MARUDHAR KESARI JAIN COLLEGE FOR WOMEN, VANIYAMBADI

PG & RESEARCH DEPARTMENT OF MATHEMATICS

CLASS : I M. C. A

SUBJECT CODE : 23PCA11

SUBJECT NAME : DISCRETE MATHEMATICS

SYLLABUS

UNIT 5

GRAPHS

Connected Graphs -Euler Graphs- Euler line-Hamiltonian circuits and paths planar graphs Complete graph-Bipartite graph-Hyper cube graph-Matrix representation of graphs

UNIT-5

GRAPH

2m

Graph Definition :

A graph G is an ordered triple $(V(G), E(G), \psi)$ consisting of (i) a non empty finite set. $V(G)$

(ii) a finite set $E(G)$ which is disjoint from $V(G)$.

(iii) an incidence function ψ that associates with each element of $E(G)$ and unordered pair of elements of $V(G)$.

The element of $V(G)$ are called the vertices (points) of G, and the elements of $E(G)$ are called the edges (lines) (~~nodes~~) of G.

If e is an edge (line) and $\psi(e) = (u, v)$ then we say that e is an edge joining u and v and the vertices u and v are called the ends of e.

Eg:-

Draw a graph $G = (V(G), E(G), \psi)$

where $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

$E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}\}$

and $\psi(G)$ (or) ψ is defined by

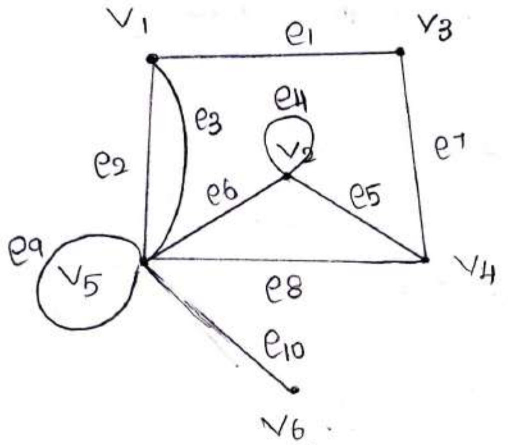
$\psi(e_1) = (v_1, v_3)$ $\psi(e_2) = (v_1, v_5)$

$\psi(e_3) = (v_1, v_5)$ $\psi(e_4) = (v_2, v_2)$ $\psi(e_5) = (v_2, v_4)$

$\psi(e_6) = (v_2, v_5)$ $\psi(e_7) = (v_3, v_4)$

$\psi(e_8) = (v_4, v_5)$ $\psi(e_9) = (v_5, v_5)$. $\psi(e_{10}) = (v_5, v_6)$

Solution :



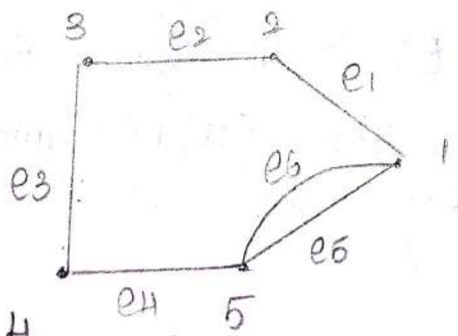
ii) $G = (V(G), E(G), \psi)$

$V(G) = \{1, 2, 3, 4, 5\}$ $E(G) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

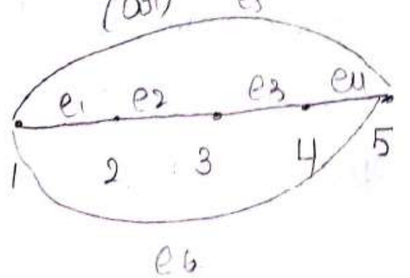
and ψ is defined by $\psi(e_1) = (1, 2)$

$\psi(e_2) = (2,3)$, $\psi(e_3) = (3,4)$, $\psi(e_4) = (4,5)$,

$\psi(e_5) = (5,1)$, $\psi(e_6) = (5,1)$.



(or)

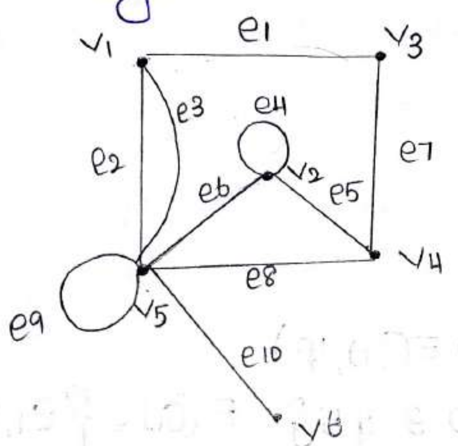


2m

Degree of a graph:

Let v be a Vertex in a graph G then a degree $d_G(v)$ of the Vertex v in G is the number of edges (lines) of G that are incident with v (each loop is Counted twice) then degree of v Can also be denoted by $deg_G(v)$.

Eg:-



$d_G(v_1) = 3$, $d_G(v_2) = 4$, $d_G(v_3) = 2$

$d_G(v_4) = 3$, $d_G(v_5) = 7$, $d_G(v_6) = 1$

~~dG~~

Loop:

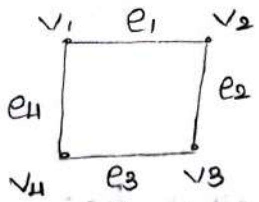
2m

If e is an edge in a graph G Such that $\psi(e) = (u,u)$ for Some Vertex $u \in V$ then e is Said to be a loop in G.

2m

Simple graph: A graph which has no loop and no Parallel edges is Said to be a Simple graph.

Eg:-



$\psi(e_1) = (v_1, v_2)$ $\psi(e_2) = v_2, v_3$ $\psi(e_3) = v_3, v_4$

$\psi(e_4) = v_1, v_4$

2m

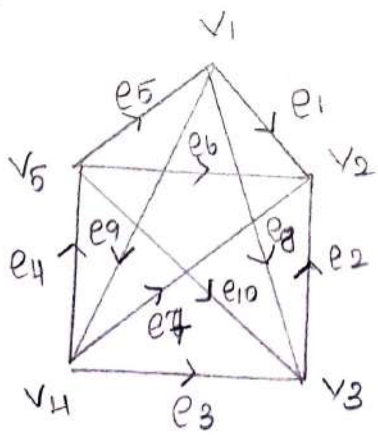
Digraph:

An ordered triple $(V(G), E(G), \psi)$ consisting of a non empty finite set $V(G)$, a finite set $E(G)$ disjoint of $V(G)$ and an incidence function ψ is said to be a directed graph (Simply digraph).

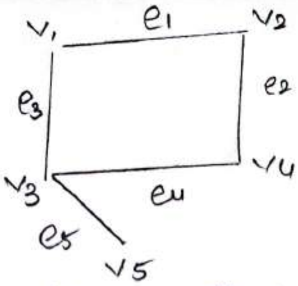
The elements of $V E(G)$ are called vertices (Points or nodes) and $E(G)$ are called edges (lines, arcs) respectively.

If e is an edge and $\psi(e) = (u, v)$ then we say that e is an edge joining u and v then and vertices u and v are called the ends of e.

The vertex u is said to be the tail (initial vertex) of e and v is said to be head (terminal vertex).



→ Directed graph



→ undirected graph (No arrows)

↳ 2m A digraph that has no Parallel edges is called a Simple digraph.

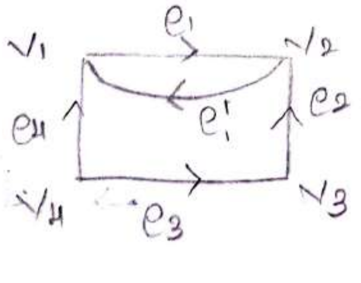
↳ 2m A digraph that has atmost one directed edge between a Pair of vertices (loops are allowed) is called asymmentric or antisymmetric

↳ 2m If G is a digraph such that whenever e is an edge with $\psi(e) = (u, v)$ then there is an edge e' with $\psi(e') = (v, u)$. then G is said to be a symmetric digraph.

↳ A digraph which is both 2m Simple and Symmetric is called a Simple Symmetric digraph.

↳ A digraph which is both 2m Simple and asymmetric is called a Simple asymmetric diagraph.

Eg:- Symmetric digraph



2m

Incident of a graph:

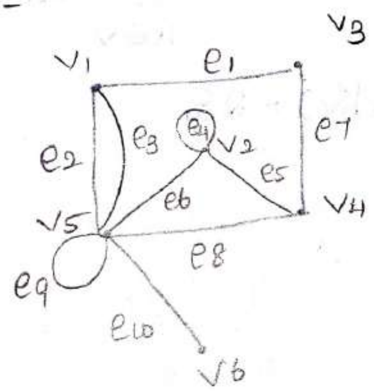
Let $G = (V(G), E(G), \psi)$ be a graph and $\psi(e) = (u, v)$ then e is said to be incident with the vertices u and v and u and v are said to be incident with e.

2m

Adjacent:

Two vertices u and v in $V(G)$ are said to be adjacent if there is an edge $e \in E(G)$ such that $\psi(e) = (u, v)$.

Eg:-



v_1 & v_3 are adjacent

v_1 & v_4 are not adjacent

v_1 & v_3 are incident with e_1

v_4 is incident with e_5, e_7, e_8 and adjacent to 3 vertices.

Theorem : 5m

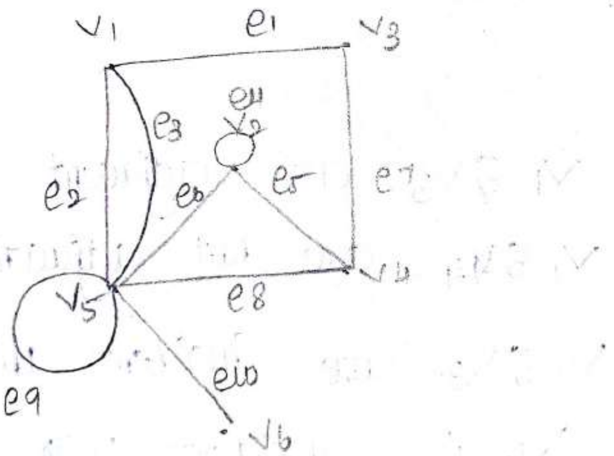
Let G be a graph then $\sum_{u \in V} d(v) = 2\epsilon$

Where $\epsilon = E(G)$ are the Sum of Points of a graph G degree of the number of lines that it twice the is $\sum_i \deg v_i = 2q$

Let G be a graph with ϵ edges and n Vertices $(v_1, v_2 \dots v_n)$ Since each edges Contributes two degrees to the Sum of degree of all vertices in G we have $\sum_{u \in V} d(v) = 2\epsilon$.

Proof :- $\sum d(v) = 2\epsilon$

Eg :-



ϵ (or) $E(G) = 10$

$\sum d(v) = 2(10)$

$\sum d(v) = 20$

5) Theorem:

In any graph the number of vertices of odd degree is even (or) in any graph G the number of points of odd degree is even.

Proof:-

V_1, V_3, V_5 (V_2, V_4, V_6)

Let V_1 and V_2 be the set of vertices of odd and even degree. then

$V = V_1 \cup V_2$

empty $\phi = V_1 \cap V_2$

$$\sum_{v \in V} d(v) = \sum_{v \in V_1} \overset{\to odd}{d(v)} + \sum_{v \in V_2} \overset{\to even}{d(v)} \to ①$$

" Let G be a graph then $\sum_{u \in V} d(v) = 2\xi$ "

$\therefore \sum d(v) = 2\xi$

$\therefore$ LHS of ① is even

Since $d(v)$ is even for $v \in V_2$ the sum $\sum_{v \in V_2} d(v)$ is even

$$① \Rightarrow \sum_{v \in V_1} d(v) = \text{even} \to ②$$

In LHS of eqn ②

each term $d(v)$ (as $v \in V_1$) is odd and so the total number of terms in the sum must be even to make the sum an even number.

Thus $|V_1|$ is even.

Hence the Proof.

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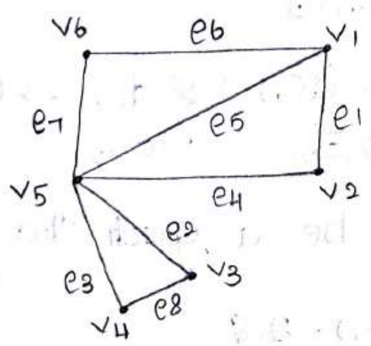
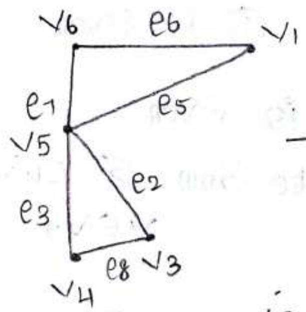
2m Subgraph :

A graph $H = (V(H), E(H), \psi)$ is Said to be a Sub graph of a graph $G = (V(G), E(G), \psi)$. If $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and the map ψ_H is restriction of ψ_G to $E(G)$.

If H is a Subgraph of G. We write $H \subseteq G$.



→ graph (G)



→ Missing edge & vertices

→ Subgraph (H)

$\therefore H \subseteq G$

$H(V) \subseteq G(V)$

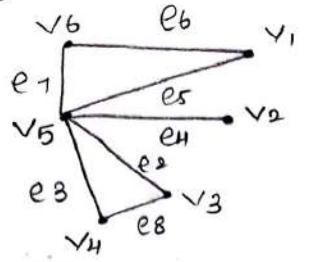
$H(E) \subseteq G(E)$

If H Containing G then G is Said to be Super graph of H.

2m Spanning Subgraph : (Not missing of vertices but missing edges)

A Subgraph H of a graph G is Said to be a Spanning Subgraph of G if $V(H) = V(G)$.

Eg:-

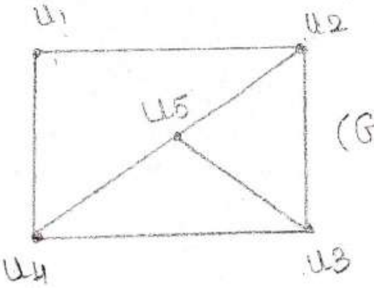


2m

Isomorphism:

Two graphs $G = (V(G), E(G), \psi_G)$ and $H = (V(H), E(H), \psi_H)$ are said to be isomorphic if there are bijection (one-one onto) $\phi : V(G) \to V(H)$ and $\theta : E(G) \to E(H)$ such that $\psi_G(e) = (u,v)$ if and only if $\psi_H(\theta(e)) = (\phi(u), \phi(v))$.
If the graph G and H are isomorphic then we write $G \cong H$.

Eg:-

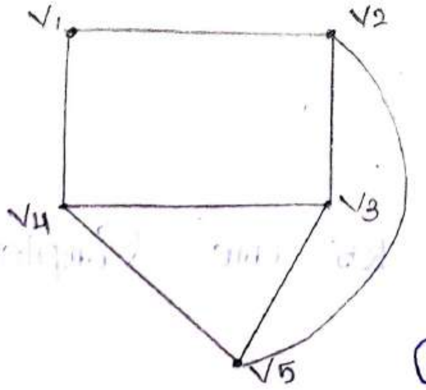


(G)

Vertices = 5

Edges = 7

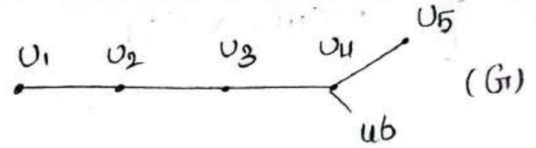
Degree (G)	Deg (H)
$u_1 = 2$	$v_1 = 2$
$u_2 = 3$	$v_2 = 3$
$u_3 = 3$	$v_3 = 3$
$u_4 = 3$	$v_4 = 3$
$u_5 = 3$	$v_5 = 3$

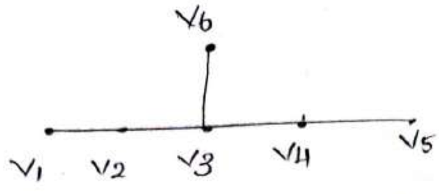


(H)

$G \cong H$

G & H are isomorphic.





G & H are not isomorphic.

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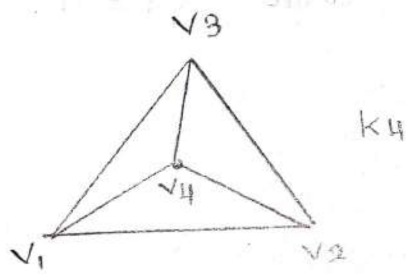
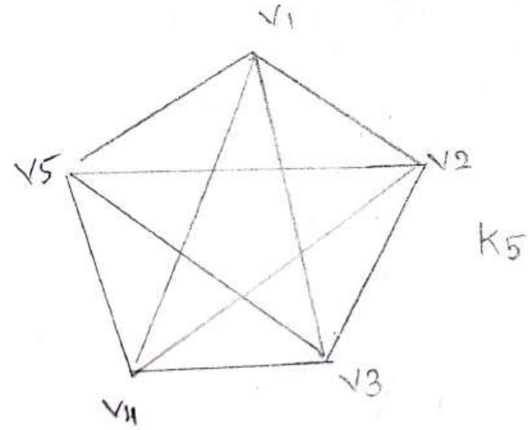
Complete graph:

A Simple graph in which every Pair of distinct vertices are adjacent is called a Complete graph. It is denoted by K_n.

Eg:-





K_4 & K_5 are Completed graph.

2m

Bipartite graph:

A graph G is Said to be bipartiate if its Vertex $V(G)$ Can be Partitioned into two SubSets V_1 and V_2 Such that every edge of e of G has one end in V_1 and other end in V_2. Such that a Partition (V_1, V_2) is called a bipartition of G.

Eg:-

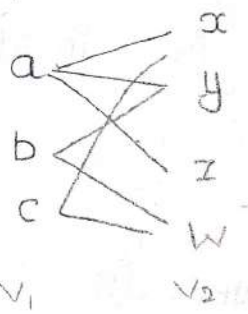
$V(G) = \{a, b, c, x, y, z, w\}$

$E(G) = \{ax, ay, az, by, bw, cx, cw\}$

Solution :-

$V_1 = \{a, b, c\}$

$V_2 = \{x, y, z, w\}$



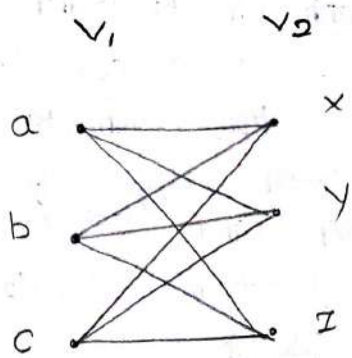
2m

Complete biparti graph:

A Simple graph G with bipartition (V_1, V_2) is Said to be Comple bipartition graph if every Vertex of V_1 is adajacent to every Vertex of V_2.

It is denoted by $K_{m,n}$

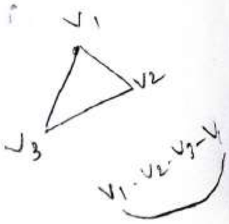
Eg:-



Cycle:

A $v_0 - v_n$ walk is closed if $v_0 = v_n$.

A closed walk $v_0, v_1, v_2 \ldots v_n = v_0$ in which $n \geq 3$ and $v_0, v_1 \ldots v_{n-1}$ are distinct is called a cycle of length n.



Connected graph:

2m

Walk:

A walk of a graph G is an alternating sequence of points (vertices) and lines (edges)

$$v_0, x_1, v_1, x_2, v_2, x_3 \ldots v_{n-1}, x_n, v_n$$

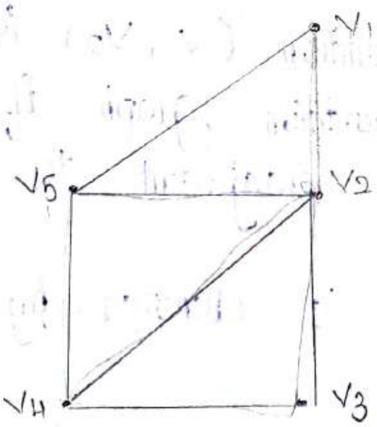
Beginning and ending with points such that x_i is incident with v_{i-1} & v_i

2m

Trails & Paths:-

A walk is called a trail if all its lines are distinct.

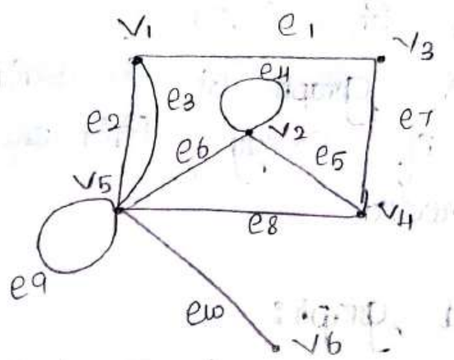
A walk is called a path if all its points are distinct.



i) $v_1, v_2, v_3, v_4, v_2, v_5$ is a walk.

ii) $v_1, v_2, v_4, v_3, v_2, \overset{e_6}{,} v_5$ is a trail but not Path.

iii) v_1, v_2, v_4, v_5 is a Path.



$v_1 e_1 v_3 e_7 v_4 e_5 v_2 e_4 v_2 e_6 v_5$ is a trail but not a Path.

$v_1 e_1 v_3 e_7 v_4 e_5 v_2 e_6 v_5$ is a Path

$v_1 - v_5$ Path

$v_1 e_2 v_5$; $v_1 e_1 v_3 e v_4 e_8 v_5$

are $v_1 - v_5$ Path

Let u and v be distinct vertices in a graph G. If there is a u,v Path in G then a u,v Path of ~~an~~ least length is called a $\boxed{\text{geodesic}}$

The length of a geodesic between u and v is called the distance between u and v & it is denoted by $d(u,v)$.

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Connected graph:

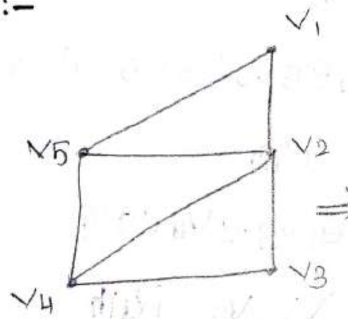
Let G be a graph two vertices u and v of G are said to be connected if $u = v$ or there is a u, v path in G. (OR)

A graph G is said to be connected if every pair of its points are connected.

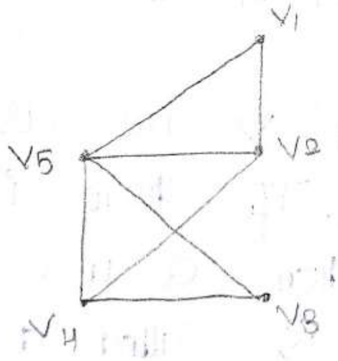
Disconnected graph:

A graph it is not connected is said to be disconnected.

Eg:-



$\Rightarrow$ Connected



$\Rightarrow$ Disconnected

Components:

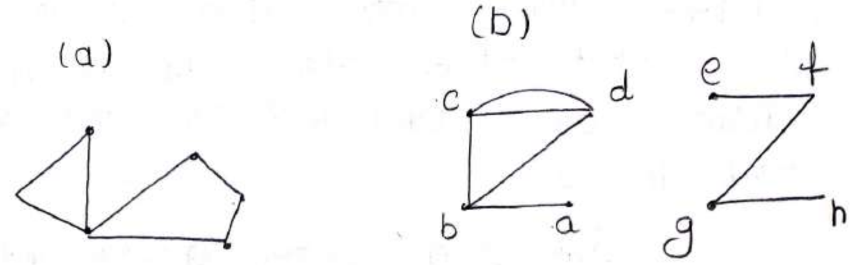
Let G_i denote the induced subgraph of G with vertex set V_i clearly the subgraph $G_1, G_2 \ldots G_i$

are Connected and are Called the Components of G.

10m

Theorem:-

A graph G is disconnected if and only a is vertex I Set BV Can be Partition into two non empty Subset V_1 and V_2. Such that there exist no edge in G. whose one End vertex in V_1 and another in V_2.



Assume that G is disconnected. Consider a vertex $u \in V$. Let $V_1 \{v \in V$ / there is u - v path in G$\}$. ~~ha~~ As G is disconnected $V_1 \neq V$.

Let $V_2 = V - V_1$ Then $V_2 \neq \phi$ and $V_1 \cap V_2 = \phi$.

We Claim that there is no edge in one end in V_1 and other end in V_2. If it is not So let $e = ab$ be an edge in G Such that $a \in V_1$ & $b \in V_2$.

Let $u x_1, x_2 \dots x_m$ be a u-a Path in G. Clearly $x_1, x_2, \dots x_n \in V_1$, $b \neq x_i$ for any i.

Thus there is no edge in G one end of e in V_1 and another end in V_2.

⇒ Conversely, assume that the vertex V can be Partitioned into two disjoint non empty Subset (V_1 and V_2) Such that there Edges no edge in G. whose one end is in V_1 and another end in V_2.

Consider a vertex $a \in V_1$ and a vertex $b \in V_2$. We Claim that there is no a-b path in G.

Suppose there is a a-b Path $x_0, x_1 \dots x_m x_{m+1}$ (where $x_0 = a$, $x_{m+1} = v$)

Let i be the least Positive integer Such that $x_i \in V_2$ (Such as i exists Since $x_{m+1} \in V_2$) $i \geq 1$.

Since $x_0 = u \in V_2$.

Also

$x_{i-1} \in V_1$

Thus there is edge in x_{i-1} and x_i (e) as end vertices and x_{i-1} belongs to V_1 and $x_i \in V_2$.

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This is a Contradiction there is no $a-b$ path in G.

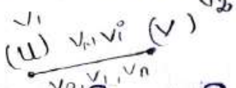
$\therefore$ G is disconnected

Theorem:

A graph G is Connected iff and only if for any Partition V into Subsets V_1 & V_2 their is a line a G joining a Point of V_1 and Point of V_2.

Proof:-

Let $V = V_1 \cup V_2$ be a Partition of V into two Subsets.

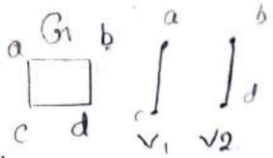


Let $u \in V_1$ & $v \in V_2$. Since G is Connected there exist $u-v$ path in G Say $u = v_0, v_1, \ldots v_n$

Let i be the least Positive integer Such that $v_i \in V_2$ (Since an i exist Since $v_n = v \in V_2$). Then $v_{i-1} \in V_1$ and v_{i-1}, v_i are adjacent.

This there is a line joining $v_{i-1} \in V_1$ & $v_i \in V_2$.

Conversly,



Suppose G is not Connected. Then G Contains at least 2 Components.

Let V_1 denote the Set of all Vertices of one Component and V_2 the remaining vertices of G.

Clearly $V = V_1 \cup V_2$ is a Partition of V and there is no line joining any Point of V_1 to any Point of V_2.

Hence the theorem.

Theorem:

Let G be an undirected graph (let G be a graph) then G is bipartite if and only if it Contains no odd Cycle.

Proof:- Necessary Part

Let G be bipartite with bipartition (x, y). Let $C = v_0 v_1 \ldots v_{k-1} v_k$, where $v_k = v_0$ be a Cycle in G.

We may assume that $v_0 \in x$. Then as $v_0 v_1$ is an edge and G is bipartite $v_1 \in y$.

As $v_1 \in y$ and $v_1 v_2$ is an edge, it follows that $v_2 \in x$.

Proceeding like this, we have $v_{2i} \in x$ and $v_{2i+1} \in y$.

As $v_k \in x$, k is even So c is an even Cycle.

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Thus G Contains no odd Cycle.

Sufficiency:

It Suffices to Prove the Converse for Connected graphs.

Let G be a Connected graph that Contains no ~~odd~~ odd Cycle. Choose an arbitary Vertex v and define a Partition (x,y) of V by defining.

$x = \{ v \in V \mid d(u,v) \text{ is even} \}$

$Y = \{ v \in V \mid d(u,v) \text{ is odd} \}$

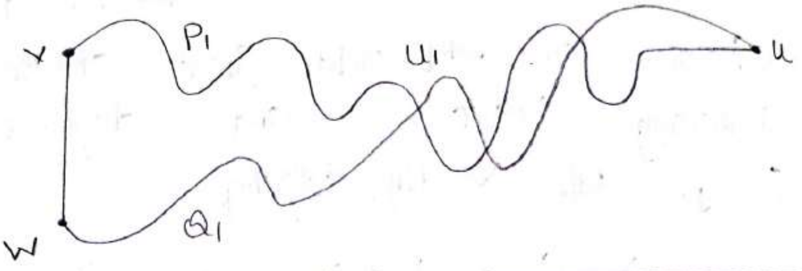
We Claim that G is bipartite graph with bipartition (x,y). Let v and $w \in x$. Let P be a Shortest u-v path and Q be a Shortest u, w path in G.

Let u_1 be the last vertex Common to P and Q. Since P & Q are Shortest Paths, (u,u_1) Sections of P and Q are Shortest $u-u_1$ paths and therefore have the Same length Says k As the lengths of the Paths P & Q are even the lengths of (u_1,v) Section P_1 of P and (v, w) Section Q_1 of Q are both either even or odd.

If vw were edge in G, then the Cycle $P_1^{-1}Q_1v$ is Cycle of odd length, Contrary to the hypothesis.

Hence no two vertices in X are adjacent, Similarly no two vertices in Y are adjacent.

Thus (X, Y) is a bipartition of the vertex set and G is bipartite.



Theorem:

A simple graph with n vertices and k components can have at most $(n-k)(n-k+1)/2$ edges.

Proof:-

We prove the result by induction on the number of components of G. Let $P(k)$: If G is a simple graph with k components, then it can have atmost $(n-k)(n-k+1)/2$ edges, where $n = |V(G)|$

vertices Com ver Com

If $k=1$, then G is a simple connected graph and hence the number of edges in $G \leq$ number of edges of K_n

$(n-1)\cdot(n/(1+1))/2$

$$\leq (n-1)\, n/2,$$

Where $n = |V(G)|$ and K_n is the complete graph of n vertices.

Thus P(1) is true $\rightarrow$ ①

Assume that P(m) is true, for Some m $\rightarrow$ ②

Let G be a Simple graph with n vertices and (m+1) Components. Let H_1 be a Component of G. Let $|V(H_1)| = n_1$. As G has m remaining Components and each Component has at least one Vertex, we have $n_1 \le n-m$. Let H_2 be the Subgraph of G induced by $V(G) - V(H_1)$. Then H_2 is a Simple graph with $n - n_1$ vertices and m Components, and by (2). H_2 Can have atmost $(n-n_1-m)\ (n-n_1-m+1)/2$ edges. As H_1 is a Connected Simple graph with n vertices, by (1), it Can have atmost $(n_1-1)\,n_1/2$ edges

Thus the number of edges in $G \le \frac{n_1(n_1-1)}{2} + (n-m-n_1)(n-m+1-n_1)/2$

$= \frac{1}{2}\left[n_1^2 - n_1 + (n-m)(n-m+1) - n_1(n-m+1+n-m) + n_1^2\right]$

$= \frac{1}{2}\left[(n-m)(n-m-1+2) - 2n_1(n-m) - 2n_1 + 2n_1^2\right]$

$= \frac{1}{2}\left[(n-m)(n-m-1) - 2(n-m)(n_1-1) + 2n_1(n_1-1)\right]$

$\le \frac{1}{2}(n-m)(n-m-1)$ as $n_1 \le n-m$.

Thus from (1) and (2), P(m+1) is also true.
By induction Principal, P(k) is true for all Positive integers k.

Let G be a graph and u and v be two distinct vertices of G. Show that if there is a $u-v$ walk in G, then there is also a $u-v$ path in G.

Solution:

It is given that there is a $u-v$ walk in G. Among all $u-v$ walks in G, find one walk with least length.

Let $u x_1 x_2 \dots x_m v$ be a $u-v$ walk with least length.

We claim that the vertices $u, x_1, x_2, \dots x_m$ and v are all distinct.

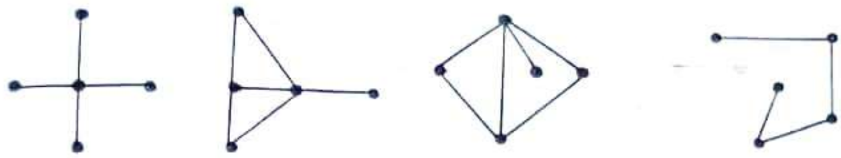
It is given that $u \neq v$. If $u = x_i$ for some i, $1 \leq i \leq m$, then $u = x_i x_{i+1} \cdots x_m v$ is a $u-v$ walk of length $\leq m$ which is a contradiction.

If $x_i = v$ for some i, then $u x_1 \dots x_i$ is a $u-v$ walk, leading to a contradiction.

Thus the vertices $u, x_1, \dots x_m, v$ are all distinct and hence $u x_1 x_2 \dots x_m v$ is a $u-v$ path.

If u and v are two distinct vertices of a digraph G, and if there is a $u-v$ directed walk in G, then there is a u-v directed path in G.

Which of the following graphs are isomorphic?



(a) (b) (c) (d)

Solution:

The graph (b) and (c) are isomorphic. Map the unique vertices of degree one, three and four of the graph (b) into the respective unique vertices of (c).

The remaining two vertices of (b) can be mapped onto the remaining two vertices of (c). The map is an isomorphism.

The graph (a) has four vertices of degree one, while other graphs have atmost two vertices of degree one.

So the graph (a) is not isomorphic to any one of the others. Of these four graphs, graph (d) is the only graph with exactly two vertices of degree one.

So the graph (d) is not isomorphic to any one of the remaining graphs.

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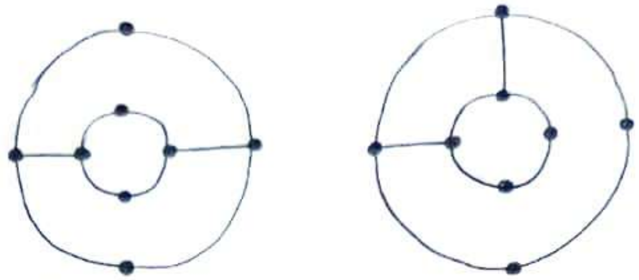
Are the two graphs given in the following figure isomorphic? why?



Solution:

They are not isomorphic to each other. In the first graph each vertex of degree three is adjacent to exactly one vertex of degree three.

But in the second graph, there are vertices of degree three which are adjacent to more than one vertex of degree three.

Prove that if a graph has exactly two vertices of odd degree, there must be a path joining these two vertices.

Solution:

Let G be a graph which has exactly two vertices of odd degree. Let u and v be the vertices of odd degree in G. If G is connected then there is a u-v path in G.

If G is not connected, let H be the component of G containing the vertex u.

We note that

(i) H is a connected graph

(ii) $\deg_H(a) = \deg_G(a)$, for every vertex a in H.

(iii) as $u \in H$ and u is an odd degree vertex in G, u is also an odd degree vertex in H

(iv) $v \in H$, for if $v \notin H$, then u is the only vertex of odd degree in H, which is a contradiction by Theorem 2.

(v) As $u, v \in H$ and H is connected, there is a $u-v$ path in H. As H is a subgraph of G, this $u-v$ path in H is also a $u-v$ path in G. Thus there is path in G joining the vertices u and v.

If a graph has n vertices and a vertex v is connected to a vertex w, then there exists a path from v to w of length no more than $(n-1)$.

Solution:

Let $v u_1 u_2 \cdots u_{m-1} w$ be a path in G from v to w. By the definition of the path, the vertices $v, u_1, u_2 \ldots u_{m-1}$ and w are all distinct.

As G contains only n vertices, it follows that $m+1 \leq n$. i.e., $m \leq n-1$.

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Prove that a Simple graph with n vertices must be Connected if it has more than $(n-1)(n-2)/2$ edges.

Solution:-

Let G be a Simple graph with more than $(n-1)(n-2)/2$ edges.

Assume that G is not Connected. Select any one of the Connected Component of G. Let V_1 be the Vertex Set of that Component. Take $V_2 = V(G) - V_1$ and $m = |V_1|$. Then

(i) $1 \leq m \leq n-1$

(ii) There is no edge Joining a Vertex of V_1 and a Vertex of V_2 and

(iii) $|V_2| = n-m \geq 1$.

So $|E(G)| = |E(G[V_1])| + |E(G[V_2])|$

$$\leq \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2}$$

$$= \frac{1}{2}[m(m-1) + (n-m)(n-m-1)]$$

$$= 1/2 [(n-m)(n-m-1) + m^2 - m]$$

$$= 1/2 [n(n-1) - nm - m(n-m-1) + m^2 - m]$$

$$= 1/2 [(n-1)(n-2) + 2(n-1) - 2nm + m^2 + m + m^2 - m]$$

$$= 1/2 [(n-1)(n-2) + 2n - 2 - 2nm + 2m^2]$$

$$= 1/2 [(n-1)(n-2) + 2n(1-m) + 2(m^2-1)]$$

$$= 1/2 [(n-1)(n-2) + 2(1-m)\{n-1-m\}]$$

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$= 1/2 \left[(n-1)(n-2) - 2(m-1)(n-m-1) \right]$

$\leq 1/2 (n-1)(n-2)$ Since $(m-1)(n-m-1) \geq 0$

for $1 \leq m \leq n-1$

Which is a contradiction as G has more than $(n-1)(n-2)/2$ edges. Hence G is connected.

Let G be a simple graph and the minimum degree $\delta(G) \geq 2$. Then G contains a cycle of length $\geq \delta + 1$.

Solution :-

Let G be a simple graph and $\delta(G) \geq 2$

Let $P : u_0 u_1 \dots u_m$ be a longest path in G.

We claim that the length of this path = $m \geq \delta(G)$. Suppose $m < \delta(G)$. As $\deg(u_m) \geq \delta(G)$, there is a vertex $v \in V(G)$ such that $u_m v$ is an edge and $v \neq u_i$, for all $i = 0, 1, 2, \dots m-1$.

Now $u_0 u_1 \dots u_m v$ is a path of length $m+1$, which is a contradiction. Thus $m \geq \delta(G)$.

Now as P is a longest path, u_0 is not adjacent to any vertex in $V(G) - \{u_1, u_2, \dots u_m\}$. (if u_0 is adjacent to a vertex w, where $w \neq u_i$ $i = 1, 2, \dots m$, then we get a new path $w u_0 u_1 \dots u_m$ of length $> m$, which is a contradiction).

As $\deg(u_0) \geq \delta(G) \geq 2$ and as $u_0 v$ is an edge implies $v \in \{u_1, u_2, \ldots u_m\}$, it follows that $u_0 u_k$ is an edge implies $v \in \{u_1, u_2, \ldots u_m\}$, it follows that $u_0 u_k$ is an edge for some $k \geq \delta(G)$. Now $u_0 u_1 \ldots u_k u_0$ is a cycle of length $k+1 \geq \delta(G)+1$.

Let G be a simple graph with n vertices. Show that if $\delta(G) \geq \left[\frac{n}{2}\right]$, then G is connected.

Proof:

Let u and v be two distinct vertices in G, we claim that there is a u-v path in G. If uv is an edge in G, then it is a u-v path.

Assume that uv is not an edge in G. Let A be the set of all vertices which are adjacent to u and B be the set of all vertices which are adjacent to v. Then $u, v \notin A \cup B$, and hence $|A \cup B| \leq n-2$.

As now $|A| = \deg(u) \geq \delta(G) \geq \left[\frac{n}{2}\right]$

Similarly $|B| \geq \left[\frac{n}{2}\right]$

Hence we get $|A| + |B| \geq n-1$. Now from $|A \cup B| + |A \cap B| = |A| + |B|$, it follows that $|A \cap B| \geq 1$.

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So $A \cap B \neq \phi$. Take a vertex $w \in A \cap B$. Then uwv is a $u-v$ path in G. Thus for every pair of distinct vertices u, v, there is a $u-v$ path in G. In otherwords, G is connected.

If u and v are distinct vertices in a directed (or undirected) graph G, then every $u-v$ walk in G contains a $u-v$ path.

[A walk contains a path P if the vertices and edges of P occur as a subsequence of the vertices and edges of W].

Solution :-

We prove the result for directed graphs. (Similar proof holds for undirected graphs).

We use the induction principle to prove that for all positive integers l the statement $P(l)$:

$P(l)$: Every $u-v$ directed walk of length $\leq l$ contains a $u-v$ path.

Let $l = 1$. If a $u-v$ directed walk W is of length ≤ 1, then its length $= 1$, and it contains only one edge. The unique edge in W is a directed edge e from u to v and hence $W: u\,e\,v$ is a $u-v$ path. So the result is true for $l = 1$. So $P(l)$ is true for $l = 1$. Assume that $P(l)$ is true for some l.

Let W be a u-v directed walk of length $r+1$. If W has no repeated vertex, then W is itself a u-v path.
If W has a repeated vertex, Select one such vertex and delete the edges and vertices between the first and last appearances of that repeated vertex to obtain a shorter u-v directed walk W' contained in W. By the induction hypothesis. W' contains a u-v path and this u-v path is contained in W.

Thus $P(r+1)$ is true.

Show that for $n \geq 1$, there are $2^{n(n-1)/2}$ simple undirected graphs with vertex set $\{v_1, v_2, \dots v_n\}$.

Sol:-

Let K_n be the complete graph with the vertex set $\{v_1, v_2, \dots, v_n\}$. Every simple graph with vertex set $\{v_1, v_2, \dots v_n\}$ can be viewed as a spanning subgraph of K_n.

For every spanning subgraph G of K_n, the edge set $E(G)$ is a subset of $E(K_n)$, and conversely for every subset E_1 of $E(K_n)$, there is a unique spanning subgraph G of K_n with $E(G) = E_1$.

Thus the number of Spanning Subgraphs of k_n

$=$ the number of Subsets of $E(k_n)$..

$= 2^{n(n-1)/2}$ as $E(k_n)$ Contains $n(n-1)/2$ elements.

Hence there are $2^{n(n-1)/2}$ Simple graphs with vertex Set $\{v_1, v_2, \dots v_n\}$.

Show that for $n \geq 1$, there are $2^{(n-1)(n-2)/2}$ Simple undirected graphs with vertex Set $\{v_1, v_2, \dots v_n\}$ Such that degree of every vertex is even.

Solution :-

Let G_{n-1} be the Set of all Simple graphs with Vertex Set $\{v_1, v_2, \dots v_{n-1}\}$. Let G_e be the Set of all Simple graphs with Vertex Set $\{v_1, v_2, \dots v_{n-1}, v_n\}$ Such that all Vertices are of even degree.

If $G \in G_{n-1}$, then the number of odd degree Vertices of G is even (by Theorem 2). Introduce an edge between every odd degree Vertex of G and v_n.

The resulting Simple graph has the Vertex Set $\{v_1, v_2, \dots v_{n-1}, v_n\}$ and its Vertices are of even degree. Thus to each $G \in G_{n-1}$, We associate a unique $G' \in G_e$.

Conversely, if $G' \in G_e$, just omit the vertex v_n and all the edges of G' which are incident with v_n, to get a simple graph $G \in G_{n-1}$. [If $G' \in G_e$, then $G' - v_n \in G_{n-1}$). Thus to each $G' \in G_e$, we associate a unique graph $G \in G_{n-1}$.

Hence there is a bijection between G_e and G_{n-1}. As G_{n-1} is the class of all simple graphs with vertex set $\{v_1, v_2, \ldots, v_{n-1}\}$, it has $2^{(n-1)(n-2)/2}$ elements.

So G_e also has $2^{(n-1)(n-2)/2}$ elements.

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(7) The adjacency Matrix:

Definition:

Let G be a graph with n vertices and no parallel edges. Let $V = \{v_1, v_2 \ldots v_n\}$ be the vertex set of G. The adjacency matrix $A = [a_{ij}]$ of G is an $n \times n$ symmetric matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between } v_i \,\&\, v_j \\ 0 & \text{if there is no edge between } v_i \text{ and } v_j. \end{cases}$$

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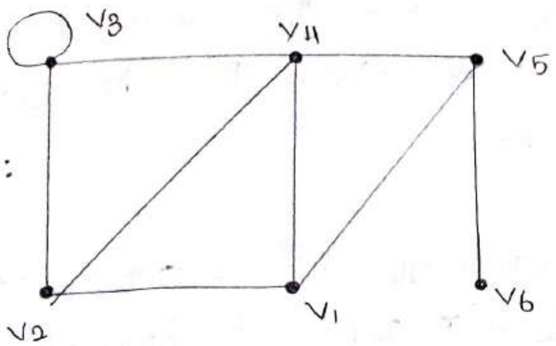
Example:

A graph G and its adjacency matrix $A(G)$.

G :



$$A(G) = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Theorem:

Let r be a positive integer. Let A be the adjacency matrix of a simple graph G. Then the ij^{th} entry in A^r is the number of different walks of length r between the vertices v_i and v_j.

Proof:-

We prove the theorem by induction on r.

Let $r=1$. Then $A^r = A$. The ij^{th} entry of A is 1 iff v_iv_j is an edge in G, otherwise it is 0. There is a $v_i - v_j$ walk of length 1 iff and only if v_iv_j is an edge.

In this case there is only one $v_i - v_j$ walk of length 1. Thus if ij^{th} entry of A = the number of $v_i - v_j$ walk of length 1 and the result is true for $r=1$.

Assume that the result is true for some $r \geq 1$. We show that the result is true for $r+1$.

Now ij^{th} entry of A^{r+1} = dot product of i th row of A^r and j^{th} column of A

$$= \sum_{k=1}^{n} ik^{th} \text{ entry of } A^r . kj^{th} \text{ entry of } A \ldots \rightarrow (1)$$

Note that if $v_k v_j$ is an edge in G, then every $v_i - v_k$ walk $v_i v_1 v_2 \ldots v_{r-1} v_k$ of length r can be extended to a $v_i - v_j$ walk $v_i v_1 v_2 \ldots v_{r-1} v_k v_j$ of length $r+1$.

Conversely if $v_i v_1 v_2 \ldots v_k v_j$ is a $v_i - v_j$ walk of length $r+1$, then $v_k v_j$ is an edge and $v_i v_1 v_2 \ldots v_k$ is a $v_i - v_k$ walk of length r.

So,

The number of $v_i - v_j$ walks of length $r+1$.

$= \sum$ (the number of $v_i - v_k$ walks of length r)

Where the Sum is taken over all k for which $v_k v_j$ is an edge.

$$= \sum_{k=1}^{n} (\text{the number of } v_i - v_k \text{ walks of length } r).(kj^{th} \text{ entry in } A)$$

$$= \sum_{k=1}^{n} (ik^{th} \text{ entry in } A^{r}).(kj^{th} \text{ entry in } A)$$

(By induction hypothesis)

$= ij^{th}$ entry in A^{r+1} by (1)

Thus if the result is true for some $r \geq 1$, then it is true for $r+1$. Hence by the Principle of induction the result is true for all Positive integers r.

Theorem:

If G is a Connected Simple graph, the distance between v_i and v_j (for $i \neq j$) is k if and only if k is the Smallest integer for which ij^{th} entry in A^k is nonzero.

Proof:-

Let v_i and v_j $(i \neq j)$ be two vertices in a Connected Simple graph. Then there is a $v_i - v_j$ walk in G. The length of a Shortest $v_i - v_j$ walk is the distance between v_i and v_j. So distance between v_i and v_j is k

$\Leftrightarrow$ there is a $(v_i - v_j)$ walk of length k and there is no $(v_i - v_j)$ walk of length less than k.

$\Leftrightarrow$ ij^{th} entry in A^r is zero for all $r < k$ and ij^{th} entry in A^k is nonzero.

$\Leftrightarrow$ k is the smallest integer such that ij^{th} entry in A^k is nonzero.

Theorem :

If A is the adjacency matrix of a graph G with n vertices and

$$M = A + A^2 + \dots + A^{n-1},$$

Then G is not connected if and only if there exists atleast one entry in matrix M that is zero.

Proof :-

Note that G has no parallel edges. Also a walk of length n or more can be reduced to a walk of length $n-1$ or less.

G is not connected $\Leftrightarrow$ there exist vertices v_i and v_j $(i \neq j)$ in G such that there is no $v_i - v_j$ walk in G.

$\Leftrightarrow$ for every $r = 1, 2, \dots n-1$, there is no $v_i - v_j$ walk of length r.

$\Leftrightarrow$ ij^{th} entry in A^r is zero for all $r = 1, 2, \dots, n-1$.

$\Longleftrightarrow$ ij^{th} entry in $M = A + A^2 + \dots + A^{n-1}$ is zero.

[In each A, ij^{th} entry is a nonnegative integer and hence ij^{th} entry in $A + A^2 + \dots + A^{n-1}$ is zero if and only if ij^{th} entry in A^r is zero for all $r = 1, 2, \dots n-1$].

Incidence Matrix:

Let G be a graph with n vertices and no self-loops. Let $V = \{v_1, v_2, \dots v_n\}$ and $E = \{e_1, e_2, \dots e_m\}$. Define an $n \times m$ matrix B as follows:

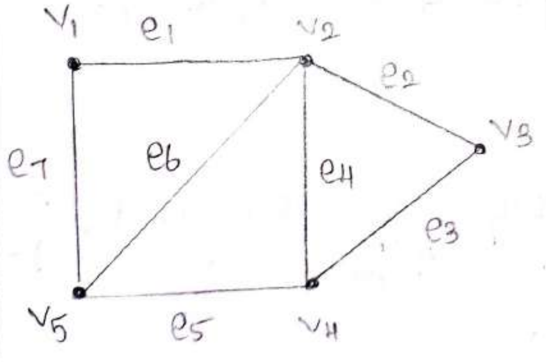
$$B_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j. \\ 0 & \text{otherwise} \end{cases}$$

The matrix B is called the vertex-edge incidence matrix or simply incidence matrix of G.

It is also written as $B(G)$.

Eg:-

A graph G and its incidence matrix.



$$B(G): \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

Adjacency Matrix of a Digraph:

Let G be a digraph with no Parallel edges. Let $V = \{v_1, v_2, \dots, v_n\}$ be its Vertex Set. An $n \times n$ matrix A defined by

$A_{ij} = 1$ if there is a directed edge from v_i to v_j in G.

0 Otherwise

is Called the adjacency matrix of the digraph G. It is denoted by $A(G)$. [$v_i v_j$ is an edge means it is a directed edge with tail v_i and head v_j].

Theorem:

Let G be a directed graph with no Parallel edges. If A is the adjacency matrix of G, and r is a Positive integer, then the ij^{th} entry in A^r Equals the number of different, directed walks of length r from v_i to v_j.

Proof:-

Same Proof of Theorem 4, is Valid for this theorem, after the following Changes.

(1) Replace 'walk' by 'directed walk'.

(2) Replace '$v_i - v_j$ walk' by 'directed walk from v_i to v_j'.

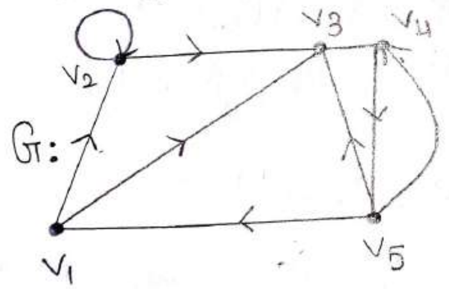
(3) Replace 'edge $v_k v_j$' by 'a directed edge from v_k to v_j'.

Corollary:

Let G be a directed graph without Parallel edges and $V = \{v_1, v_2, \dots, v_n\}$. Let $X = A + A^2 + \dots + A^n$, where $A =$ the adjacent matrix of G. Then the ij^{th} entry of X is the number of directed walks of length $\leq n$ from the vertex v_i to the vertex v_j.

Example:-

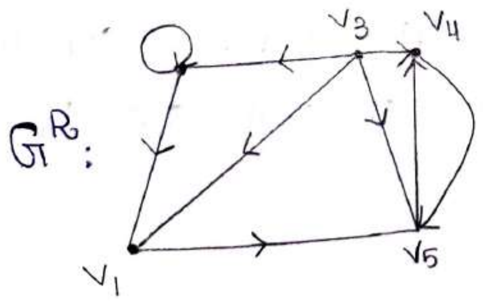
Consider the digraph G. Its adjacency matrix $A(G)$ is given. The reversal (Converse) of G and its adjacency matrix are given.



(a)

$$A(G): \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(b)



$$A(G^R) = A^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(c)

Warshall Algorithm :

We recall have seen that a relation on a set V can be represented by a directed graph, and we have also seen that a given directed graph (without parallel edges) induces a relation R on the vertex set of the graph such that the given digraph is the digraph of R.

Now consider a digraph G which has no parallel edges. The edge set E(G) can be interpreted as a relation R, on the vertex set V(G). The relation matrix of the relation R is the adjacency matrix A (G) of the digraph G.

The transitive closure of R is given by

$$R^{\infty} = R \cup R^2 \cup \dots \cup R^n,$$

where $n = |V(G)|$. (R^{∞} can also denoted by R^{+}). As the relation matrix of R^k is $A^{(k)}$, where $A = A(G)$, the relation matrix of R^{∞} is given by

$$A^{+} = A \vee A^{(2)} \vee \dots \vee A^{(n)} = P$$

Thus the matrix A^{+} is same as the path matrix P.

Hence the path matrix P can be obtained by using warshall algorithm.

Algorithm Warshall:

Given the adjacency matrix A of a digraph (without parallel edges), the following steps produce the path matrix P (or A^+).

1. $P \leftarrow A$.
2. $k \leftarrow 1$.
3. $i \leftarrow 1$.
4. $P_{ij} \leftarrow P_{ij} \vee (P_{ik} \wedge P_{kj})$ for all j from 1 to n.
5. $i \leftarrow i+1$. If $i \leq n$, go to Step 4.
6. $k \leftarrow k+1$. If $k \leq n$, go to Step 3; otherwise, halt.

Algorithm Minima:

Start with the adjacency matrix. Replace the zero elements in the adjacency matrix by infinity or by some very large number. Let the resulting matrix be D. The matrix C produced by the following steps gives the minimum length of the paths between the nodes.

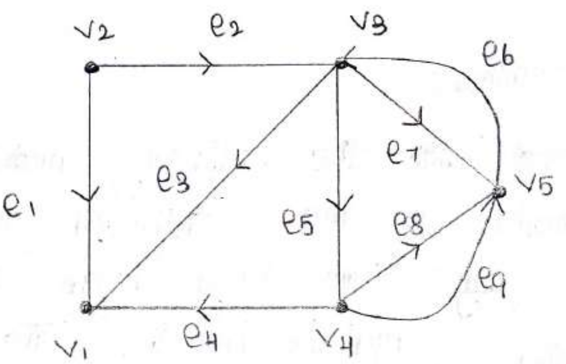
1. $C \leftarrow D$
2. $k \leftarrow 1$.
3. $i \leftarrow 1$.
4. $C_{ij} \leftarrow \min(C_{ij}, C_{ik} + C_{kj})$ for all j from 1 to n
5. $i \leftarrow i+1$. If $i \leq n$, go to Step 4.
6. $k \leftarrow k+1$. If $k \leq n$, go to Step 3; otherwise, halt.

Incidence Matrix of a digraph:

The incidence matrix of a digraph with the vertex set $\{v_1, v_2, \dots v_n\}$, the edge set $\{e_1, e_2, \dots e_m\}$ and with no self-loop, is an $n \times m$ matrix $B = (b_{ij})$ defined by

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the tail of the edge } e_j \\ -1 & \text{if } v_i \text{ is the head of the edge } e_j. \\ 0 & \text{otherwise.} \end{cases}$$

A digraph G and its incidence matrix B(G).



$$B(G) = \begin{pmatrix} -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \end{pmatrix}$$

Worked Examples :-

1. If G is a simple undirected graph with vertex set $V = \{v_1, v_2, \dots v_k\}$ and adjacency matrix A, show that ii^{th} entry in A^2 is the degree of the vertex v_i, for all $i = 1, 2, \dots n$.

Solution :-

The ii^{th} entry in A^2 = $\sum_{k=1}^{n} a_{ik} a_{ki}$

$= \sum a_{ik} a_{ki}$, the sum is taken over all k for which $a_{ik} \neq 0$.

$= \sum a_{ik} a_{ki}$, the sum is taken over all k for which there is an edge between v_i and v_k.

$= \deg(v_i)$ Since $a_{ik} = a_{ki} = 1$, iff there is an edge between v_i and v_k.

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2. Let G be a digraph whose underlying undirected graph has no parallel edges. From the adjacency matrix of G, how will you obtain the adjacency matrix for the underlying undirected graph ?

Sol :-

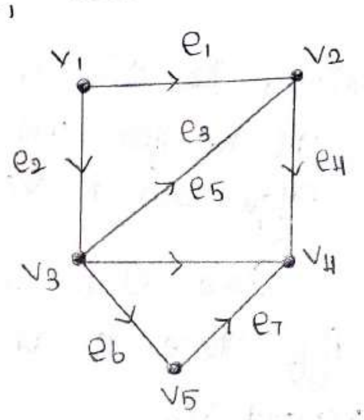
Let $A = (a_{ij})$ be the adjacency matrix of the digraph G.

There is an edge between the vertices v_i and v_j in the undirected graph if and only if either v_iv_j or v_jv_i is a directed graph edge in G.

So if $B = (b_{ij})$ is the adjacency matrix of undirected graph, then $b_{ij} = \max\{a_{ij}, a_{ji}\}$.

So $B = (b_{ij})$, where $b_{ij} = \max\{a_{ij}, a_{ji}\}$ is the required matrix.

3. Find the incidence matrix for the directed graph given below: If the graph is not directed, how will this matrix change.

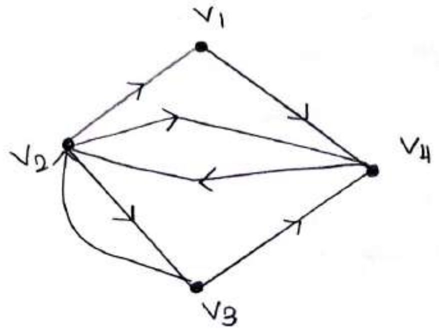


<u>Solution :-</u>

$$\begin{array}{c c} & \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$$

is the incidence matrix of the digraph. If the graph is not directed, the change -1 entries into 1.

4. Obtain the adjacency matrix A of digraph given below: Find the Elementary Paths of length 1 and 2 from v_1 to v_4.



Solution :-

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Elementary paths of length 1 from v_1 to v_4 Should be the directed edge v_1v_4. It exists.

An Elementary path of length 2 from v_1 to v_4 Should be of the from $v_1 x v_4$, where v_1x and xv_4 are directed edges.

There is no the elementary path of length 2 from v_1 to v_4.

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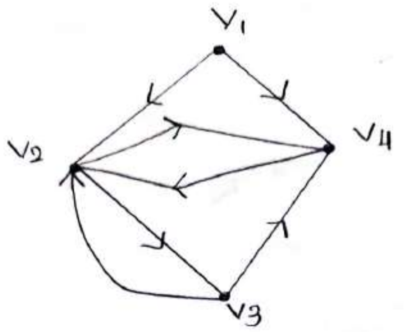
5. For a Simple digraph $G = (V, E)$ with adjacency matrix A, its distance matrix is given by

$d_{ii} = 0$, for all $i = 1, 2, \ldots n$.

$d_{ij} = k$, if k is the Smallest Positive integer for which $a_{ij}^{(k)} \neq 0 = \infty$; if no Such k exists.

Determine the distance matrix of the digraph G given below:



Solution :-

Its adjacency matrix A is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Calculate $A^{(2)}$, $A^{(3)}$ (As $n=4$, it is enough to find upto $A^{(3)}$)

$$A^{(2)} = A \wedge A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$A^{3} = A^{2} \wedge A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Let $\mathcal{D}$ be the distance matrix.

Then

$$d_{11} = 0 ; d_{12} = 1 \text{ as } a_{12}^{(1)} \neq 0$$

$$d_{13} = 2 \text{ as } a_{12}^{(1)} = 0,\ a_{12}^{(2)} \neq 0$$

$$d_{14} = 1.$$

$$d_{21} = \infty \text{ as } a_{21}^{(1)} = a_{21}^{(2)} = a_{21}^{(3)} = 0$$

Similarly

$$d_{22} = 0 ; d_{23} = 1 ; d_{24} = 1$$

$$d_{31} = \infty ; d_{32} = 1 ; d_{33} = 0 ; d_{34} = 1$$

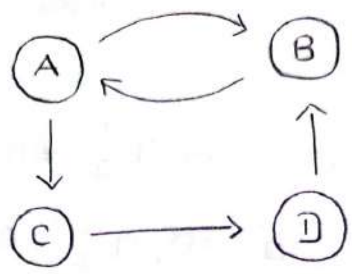
$$d_{4} = \infty ; d_{42} = 1 ; d_{43} = 2 ; d_{44} = 0$$

$(d_{11} = d_{22} = d_{33} = d_{44} = 0$ by definition of $\mathcal{D})$

Thus $\mathcal{D} = \begin{pmatrix} 0 & 1 & 2 & 1 \\ \infty & 0 & 1 & 1 \\ \infty & 1 & 0 & 1 \\ \infty & 1 & 2 & 0 \end{pmatrix}$

Note that $d_{ij} = 1$ means $v_i v_j$ is a directed edge in G).

6. Consider the following digraph. Use its adjacency matrix to find how many paths of length 3 exists from A to B.



(Note : In this Problem path means a directed walks).

Solution :-

First find the adjacency matrix A.

$$A(G) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

To find the number of directed walks of length 3 from A to B, We have to find A^3 (not $A^{(3)}$) and the term (1,2) in A^3.

$$A^2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

As $(1,2)^{th}$ Entry in $A^{(3)}$ is 2, there are two directed walks of length 3 from v_1 to v_2, i.e., from A to B.

They are $A \to B \to A \to B$, $A \to C \to D \to B$.