ 1000^{th} place can be filled in 9 ways

100^{th} place can be filled in 9 ways.

10^{th} place can be filled in 8 ways

1^{th} place can be filled in 7 ways.

$\therefore$ Total no of ways $= 9 \times 9 \times 8 \times 7$

$= 4536$ ways.

ii) If repetition is allowed

1000^{th} place can be filled in 9 ways

100^{th} place can be filled in 10 ways

10^{th} place can be filled in 10 ways

Unit place can be filled in 10 ways

$\therefore$ Total no of ways $= 9 \times 10 \times 10 \times 10$

$= 9000$ ways.

How many odd no of 4 digits can be formed out of the digits 1, 2, ... 9 if repetition of digits i) not allowed ii) allowed.

Sol :-

There are 5 odd places (1, 3, 5, 7, 9)

(i) ∴ The unit place can be filled in 5 ways

The 10^{th} place can be filled in 8 ways.

The 100^{th} place can be filled in 7 ways.

The 1000^{th} place can be filled in 6 ways.

∴ Total no of ways = $5 \times 8 \times 7 \times 6$

= 1680 ways.

(ii) If repetitions is allowed.

The unit place can be filled in 5 ways

The 10^{th} place can be filled in 9 ways

The 100^{th} place can be filled in 9 ways

The 1000^{th} place can be filled in 9 ways

∴ Total no of ways = $5 \times 9 \times 9 \times 9$

= 3645 ways.

How many odd no of 4 digits can be formed out of the digits 0, 1, 2, ... 9 if repetition is i) not allowed ii) allowed.

Sol :-

There are 5 odd Places (1,3,5,7,9)

(i)

0 Can't be placed in the unit place.

Unit Place Can be filled in 5 ways

1000th place Can be filled in 8 ways

100th place Can be filled in 8 ways

10th place Can be filled in 7 ways

$\therefore$ Total no of ways = $5 \times 8 \times 8 \times 7$

= 2240 ways

(ii) If repetition is allowed

unit Placed Can be filled in 5 ways

1000th place Can be filled in 9 ways

100th place Can be filled in 10 ways

10th place Can be filled in 10 ways.

$\therefore$ Total no of ways = $5 \times 9 \times 10 \times 10$

= 4500 ways.

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H.W

i) How many even no of 4 digits can be formed out of the 1, 2, ... 9 if the repetition of the digits
i) not allowed ii) allowed.

1000^{th} 100^{th} 10^{th} unit
___ ___ ___ ___

(i)

even No (2, 4, 6, 8)

Unit Place = 4 ways
10^{th} = 8 ways
100^{th} = 7 ways
1000^{th} = 6 ways

Total = $4 \times 8 \times 7 \times 6$
= 1344 ways

(ii) Repetition allowed

Unit Place = 4 ways
10^{th} = 8 ways
100^{th} = 9 ways
1000^{th} = 9 ways

= $4 \times 9 \times 9 \times 9$

= 2916 ways

H.W

ii) How many even no of 4 digits can be formed out of 0, 1, ... 9 if the repetition of the digits

(i) not allowed (ii) allowed

(i) even no (2, 4, 6, 8)

0 can't be placed in unit place

unit place = ~~9~~ 4 ways.

1000^{th} = 8 ways

100^{th} = 8 ways

10^{th} = 7 ways

Total = $4 \times 8 \times 8 \times 7$

= 1792 ways

(ii)

unit = 4 ways

1000^{th} = 9 ways

100^{th} = 10 ways

10^{th} = 10 ways

= $4 \times 9 \times 10 \times 10$ ways

= 3600 ways.

Find out the no of arrangement of 5 boys & 5 girls in a row. So that no two boys & no two girls sit together.

Sol:-

If the arrangement start with boys

BG BG BG BG BG

5 boys can be arranged in odd places = $5P_5$ ways

5 girls can be arranged in even places = $5P_5$ ways

$\therefore$ Total = $5P_5 \times 5P_5$

$= \overset{14400}{28800}$ ways.

If the arrangement start with girls

GB GB GB GB GB

5 girls can be arranged in odd places = $5P_5$ ways

5 boys can be arranged in even place = $5P_5$ ways

$\therefore$ Total = $5P_5 \times 5P_5$

$= \overset{14400}{\cancel{28800}}$ ~~ways~~

$\therefore$ The total no of way = (14400 + 14400) = 28800

A family of 4 brother & 3 sis are to be arranged for a photograph in one row. In how many ways can they seated if all the sister sit together.

Sol :-

3 sis together = 1 unit
4 bro together = 4 unit

Totally = 5 unit

∴ The 5 unit Can be arrange in 5 ways
The 3 Sis Can be arrange in 3 ways

∴ The total no of arrangements = $5! \times 3!$
= 720 ways.

There are 6 books on eco, 3 on maths and 2 on accounts. In how many ways can they be arranged on a Shelf if the books of the Same Subject are always together.

Sol :-

6 eco book Consider as 1 unit.
3 Maths book Consider as 1 unit
2 account book Consider as 1 unit.

∴ 3 units Can be arranged in 3! ways

6 eco Can be arranged in 6 ways
3 maths book Can be arranged in 3 ways
2 account book Can be arranged in 2 ways

∴ Total no of ways = $3! \times 6! \times 3! \times 2!$
= 51840 ways

In how many ways can the letter of the word MOBILE be arranged so that the consonents always occupy the odd places.

Sol:-

There are 6 letters in the word MOBILE

3 are Vowels & 3 are Consonents

3 Vowels can be arranged in 3 odd places $= 3P_3$

3 Consonents can be arranged in 3 even places $= 3P_3$

$\therefore$ Total no of ways $= 3P_3 \times 3P_3$

$= 36$ ways

Combination:

A combination of n object taken ($r \le n$) at a time is any selection of r of there objects when order does not count.

$$nC_r = \frac{n!}{r!(n-r)!}$$

Find

$$3C_2 = \frac{3!}{2!(3-2)!}$$

$$= \frac{3!}{2!1!}$$

1.

$$= \frac{3\times\cancel{2!}}{\cancel{2!}}$$

$$=3$$

Properties:

1. $nC_r = nC_{n-r}$

L.H.S $\quad nC_r = \frac{n!}{r!(n-r)!}$

R.H.S $\quad nC_{n-r} = \frac{n!}{(n-r)!(\cancel{n}-\cancel{n}+r)!}$

$$= \frac{n!}{(n-r)!\,r!}$$

$\therefore$ L.H.S = R.H.S

2. $\frac{nC_r}{nC_{r-1}} = \frac{n-r+1}{r}$

$$nC_r = \frac{n!}{r!(n-r)!}$$

$$nC_{r-1} = \frac{n!}{(r-1)!(n-r+1)!}$$

$$\frac{nC_r}{nC_{r-1}} = \frac{\frac{\cancel{n!}}{r!(n-r)!}}{\frac{\cancel{n!}}{(r-1)!(n-r+1)!}}$$

$$= \frac{(r-1)!(n-r+1)!}{r!(n-r)!}$$

$$= \frac{\cancel{(r-1)}!\,(n-r+1)\,\cancel{(n-r)}!}{r\,\cancel{(r-1)}!\,\cancel{(n-r)}!}$$

$$= \frac{n-r+1}{r}$$

---

3. P.T $nC_r + nC_{r-1} = (n+1)C_r$

$$nC_r = \frac{n!}{r!\,(n-r)!}$$

$$nC_{r-1} = \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$nC_r + nC_{r-1} = \frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$= \frac{n!\,(r-1)!\,(n-r+1)! + n!\,r!\,(n-r)!}{r!\,(n-r)!\,(r-1)!\,(n-r+1)!}$$

$$= \frac{n!\,(r-1)!\,(n-r+1)\,(n-r)! + n! + r!\,(n-r)!}{r!\,(n-r)!\,(r-1)!\,(n-r+1)!}$$

$$= \frac{n!\,\cancel{(n-r)}!\,[(r-1)!\,(n-r+1) + r!]}{r!\,\cancel{(n-r)}!\,(r-1)!\,(n-r+1)!}$$

$$= \frac{n!\,[(r-1)!\,(n-r+1) + r\,(r-1)!]}{r!\,(r-1)!\,(n-r+1)!}$$

$$= \frac{n!\,\cancel{(r-1)}!\,[(n-\cancel{r}+1)+\cancel{r}]}{r!\,\cancel{(r-1)}!\,(n-r+1)!}$$

$$= \frac{n!\,(n+1)}{r!\,((n+1)-r)}$$

$$= \frac{(n+1)!}{r!\,((n+1)-r)!}$$?

$$= (n+1)\,C_r$$

If $16C_r = 16C_{r+2}$ find rC_3

$16C_r = 16C_{r+2}$

$r = r+2$ (or) $\quad {}^nC_r = \frac{n!}{r!\,(n-r)!} \quad {}^nC_r = \frac{n!}{r!\,(n-r)!}$

not Possible $\quad \frac{16!}{r!\,(16-r)!}$

$r = n - r \Rightarrow$

$= 16 - (r+2)$

$= 14 - r$

$2r = 14$

$r = 14/2$

$\boxed{r = 7}$

$$rC_3 = 7C_3 = \frac{7\times 6\times !}{3!\,4!}$$

$$= \frac{7\times 6\times 5\times \cancel{4!}}{3!\,\cancel{4!}}$$

$$= \frac{7\times 6\times 5}{3\times 2\times 1}$$

$= 35$

If $4 \cdot {}^nC_2 = {}^{(n+2)}C_3$ find n.

$4 \cdot {}^nC_2 = {}^{(n+2)}C_3$

$$4 \cdot \frac{n!}{2!\,(n-2)!} = \frac{(n+2)!}{3!\,(n+2-3)!}$$

$$\cancel{4}^{2} \cdot \frac{n(n-1)\cancel{(n-2)!}}{\cancel{2}!\,\cancel{(n-2)!}} = \frac{(n+2)(n+1)\,n\,\cancel{(n-1)!}}{3!\,\cancel{(n-1)!}}$$

$$2\cancel{n}(n-1) = \frac{(n+2)(n+1)\cancel{n}}{6}$$

$12(n-1) = (n+2)(n+1)$

$12n - 12 = n^2 + n + 2n + 2$

$12n - 12 = n^2 + 3n + 2$

$n^2 + 3n + 2 - 12n + 12 = 0$

$n^2 - 9n + 14 = 0$

$(n-2)(n-7) = 0$

$n = 2, 7$

14

$-7 \quad -2$

-9

If ${}^nC_{10} = {}^nC_6$ find ${}^nC_{11}$

${}^nC_{10} = {}^nC_6$

$10 = 6$ (Not Possible)

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$$= \frac{16\times15\times14\times13\times12\times\cancel{11!}}{\cancel{11!}\ 5!}$$

$$= \frac{\overset{2}{\cancel{\overset{4}{\cancel{16}}}}\times\overset{3}{\cancel{15}}\times14\times13\times12}{\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times1}$$

$= 2\times14\times13\times12$

$= 4368$

A Company has 7 Chartered accountants, 6 Engineers, 3 Scientists in their managerial Cadre. In how many ways can they form a Committee if the Committee must contain 2 members from different discipline.

Sol :-

There are 7 CA
6 Engineers
3 Sciendist

We have to Select 2 from each group 2 CA Can be Selected from 7 CA

$= {}^7C_2$ ways

2 Engineers Can be Selected from 6 engineer's =

${}^6C_2$ ways

2 Scientist Can be Selected from 3 Sciendist.

${}^3C_2$ ways

$\therefore$ Total no of ways of Selection $= {}^7C_2 \times {}^6C_2 \times {}^3C_2$

$= 21\times15\times3$

$= 945$ ways

H.W $nC_2 = nC_3$ find n

$2 = n - 3$ $\boxed{r = n - r}$

$n = 2+3$

$\boxed{n = 5}$

$2 = 3$ (Not Possible)

$28C_{2r} : 24C_{2r-1} = 225 : 11$ find the value of r.

Sol :-

$$\frac{28C_{2r}}{24C_{2r-1}} = \frac{225}{11}$$

$$\frac{\frac{28}{2r}\;\frac{27}{2r-1}\;\cdots\;{}^{24}C_{2r-1}}{24C_{2r-1}} = \frac{225}{11}$$

$$2r(2r-1)(2r-2)(2r-3) = \frac{28.27.26.25}{225}$$

There are 6 vacancies in an office. If 8 men & 5 women offer themselves. In how many ways can the posts be filled. If the condition are such that the vacancies should go half to men & half to women.

Sol:-

3 men can be selected from 8 men $= 8C_3$

3 women can be selected from 8 women $= 5C_3$

$\therefore$ Total no of ways $= 8C_3 \times 5C_3$

$= 560$ ways.

Permutation: (Sum)

1. How many ways can letter of word NAGERKOIL.

$9!$

2. How of many of them begin with NA.

$7!$

3. In how many of them 4 vowels come together.

$6! \times 4!$

4. How many of them begin with 4 vowels

$4! \times 5!$

In how many ways can the letter STRANGE be arranged so that the vowels may appear the odd place? $4P_2 \times 5P_5$