MARUDHAR KESARI JAIN COLLEGE FOR WOMEN, VANIYAMBADI

PG & RESEARCH DEPARTMENT OF MATHEMATICS

CLASS : I B.Sc CHEMISTRY

SUBJECT CODE : MATHEMATICS I

SUBJECT NAME : 23UEMA10C

SYLLABUS

UNIT- I

Summation of series

Binomial series -Exponential series - Logarithmic series -Simple Problems.

Binomial series

when n is a rational number,

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

$\forall x$, such that $-1 < x < 1$.

Results:

1. $(1-x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots$

2. $(1-x)^{-n} = 1 - \frac{(-n)}{1}x + \frac{(-n)((-n)-1)}{1\cdot 2}x^2 - \cdots$

when $-1 < x < 1$ and n is a positive integer.

3. $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

4. $\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

5. $\frac{1}{(1-x)^3} = (1-x)^{-3} = \frac{1}{1\cdot 2}\left[1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + 4\cdot 5x^3 + \cdots\right]$

6. $\frac{1}{(1-x)^4} = (1-x)^{-4} = \frac{1}{1\cdot 2\cdot 3}\left[1\cdot 2\cdot 3 + 2\cdot 3\cdot 4\cdot x + 3\cdot 4\cdot 5\cdot x^2 + \cdots\right]$

7. $\frac{1}{(1-x)^n} = (1-x)^{-n} = \frac{1}{1\cdot 2\cdot 3\cdots(n-1)}\left[1\cdot 2\cdots(n-1) + 2\cdot 3\cdot 4\cdots n\cdot x + \cdots\right]$

8. $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

9. $\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

10. $\frac{1}{(1+x)^3} = (1+x)^{-3} = \frac{1}{1\cdot 2}\left[1\cdot 2 - 2\cdot 3x + 3\cdot 4x^2 - \cdots\right]$

11. $\frac{1}{(1+x)^4} = (1+x)^{-4} = \frac{1}{1\cdot2\cdot3}\left[1\cdot2\cdot3 - 2\cdot3\cdot4\,x + 3\cdot4\cdot5\,x^2 - \cdots\right]$

12. $\frac{1}{(1+x)^n} = (1+x)^{-n} = \frac{1}{1\cdot2\cdot3\cdots n}\left[1\cdot2\cdot3\cdots(n-1) - 2\cdot3\cdots n\,x + \cdots\right]$

when n is positive number.

Problems

1. V.Q Find the Co-eff of x^n in the expansion of $\frac{1}{1-x^2}$

Soln:

$$\frac{1}{1-x^2} = (1-x^2)^{-1}$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\cdots+x^n+\cdots$$

Put $x = x^2$

$$(1-x^2)^{-1} = 1+x^2+(x^2)^2+(x^2)^3+\cdots(x^2)^n+\cdots$$

$$= 1+x^2+x^4+x^6+\cdots+x^{2n}+\cdots$$

$\therefore$ Co-eff of $x^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

2. V.Q Find the co-eff of x^{2n} in the expansion of $(1-x^2)^{-1}$.

Soln:

$$(1-x^2)^{-1}$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\cdots+x^n+\cdots$$

Put $x = x^2$ in formula

$$(1-x^2)^{-1} = 1+x^2+(x^2)^2+(x^2)^3+\cdots+(x^2)^n+\cdots$$

$= 1 + x^2 + x^4 + x^6 + \cdots + (x^{2n}) + \cdots$

$\therefore$ co-eff of $x^{2n} = 1$

3. Find the co-eff of x^2 in expansion of $(1+x)^{-3}$

U.Q

Soln:

$(1+x)^{-3} = \frac{1}{1\cdot 2}[1\cdot 2 - 2\cdot 3x + 3\cdot 4x^2 + \cdots]$

co-eff of $x^2 = \frac{1}{1\cdot 2}(3\cdot 4)$

co-eff of $x^2 = 6$

4. Find co-eff of x^n in $\frac{1}{1-2x} + \frac{1}{1-3x}$

U.Q Soln:

$\frac{1}{1-2x} = (1-2x)^{-1}$; $\frac{1}{1-3x} = (1-3x)^{-1}$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

Put $x = 2x$ & $x = 3x$

$(1-2x)^{-1}(1-3x)^{-1} = [1 + 2x + (2x)^2 + (2x)^3 + \cdots + (2x)^n + \cdots]$
$+ [1 + 3x + (3x)^2 + (3x)^3 + \cdots + (3x)^n + \cdots]$

U.Q $\therefore$ co-eff of $x^n = 2^n + 3^n$

5. Find the sums of the following series

i) $1 + 2(1/2) + 3(1/2)^2 + \cdots + \infty$

ii) $1 + 2(1/3) + 3(1/3)^2 + 4(1/3)^3 + \cdots \infty$

Soln:

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Put $x = 1/2$

$$1 + 2(1/2) + 3(1/2)^2 + \cdots = (1 - 1/2)^{-2}$$

$$= \left(\frac{2-1}{2}\right)^{-2}$$

$$= (1/2)^{-2}$$

$$= (1/1/4)$$

$$= 4$$

ii) $1 + 2(1/3) + 3(1/3)^2 + \cdots = (1 - 1/3)^{-2}$

$$= \left(\frac{3-1}{3}\right)^{-2}$$

$$= \frac{1}{(2/3)^2}$$

$$= \left(\frac{1}{4/9}\right)$$

$$= 9/4$$

$1 + 2(1/3) + 3(1/3)^2 + \cdots$

6. Find the coeff of x^n in expansion of $(2+3x)^{-1}$ in ascending power of x.

Soln:

$$(2+3x)^{-1} = 2^{-1}\left(1 + \frac{3x}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(1 + \frac{3}{2}x\right)^{-1}$$

W.K.T $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$]

$$\frac{1}{2}\left[1+\frac{3}{2}x\right]^{-1} = \frac{1}{2}\left[1-\frac{3}{2}x+\left(\frac{3}{2}x\right)^2-\left(\frac{3}{2}x\right)^3-\ldots\right.$$

$$\left.+(-1)^n\left(\frac{3}{2}x\right)^n+\ldots\right]$$

$$=\frac{1}{2}\left[1-\frac{3}{2}x+\frac{3^2}{2^2}x^2+\ldots+(-1)^n\frac{3^n}{2^n}x^n+\ldots\right]$$

$\therefore$ Co-eff of $x^n = \frac{1}{2}\left[(-1)^n\left(\frac{3^n}{2^n}\right)\right]$

$\therefore$ co-eff of $x^n = (-1)^n\frac{3^n}{2^{n+1}}$

T. Find the co-eff of x^n in expansion of $[1+2x+3x^2+4x^3+\ldots]^2$

Soln:

$$[1+2x+3x^2+4x^3+\ldots]^2 = [(1-x)^{-2}]^2$$

$$= \left[\frac{1}{(1-x)^2}\right]^2$$

$$= \frac{1}{(1-x)^4}$$

$$= (1-x)^{-4}$$

W.K.T

$$(1-x)^{-4} = \frac{1}{1\cdot2\cdot3}\left[1\cdot2\cdot3+2\cdot3\cdot4\cdot x+3\cdot4\cdot5\cdot x^2+\ldots\right.$$

$$\left.+(n+1)(n+2)(n+3)x^n+\ldots\right]$$

$\therefore$ Co-eff of $x^n = \frac{1}{1\cdot2\cdot3}\left[(n+1)(n+2)(n+3)\right]$

$\therefore$ Co-eff of $x^n = \frac{1}{6}\left[(n+1)(n+2)(n+3)\right]$

8. Write the $(n+1)^{th}$ term in expansion of $(3-2x)^{-2}$, where x is small.

Soln:

$$(3-2x)^{-2} = \frac{1}{(3-2x)^2}$$

$$= \frac{1}{3^2\left[1-\frac{2}{3}x\right]^2}$$

$$= \frac{1}{9}\left[1-\frac{2}{3}x\right]^{-2}$$

$$(1-x)^{-2} = 1+2x+3x^2+\cdots+(n+1)x^n+\cdots$$

$$\frac{1}{9}\left[1-\frac{2}{3}x\right]^{-2} = \frac{1}{9}\left[1+2\left(\frac{2}{3}x\right)+3\left(\frac{2}{3}x\right)^2+\cdots+(n+1)\left(\frac{2}{3}x\right)^n+\cdots\right]$$

$$= \frac{1}{9}\left[1+2\left(\frac{2}{3}\right)x+3\left(\frac{2^2}{3^2}x^2\right)+\cdots+(n+1)\left(\frac{2^n}{3^n}x^n\right)+\cdots\right]$$

$(n+1)^{th}$ term $= \left[\frac{2^n}{3^n}x^n\right]\frac{1}{9}$

$\therefore (n+1)^{th}$ term $= \frac{2^n}{3^{n+2}}(n+1)x^n = \frac{2^n}{3^{n+2}}(n+1)x^n$.

9. Find the co-efficient of x^2 in the expansion of $(1+\frac{2}{3}x)^{3/2}$

Soln:

W.K.T

$$(1+x)^n = 1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\cdots$$

Put $x = \frac{2}{3}x$, $n = 3/2$

$$\left(1+\frac{2}{3}x\right)^{3/2} = \left[1+\frac{\frac{3}{2}}{1}\left(\frac{2}{3}\right)x+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{1\cdot 2}\left(\frac{2}{3}x\right)^2+\cdots\right]$$

Co-efficient of $x^2 = \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{1\cdot 2}\ \frac{4}{9}$

$= \frac{\frac{3}{2}\left(\frac{1}{2}\right)}{2} \times \frac{4^2}{9}$

$= \frac{1}{4} \times \frac{2}{3}$

$\therefore$ Co-eff of $x^2 = \frac{1}{6}$

10. Find the co-eff of x^6 in the expansion of $\frac{1}{(1-x^2)^3}$

Soln:

W.K.T

$(1-x)^{-3} = \frac{1}{1\cdot 2}[1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + 4\cdot 5x^3 + 5\cdot 6x^4 + 6\cdot 7x^5 + 7\cdot 8x^6 + \cdots]$

Put $x = x^2$

$(1-x^2)^{-3} = \frac{1}{1\cdot 2}[1\cdot 2 + 2\cdot 3(x^2) + 3\cdot 4(x^2)^2 + 4\cdot 5(x^2)^3 + 5\cdot 6(x^2)^4 + 6\cdot 7(x^2)^5 + 7\cdot 8(x^2)^6 + \cdots]$

$= \frac{1}{1\cdot 2}[1\cdot 2 + 2\cdot 3x^2 + 3\cdot 4x^4 + 4\cdot 5x^6 + 5\cdot 6x^8 + 6\cdot 7x^{10} + 7\cdot 8x^{12} + \cdots]$

Co-eff of $x^6 = \frac{1}{1\cdot 2}[4\cdot 5]$

$\therefore$ Co-eff of $x^6 = 10$

11. If x is small, what is the value of a if $\sqrt{x^2+4} - \sqrt{x^2+1} = 1 - ax^2$ nearly.

U.Q

Soln:

$$\sqrt{x^2+4} - \sqrt{x^2+1} = (x^2+4)^{1/2} - (x^2+1)^{1/2}$$

$$= 4^{1/2}\left(1 + \frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

$$= 2\left(1+\frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

W.K.T

$$(1+x)^n = \left[1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \cdots\right]$$

Put $x = \frac{x^2}{4}$; $n = 1/2$

$$= 2\left[1 + \frac{1/2}{1}\left(\frac{x^2}{4}\right) + \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{x^2}{4}\right)^2 + \cdots\right]$$

$$- \left[1 + \frac{1}{2}x^2 + \frac{1}{2}\left(\frac{1}{2}-1\right)(x^2)^2 + \cdots\right]$$

$$\simeq 2\left[1 + \frac{1}{2}\left(\frac{x^2}{4}\right)\right] - \left[1 + \frac{1}{2}x^2\right] \text{ nearly}$$

$$= 2\left[1 + \frac{x^2}{8}\right] - 1 - \frac{1}{2}x^2$$

$$= 2 + \frac{2x^2}{4} - 1 - \frac{1}{2}x^2$$

$$= 1 + \frac{x^2}{4} - \frac{x^2}{2}$$

$$= 1 + \frac{x^2 - 2x^2}{4}$$

$$= 1 - \frac{x^2}{4}$$

$\therefore \sqrt{x^2+4} - \sqrt{x^2+1} = 1 - \frac{x^2}{4}$

$a = \frac{1}{4}$

12. When x is small P.T

$(1-x)^{-1/2} + (1+x)^{1/2} = 2 + x + \frac{x^2}{4}$ (nearly)

Soln:

W.K.T

$(1+x)^{-n} = 1 - \frac{(-n)x}{1} + \frac{(-n)((-n)-1)}{\quad} x^2 + \cdots$

$(1-x)^{-n} = 1 + \frac{n}{1}x + \frac{n(n+1)}{1\cdot 2}x^2 + \cdots$

$(1+x)^{+n} = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \cdots$

$(1-x)^{-1/2} + (1+x)^{1/2} = \left[1 + \frac{1}{2}x + \frac{1/2(1/2+1)}{1\cdot 2}x^2 + \cdots\right]$

$+ \left[1 + \frac{1/2}{1}x + \frac{1/2(1/2-1)}{1\cdot 2}x^2 + \cdots\right]$

$= \left[1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{3}{2}\right)}{1\cdot 2}x^2 + \cdots\right] + \left[1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1\cdot 2}x^2 + \cdots\right]$

$= \left[1 + \frac{1}{2}x + \frac{3}{4\times 2}x^2\right] + \left[1 + \frac{1}{2}x - \frac{\frac{1}{4}}{2}x^2\right]$

$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + 1 + \frac{1}{2}x - \frac{1}{8}x^2$ (nearly)

$= 2 + x + \frac{2}{8}x^2$ (nearly)

$(1-x)^{-1/2} + (1+x)^{1/2} = 2 + x + \frac{x^2}{4}$ (nearly)

13. When x is small, $P.T \sqrt{x^2+4} - \sqrt{x^2+1} = 1 - \frac{1}{4}x^2 + \frac{7}{64}x^4$ nearly.

H.W

Soln:

$$\sqrt{x^2+4} - \sqrt{x^2+1} = (x^2+4)^{1/2} - (x^2+1)^{1/2}$$

$$= 4^{1/2}\left(1+\frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

$$= 2\left(1+\frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

W.K.T

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

Put $x = \frac{x^2}{4}$ and $x = x^2$; $n = 1/2$

$$= 2\left[1 + \frac{1}{2}\left(\frac{x^2}{4}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}\left(\frac{x^2}{4}\right)^2 + \cdots\right]$$

$$- \left[1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}(x^2)^2 + \cdots\right]$$

$$= 2\left[1 + \frac{x^2}{8} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\frac{x^4}{16} + \cdots\right]$$

$$- \left[1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^4 + \cdots\right]$$

$$= 2\left[1 + \frac{x^2}{8} - \frac{1}{8}\frac{x^4}{16}\right] - \left[1 + \frac{1}{2}x^2 - \frac{1}{8}x^4\right]$$

$$= 2 + \frac{2x^2}{8} - \frac{2}{8}\frac{x^4}{16} - 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4$$

$$= 1 + \frac{x^2}{4} - \frac{2x^2}{4} + \left(\frac{8-2}{64}\right)x^4 \text{ nearly}$$

$$= 1 - \frac{1}{4}x^2 + \frac{7}{64}x^4 \text{ nearly}$$

14. Find the co-efficient of x^n in the expansion of $(1+x+x^2+x^3+\cdots)^{-n}$

H.W

Soln:

$$(1+x+x^2+x^3+\cdots)^{-n} = ((1-x)^{-1})^{-n}$$

$$= (1-x)^n$$

$$= 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \cdots$$

$$\frac{(-1)^n\, n(n-1)(n-2)\cdots 2\cdot 1}{1\cdot 2\cdots(n-2)(n-1)n}x^n$$

$$= 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \cdots + (-1)^n x^n$$

$\therefore$ Co-eff of $x^n = (-1)^n$.

SUMMATION: Binomial Series

The formula for finding the sums of Binomial Series is

$$(1-x)^{-P/q} = 1 + \frac{P}{1}\left(\frac{x}{q}\right) + \frac{P(P+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2 + \cdots$$

15. Sum the series $1 + \frac{1}{3} + \frac{1\cdot 3}{3\cdot 6} + \frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9} + \cdots$

U.Q

Soln:

W.K.T

$$(1-x)^{-P/q} = 1 + \frac{P}{1}\left(\frac{x}{q}\right) + \frac{P(P+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2 + \frac{P(P+q)(P+2q)}{1\cdot 2\cdot 3}\left(\frac{x}{q}\right)^3 + \cdots$$

Denote the sum by S.

Let

$$S = 1 + \frac{1}{3} + \frac{1\cdot 3}{3\cdot 6} + \frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9} + \cdots$$

$$\frac{1\cdot 3}{1\cdot 3\ 2\cdot 3} \qquad \frac{1\cdot 3\ 2\cdot 3\ 3\cdot 3}{1\cdot 3\ 2\cdot 3\ 3\cdot 3}$$

$p = 1, \quad q = 2$

$p + q = 3$

$1 + q = 3$

$q = 3 - 1 = 2$

$$S = 1 + \frac{1}{1}\left(\frac{1}{3}\right) + \frac{1\cdot 3}{1\cdot 2}\left(\frac{1}{3}\right)^2 + \frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^3 + \cdots$$

Here $\frac{x}{q} = \frac{1}{3}$

$x = \frac{q}{3}$

$x = 2/3$

$$\therefore (1-x)^{-p/q} = \left(1 - \frac{2}{3}\right)^{-1/2}$$

$$= \left(\frac{3-2}{3}\right)^{-1/2}$$

$$= \left(\frac{1}{3}\right)^{-1/2}$$

$$= (3^{-1})^{-1/2}$$

$$= 3^{1/2}$$

$$\boxed{S = \sqrt{3}}$$

Q.16. Sum the series $\frac{1}{10} + \frac{1\cdot 4}{10\cdot 20} + \frac{1\cdot 4\cdot 7}{10\cdot 20\cdot 30} + \cdots$

Soln:

Denote the sum by S, Then

$$S = \frac{1}{10} + \frac{1\cdot 4}{10\cdot 20} + \frac{1\cdot 4\cdot 7}{10\cdot 20\cdot 30} + \cdots$$

$1\cdot 10 \quad 1\cdot 10 \; 2\cdot 10 \quad 1\cdot 10 \; 2\cdot 10 \; 3\cdot 10$

$$= \frac{1}{1}\left(\frac{1}{10}\right) + \frac{1\cdot 4}{1\cdot 2}\left(\frac{1}{10}\right)^2 + \frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}\left(\frac{1}{10}\right)^3 + \cdots$$

W.K.T

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2 + \cdots$$

Add and Sub 1 in R.H.S

$$= \left[1 + \frac{1}{1}\left(\frac{1}{10}\right) + \frac{1\cdot 4}{1\cdot 2}\left(\frac{1}{10}\right)^2 + \frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}\left(\frac{1}{10}\right)^3 + \cdots\right] - 1$$

$p = 1$; $p+q = 4$

$q = 4-1 = 3$

$\frac{x}{q} = \frac{1}{10}$

$\frac{x}{3} = \frac{1}{10} \Rightarrow x = \frac{3}{10}$

$$S = \left(1 - \frac{3}{10}\right)^{-1/3} - 1$$

$$= \left(\frac{10-3}{10}\right)^{-1/3} - 1$$

$$= \left(\frac{7}{10}\right)^{-1/3} - 1$$

$$= \frac{1}{(7/10)^{1/3}} - 1$$

$$\boxed{S = \left(\frac{10}{7}\right)^{1/3} - 1}$$

4.Q

17 Sum the series

$$\frac{5}{1\cdot2}\left(\frac{1}{3}\right)+\frac{5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^2+\frac{5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}\left(\frac{1}{3}\right)^3+\cdots$$

Soln:

Denote the sum by S; then

$$S=\frac{5}{1\cdot2}\left(\frac{1}{3}\right)+\frac{5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^2+\frac{5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}\left(\frac{1}{3}\right)^3+\cdots$$

W.K.T

$$(1-x)^{-p/q}=1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2+\cdots$$

Multiplying by 3 on both sides.

$$3S=\frac{3\cdot5}{1\cdot2}\left(\frac{1}{3}\right)+\frac{3\cdot5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^2+\frac{3\cdot5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}\left(\frac{1}{3}\right)^3+\cdots$$

Multiplying by $\frac{1}{3}$ on both sides

$$\left(\frac{1}{3}\right)3S=\frac{3\cdot5}{1\cdot2}\left(\frac{1}{3}\right)^2+\frac{3\cdot5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^3+\frac{3\cdot5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}\left(\frac{1}{3}\right)^4+\cdots$$

Add and subtract $1+\frac{3}{1}\left(\frac{1}{3}\right)$ in R.H.S

$$S=\left[1+\frac{3}{1}\left(\frac{1}{3}\right)+\frac{3\cdot5}{1\cdot2}\left(\frac{1}{3}\right)^2+\frac{3\cdot5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^3+\cdots\right]-\left[1+\frac{3}{1}\left(\frac{1}{3}\right)\right]$$

$p=3$; $p+q=5$, $q=2$

$\frac{x}{q}=\frac{1}{3}$

$x=\frac{2}{3}$

$$S = (1-x)^{-p/q} - (1+1)$$

$$= \left(1-\frac{2}{3}\right)^{-3/2} - 2$$

$$= \left(\frac{3-2}{3}\right)^{-3/2} - 2$$

$$= \left(\frac{1}{3}\right)^{-3/2} - 2$$

$$= (3)^{3/2} - 2$$

$$\boxed{S = 3\sqrt{3} - 2}$$

18. Sum to infinity the series

$$\frac{7}{9} + \frac{7\cdot 9}{9\cdot 12} + \frac{7\cdot 9\cdot 12}{9\cdot 12\cdot 15} + \cdots$$

Soln.

Denote the sum by S, then

$$S = \frac{7}{9} + \frac{7\cdot 9}{9\cdot 12} + \frac{7\cdot 9\cdot 12}{9\cdot 12\cdot 15} + \cdots$$

$$= \frac{7}{3}\left(\frac{1}{3}\right) + \frac{7\cdot 9}{3\cdot 4}\left(\frac{1}{3}\right)^2 + \frac{7\cdot 9\cdot 12}{3\cdot 4\cdot 5}\left(\frac{1}{3}\right)^3 + \cdots$$

Multiplying by both sides $\frac{1}{1\cdot 2}$

$$\frac{S}{1\cdot 2} = \frac{7}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right) + \frac{7\cdot 9}{1\cdot 2\cdot 3\cdot 4}\left(\frac{1}{3}\right)^2 + \frac{7\cdot 9\cdot 12}{1\cdot 2\cdot 3\cdot 4\cdot 5}\left(\frac{1}{3}\right)^3 + \cdots$$

Multiplying both sides by $3\cdot 5$

$$\frac{3\cdot 5}{1\cdot 2} S = \frac{3\cdot 5\cdot 7}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right) + \frac{3\cdot 5\cdot 7\cdot 9}{1\cdot 2\cdot 3\cdot 4}\left(\frac{1}{3}\right)^2 + \cdots$$

Multiplying by $(1/3)^2$ on both sides

$$\frac{15}{2}\left(\frac{1}{3}\right)^2 = \frac{3\cdot5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^3 + \frac{3\cdot5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}\left(\frac{1}{3}\right)^4 + \cdots$$

Add and subtract by $1 + \frac{3}{1}\left(\frac{1}{3}\right) + \frac{3\cdot5}{1\cdot2}\left(\frac{1}{3}\right)^2$ in

R.H.S

$$\frac{15}{2}\times\frac{1}{9} = \left[1 + \frac{3}{1}\left(\frac{1}{3}\right) + \frac{3\cdot5}{1\cdot2}\left(\frac{1}{3}\right)^2 + \frac{3\cdot5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^3 + \cdots\right]$$

$$= \left[1 + \frac{3}{1}\left(\frac{1}{3}\right) + \frac{3\cdot5}{1\cdot2}\left(\frac{1}{3}\right)^2\right]$$

$$\frac{5}{6}S = \left[1 + \frac{3}{1}\left(\frac{1}{3}\right) + \frac{3\cdot5}{1\cdot2}\left(\frac{1}{3}\right)^2 + \frac{3\cdot5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^3 + \cdots\right]$$

$$- \left[1 + 1 + \frac{15}{2}\times\frac{1}{9}\right]$$

$p=3$; $p+q=5$; $\frac{x}{q} = \frac{1}{3}$

$q = 5-3$ $\quad x = \frac{2}{3}$

$q = 2$

$$\frac{5}{6}S = \left(1-\frac{2}{3}\right)^{-3/2} - \left[2 + \frac{5}{6}\right]$$

$$= \left(\frac{3-2}{3}\right)^{-3/2} - \left(\frac{12+5}{6}\right]$$

$$= \left(\frac{1}{3}\right)^{-3/2} - \frac{17}{6}$$

$$= (3)^{3/2} - \frac{17}{6}$$

$$\frac{5}{6}S = 3\sqrt{3} - \frac{17}{6}$$

$$S = \left(3\sqrt{3} - \frac{17}{6}\right)\frac{6}{5}$$

$$= \frac{3\sqrt{3}}{5}\times6 - \frac{17}{6}\times\frac{6}{5}$$

$$S = \frac{18\sqrt{3}}{5} - \frac{17}{5}$$

H.W

19. Sum the series $\frac{1}{3\cdot6} + \frac{1\cdot3}{3\cdot6\cdot9} + \frac{1\cdot3\cdot5}{3\cdot6\cdot9\cdot12} + \cdots$

Soln:

Denote the sum by S, then

W.K.T

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2 + \cdots$$

$$S = \frac{1}{3\cdot6} + \frac{1\cdot3}{3\cdot6\cdot9} + \frac{1\cdot3\cdot5}{3\cdot6\cdot9\cdot12} + \cdots$$

$$= \frac{1}{1\cdot2}\left(\frac{1}{3}\right)^2 + \frac{1\cdot3}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^3 + \frac{1\cdot3\cdot5}{1\cdot2\cdot3\cdot4}\left(\frac{1}{3}\right)^4 + \cdots$$

multiply both sides by -1

$$-S = \left[-\frac{1\cdot1}{1\cdot2}\left(\frac{1}{3}\right)^2 + \frac{-1\cdot1\cdot3}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^3 + \frac{-1\cdot1\cdot3\cdot5}{1\cdot2\cdot3\cdot4}\left(\frac{1}{3}\right)^4 + \cdots\right]$$

Add & subtract $1 + \frac{-1}{1}\left(\frac{1}{3}\right)$ in R.H.S

$$-S = 1 + \frac{-1}{1}\left(\frac{1}{3}\right) + \frac{-1\cdot1}{1\cdot2}\left(\frac{1}{3}\right)^2 + \cdots\Big] - \left[1 + \frac{-1}{1}\left(\frac{1}{3}\right)\right]$$

$p = -1$; $p+q = 1$

$-1+q = 1 \Rightarrow q = 2$

$\frac{x}{q} = \frac{1}{3} \Rightarrow x = \frac{2}{3}$

$$-S = (1-x)^{-p/q} - \left(1 - \frac{1}{3}\right)$$

$$= \left(1 - \frac{2}{3}\right)^{1/2} - \frac{2}{3}$$

$$= \left(\frac{3-2}{3}\right)^{1/2} - \frac{2}{3}$$

$$= \left(\frac{1}{3}\right)^{1/2} - \frac{2}{3}$$

$$-S = \frac{1}{\sqrt{3}} - \frac{2}{3} \Rightarrow \boxed{S = \frac{2}{3} - \frac{1}{\sqrt{3}}}$$

20. S.T $\sqrt{8} = 1 + \frac{3}{4} + \frac{3\cdot5}{2\cdot4^2} + \frac{3\cdot5\cdot7}{2\cdot3\cdot4^3} + \cdots$

Soln:

R.H.S

$\Rightarrow 1 + \frac{3}{4} + \frac{3\cdot5}{2\cdot4^2} + \frac{3\cdot5\cdot7}{2\cdot3\cdot4^3} + \cdots$

$\Rightarrow 1 + \frac{3}{1}(\frac{1}{4}) + \frac{3\cdot5}{1\cdot2}(\frac{1}{4})^2 + \frac{3\cdot5\cdot7}{1\cdot2\cdot3}(\frac{1}{4})^3 + \cdots$

$p=3;\ p+q=5$

$q=5-3$

$q=2$

$\frac{x}{q} = \frac{1}{4}$

$x = \frac{q}{4} = \frac{2}{4} = \frac{1}{2}$

$\Rightarrow (1-x)^{-p/q}$

$\Rightarrow (1-\frac{1}{2})^{-3/2}$

$\Rightarrow (\frac{2-1}{2})^{-3/2}$

$\Rightarrow (\frac{1}{2})^{-3/2}$

$\Rightarrow (2)^{3/2}$

$\Rightarrow (2^3)^{1/2} = (8)^{1/2}$

$\Rightarrow \sqrt{8} = $ L.H.S

L.H.S = R.H.S

21. Sum the series $1 + \frac{2}{1}(\frac{1}{72}) + \frac{2\cdot5}{1\cdot2}(\frac{1}{72^2}) + \cdots$

Soln:

Denote the sum by S, then

$S = 1 + \frac{2}{1}(\frac{1}{72}) + \frac{2\cdot5}{1\cdot2}(\frac{1}{72^2}) + \frac{2\cdot5\cdot8}{1\cdot2\cdot3}(\frac{1}{72^3}) + \cdots$

$= 1 + \frac{2}{1}(\frac{1}{49}) + \frac{2\cdot5}{1\cdot2}(\frac{1}{49})^2 + \frac{2\cdot5\cdot8}{1\cdot2\cdot3}(\frac{1}{49})^3 + \cdots$

$p=2$; $p+q=5$

$2+q=5$

$q=5-2$

$q=3$

$\frac{x}{q}=\frac{1}{49}$

$x=\frac{3}{49}$

$S=(1-x)^{-p/q}$

$=\left(1-\frac{3}{49}\right)^{-2/3}$

$S=\left(\frac{49-3}{49}\right)^{-2/3}=\left(\frac{46}{49}\right)^{-2/3}=\left(\frac{49}{46}\right)^{2/3}$

22. Sum the series $\frac{1}{3}+\frac{1\cdot3}{3\cdot6}+\frac{1\cdot3\cdot5}{3\cdot6\cdot9}+\cdots$

Soln.

Denote the sum. by S,

$S=\frac{1}{3}+\frac{1\cdot3}{3\cdot6}+\frac{1\cdot3\cdot5}{3\cdot6\cdot9}+\cdots$

$1\cdot3\quad 2\cdot3\quad 1\cdot3\quad 2\cdot3\quad 3\cdot3$

$S=\frac{1}{1}(1/3)+\frac{1\cdot3}{1\cdot2}(1/3)^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}(1/3)^3+\cdots$

Add & sub. 1

$S=\left[1+\frac{1}{1}(1/3)+\frac{1\cdot3}{1\cdot2}(1/3)^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}(1/3)^3+\cdots\right]-1$

$p=1$; $p+q=3$

$1+q=3$

$q=3-1$

$q=2$

$\frac{x}{q}=\frac{1}{3}$

$x=2/3$

$S=(1-x)^{-p/q}-1$

$=\left(1-\frac{2}{3}\right)^{-1/2}-1$

$=\left(\frac{3-2}{3}\right)^{-1/2}-1=(1/3)^{1/2}-1$

$\boxed{S=\sqrt{3}-1}$

U.Q

23. Find the sum to infinity series

$$\frac{15}{16} + \frac{15 \cdot 21}{16 \cdot 24} + \frac{15 \cdot 21 \cdot 27}{16 \cdot 24 \cdot 32} + \cdots$$

Soln:

W.K.T

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1 \cdot 2}\left(\frac{x}{q}\right)^2 + \cdots$$

Denote the sum by S

$$S = \frac{15}{\underset{2 \cdot 8}{16}} + \frac{15 \cdot 21}{\underset{2 \cdot 8 \quad 3 \cdot 8}{16 \cdot 24}} + \frac{15 \cdot 21 \cdot 27}{\underset{2 \cdot 8 \quad 3 \cdot 8 \quad 4 \cdot 8}{16 \cdot 24 \cdot 32}} + \cdots$$

$$= \frac{5}{2}\left(\frac{3}{8}\right) + \frac{5 \cdot 7}{2 \cdot 3}\left(\frac{3}{8}\right)^2 + \frac{5 \cdot 7 \cdot 9}{2 \cdot 3 \cdot 4}\left(\frac{3}{8}\right)^3 + \cdots$$

multiply by $\frac{3}{1}$ on b.s

$$3S = \frac{3 \cdot 5}{1 \cdot 2}\left(\frac{3}{8}\right) + \frac{3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3}\left(\frac{3}{8}\right)^2 + \frac{3 \cdot 5 \cdot 7 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{3}{8}\right)^3 + \cdots$$

multiplying by $\frac{3}{8}$ on b.s

$$\frac{9S}{8} = \frac{3 \cdot 5}{1 \cdot 2}\left(\frac{3}{8}\right)^2 + \frac{3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3}\left(\frac{3}{8}\right)^3 + \frac{3 \cdot 5 \cdot 7 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{3}{8}\right)^4 + \cdots$$

Add & sub $1 + \frac{3}{1}\left(\frac{3}{8}\right)$ on R.H.S

$$\frac{9S}{8} = \left[1 + \frac{3}{1}\left(\frac{3}{8}\right) + \frac{3 \cdot 5}{1 \cdot 2}\left(\frac{3}{8}\right)^2 + \frac{3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3}\left(\frac{3}{8}\right)^3 + \cdots\right] - \left[1 + \frac{3}{1}\left(\frac{3}{8}\right)\right]$$

$p = 3$, $p + q = 5$; $\frac{x}{q} = \frac{3}{8}$

$q = 5 - 3$; $x = \frac{3}{8} \times 2 = \frac{3}{4}$

$q = 2$

$$\frac{9S}{8} = \left(1 + \frac{3}{4}\right)^{-3/2} - \left[1 + \frac{9}{8}\right]$$

$$= \left(\frac{4-3}{4}\right)^{-3/2} - \frac{8+9}{8}$$

$$= \left(\frac{1}{4}\right)^{-3/2} - \frac{17}{8}$$

$$= 4^{3/2} - \frac{17}{8}$$

$$= 4\sqrt{4} - \frac{17}{8} = 4 \times 2 - \frac{17}{8}$$

$$\frac{9S}{8} = 8 - \frac{17}{8}$$

$$= \left(8 - \frac{17}{8}\right) \times \frac{8}{9}$$

$$= \frac{64}{9} - \frac{17}{8} \times \frac{8}{9}$$

$$= \frac{64}{9} - \frac{17}{9}$$

$$S = \frac{47}{9}$$

24. Sum the series $1 - \frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 8} - \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} + \cdots$

Soln:

Denote the sum by S. Then

$$S = 1 - \frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 8} - \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} + \cdots$$

$$1 \cdot 4 \quad 2 \cdot 4 \qquad 1 \cdot 4 \quad 2 \cdot 4 \quad 3 \cdot 4$$

W.K.T

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1 \cdot 2}\left(\frac{x}{q}\right)^2 + \cdots$$

$$S = 1 + \frac{1}{1}\left(\frac{-1}{4}\right) + \frac{1 \cdot 3}{1 \cdot 2}\left(\frac{-1}{4}\right)^2 + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3}\left(\frac{-1}{4}\right)^3 + \cdots$$

$p = 1$, $p + q = 3$

$1 + q = 3$

$q = 3 - 1 = 2$

$\frac{x}{q} = \frac{-1}{4}$

$x = \frac{-q}{4} = \frac{-2}{4} = -\frac{1}{2}$

$S = (1-x)^{-p/q}$

$= (1+\frac{1}{2})^{-1/2}$

$= \left(\frac{2+1}{2}\right)^{-1/2} = (3/2)^{-1/2}$

$$\boxed{S = \left(\frac{2}{3}\right)^{1/2}}$$

25. Sum the series $\frac{1\cdot4}{5\cdot10} - \frac{1\cdot4\cdot7}{5\cdot10\cdot15} + \frac{1\cdot4\cdot7\cdot10}{5\cdot10\cdot15\cdot20} - \cdots$

Soln:

Denote the sum by S, then

$$S = \frac{1\cdot4}{5\cdot10} - \frac{1\cdot4\cdot7}{5\cdot10\cdot15} + \frac{1\cdot4\cdot7\cdot10}{5\cdot10\cdot15\cdot20} - \cdots$$

$1.5 \quad 2.5 \quad 1.5\ 2.5\ 3.5 \quad 1.5\ 2.5\ 3.5\ 4.5$

W.K.T

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2 + \cdots$$

$$S = \frac{1\cdot4}{1\cdot2}\left(\frac{-1}{5}\right)^2 + \frac{1\cdot4\cdot7}{1\cdot2\cdot3}\left(\frac{-1}{5}\right)^3 + \frac{1\cdot4\cdot7\cdot10}{1\cdot2\cdot3\cdot4}\left(\frac{-1}{5}\right)^4 + \cdots$$

Add & sub $1 + \frac{1}{1}(-\frac{1}{5})$ in R.H.S

$$S = \left[1 + \frac{1}{1}\left(\frac{-1}{5}\right) + \frac{1\cdot4}{1\cdot2}\left(\frac{-1}{5}\right)^2 + \frac{1\cdot4\cdot7}{1\cdot2\cdot3}\left(\frac{-1}{5}\right)^3 + \cdots\right] - \left[1 + \frac{1}{1}\left(\frac{-1}{5}\right)\right]$$

$p = 1, \quad p+q = 4, \quad \frac{x}{q} = \frac{-1}{5}$

$q = 4-1, \quad x = -3/5$

$q = 3$

$$S = (1-x)^{-p/q} - \left(1 - \frac{1}{5}\right)$$

$$= \left(1+\frac{3}{5}\right)^{-1/3} - \frac{4}{5}$$

$$= \left(\frac{5+3}{5}\right)^{-1/3} - \frac{4}{5}$$

$$= \left(\frac{8}{5}\right)^{-1/3} - 4/5$$

$$\boxed{S = \left(\frac{5}{8}\right)^{1/3} - 4/5}$$

26. Sum the series

$$1-\frac{1}{2}(\tfrac{1}{2})+\frac{1\cdot3}{2\cdot4}(\tfrac{1}{2})^2-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}(\tfrac{1}{2})^3+\cdots$$

Soln:

Denote the sum by S. Then

$$S=1-\frac{1}{2}(\tfrac{1}{2})+\frac{1\cdot3}{2\cdot4}(\tfrac{1}{2})^2-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}(\tfrac{1}{2})^3+\cdots$$

$$S=1+\frac{1}{1}(-\tfrac{1}{4})+\frac{1\cdot3}{1\cdot2}(-\tfrac{1}{4})^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}(-\tfrac{1}{4})^3+\cdots$$

$p=1$, $p+q=3$

$q=3-1$

$q=2$

$\frac{x}{q}=-\frac{1}{4}$

$x=-\frac{2}{4\cdot2}=-\frac{1}{2}$

$S=(1-x)^{-p/q}$

$=(1+\frac{1}{2})^{-1/2}$

$=(\frac{2+1}{2})^{-1/2}$

$=(3/2)^{-1/2}$

$S=(\frac{2}{3})^{1/2}$

EXPONENTIAL SERIES

Results:

For all values of x

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

2. $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

3. $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

4. $\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

5. $e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

6. $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$

7. $\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots$

8. $\frac{e - e^{-1}}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots$

Problems:

1. Sum the series $\frac{1+3x}{1!} + \frac{(1+3x)^2}{2!} + \frac{(1+3x)^3}{3!} + \cdots$

Soln:

Let $y = 1 + 3x$

$$S = \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

Add and Sub 1

$$S = \left[1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots\right] - 1$$

$S = e^y - 1$

$$\boxed{S = e^{1+3x} - 1}$$

2. Sum the series $1 - \log_e 2 + \frac{(\log e^2)^2}{2!} - \frac{(\log e^2)^3}{3!} + \cdots$

V.Q

Soln:

Let $x = \log_e 2$

$$S = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$= e^{-x}$$

$$= e^{-\log_e 2} \qquad e^{\log x} = x$$

$$= e^{\log_e 2^{-1}}$$

$$= e^{\log_e 2^{-1}}$$

$$= 2^{-1}$$

$$S = \frac{1}{2}$$

3. S.T $\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$

V.Q

Soln:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$$

$$e - e^{-1} = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots\right)$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots - 1 + \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \cdots$$

$$= 2\frac{1}{1!} + 2\frac{1}{3!} + 2\frac{1}{5!} + \cdots$$

$= 2\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right)$

$$\frac{e - e^{-1}}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

$$\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

4. S.T $\dfrac{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots} = \dfrac{e^2+1}{e^2-1}$

U.Q

Soln:-

L.H.S

$$\frac{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots} = \frac{\frac{e + e^{-1}}{2}}{\frac{e - e^{-1}}{2}}$$

$$= \frac{e + e^{-1}}{2} \times \frac{2}{e - e^{-1}}$$

$$= \frac{e + e^{-1}}{e - e^{-1}} = \frac{e + \frac{1}{e}}{e - \frac{1}{e}}$$

$$= \frac{\frac{e^2+1}{e}}{\frac{e^2-1}{e}} = \frac{e^2+1}{e} \times \frac{e}{e^2-1}$$

$$= \frac{e^2+1}{e^2-1} = \text{R.H.S}$$

L.H.S = R.H.S

5. S.T $\dfrac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots} = \dfrac{e-1}{e+1}$

U.D

Soln:

W.K.T

$$\frac{e+e^{-1}}{2} = 1+\frac{1}{2!}+\frac{1}{4!}+\cdots$$

$$\frac{e-e^{-1}}{2} = \frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots$$

L.H.S

$$= \frac{\left[\frac{e+e^{-1}}{2}\right]-1}{\left[\frac{e-e^{-1}}{2}\right]}$$

$$= \frac{\left[\frac{e+\frac{1}{e}}{2}\right]-1}{\frac{e-\frac{1}{e}}{2}} = \frac{\frac{e^2+1}{2e}-1}{\frac{e^2-1}{2e}} = \frac{e^2+1-2e}{2e} \times \frac{2e}{e^2-1}$$

$(a-b)^2 = a^2+b^2-2ab$

$$= \frac{(e-1)^2}{e^2-1}$$

$a^2-b^2 = (a+b)(a-b)$

$$= \frac{(e-1)^2}{(e+1)(e-1)}$$

$$= \frac{e-1}{e+1}$$

6. S.T $\dfrac{1+\frac{1}{2!}+\frac{2}{3!}+\frac{2^2}{4!}+\cdots}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots} = e/2.$

Soln:

L.H.S

$$= \frac{1 + \frac{1}{2!} + \frac{2}{3!} + \frac{2^2}{4!} + \cdots}{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots}$$

Multiply and divided by 2^2 on the numerator

$$= \frac{\frac{1}{2^2}\left[2^2 + \frac{2^2}{1!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \cdots\right]}{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots}$$

Write the first term $\frac{2^2}{1!}$ as $(1 + \frac{2}{1!}) + 1$.

$$= \frac{\frac{1}{2^2}\left[1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots\right] + 1}{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots}$$

$$= \frac{1}{2^2} \frac{(e^2 + 1)}{\frac{1}{2}(e + e^{-1})}$$

$$= \frac{1}{2^2}\left(\frac{e^2+1}{e+\frac{1}{e}}\right) = \frac{1}{2}\left(\frac{e^2+1}{\frac{e^2+1}{e}}\right)$$

$$= \frac{e^2+1}{2} \times \frac{e}{e^2+1}$$

$$= \frac{e}{2} = \text{R.H.S}$$

L.H.S = R.H.S

H.W 7. S.T $1+\frac{x\log_e a}{1!}+\frac{(x\log_e a)^2}{2!}+\frac{(x\log_e a)^3}{3!}+\cdots = a^x$

Soln:

$$S = 1+\frac{x\log_e a}{1!}+\frac{(x\log_e a)^2}{2!}+\frac{(x\log_e a)^3}{3!}+\cdots$$

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$S = e^{x\log_e a}$$

$$= e^{\log_e a^x}$$

$$S = a^x$$

8. S.T $\log_e 2+\frac{(\log_e 2)^2}{2!}+\frac{(\log_e 2)^3}{3!}+\cdots = 1$

Soln:

$$S = \log_e 2+\frac{(\log_e 2)^2}{2!}+\frac{(\log_e 2)^3}{3!}+\cdots$$

Add & Sub 1 on R.H.S

$$S = \left[1+\log_e 2+\frac{(\log_e 2)^2}{2!}+\frac{(\log_e 2)^3}{3!}+\cdots\right]-1$$

Let $x = \log_e 2$

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots$$

$$S = e^x - 1$$

$$= e^{\log_e 2} - 1$$

$$= 2-1$$

$$\boxed{S = 1}$$

9. S.T $\log_e 3 - \frac{(\log_e 3)^2}{2!} + \frac{(\log_e 3)^3}{3!} + \cdots = \frac{2}{3}$

Soln:

$$S = \log_e 3 - \frac{(\log_e 3)^2}{2!} + \frac{(\log_e 3)^3}{3!} - \cdots$$

Multiply by -1 on both sides

$$-S = -\log_e 3 + \frac{(\log_e 3)^2}{2!} - \frac{(\log_e 3)^3}{3!} + \cdots$$

Add & sub by 1 on R.H.S

$$-S = [1 - \log_e 3 + \frac{(\log_e 3)^2}{2!} - \frac{(\log_e 3)^3}{3!} + \cdots] - 1$$

Let $x = \log_e 3$

$$-S = \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right] - 1$$

$$= e^{-x} - 1 \qquad \text{W.K.T } e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$= e^{-\log_e 3} - 1$$

$$= e^{\log_e 3^{-1}} - 1$$

$$= 3^{-1} - 1$$

$$= \frac{1}{3} - 1$$

$$= \frac{1-3}{3}$$

$$-S = -\frac{2}{3}$$

$$\boxed{S = \frac{2}{3}}$$

10. S.T $1+\frac{(\log_e n)^2}{2!}+\frac{(\log_e n)^4}{4!}+\cdots=\frac{n^2+1}{2n}$

Soln:

$$S=1+\frac{(\log_e n)^2}{2!}+\frac{(\log_e n)^4}{4!}+\cdots$$

Let $x=\log_e n$, then

$$S=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$$

Here

$$\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$$

$$S=\frac{e^x+e^{-x}}{2}$$

$$=\frac{e^{\log_e n}+\frac{1}{e^{\log_e n}}}{2}=\frac{n+\frac{1}{n}}{2}$$

$$=\frac{n^2+1}{2}$$

11. Sum the series $1+\frac{3^2}{2!}+\frac{3^4}{4!}+\frac{3^6}{6!}+\cdots$

Soln:

$$S=1+\frac{3^2}{2!}+\frac{3^4}{4!}+\frac{3^6}{6!}+\cdots$$

put $x=3$

$$S=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots$$

W.K.T

$$\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots$$

$$S=\frac{e^x+e^{-x}}{2}$$

$$= \frac{e^x + \frac{1}{e^x}}{2} = \frac{e^3 + \frac{1}{e^3}}{2}$$

$$= \frac{e^6 + 1}{2e^3}$$

12. Find the Co-eff of x^n in e^{ax+b}

Soln:

$$e^{ax+b} = e^{ax} \cdot e^b \qquad e^{m+n} = e^m \cdot e^n$$

$$= e^b \left[1 + \frac{ax}{1!} + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \cdots + \frac{(ax)^n}{n!} + \cdots \right]$$

$$= e^b \left[1 + \frac{ax}{1!} + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \cdots + \frac{a^nx^n}{n!} + \cdots \right]$$

Co-eff of $x^n = e^b \left(\frac{a^n}{n!} \right)$

Co-efficient of x^n in exponential series

1. Find the Co-eff of x^n in $\frac{1+2x+3x^2}{e^x}$

Soln:

$$\frac{1+2x+3x^2}{e^x} = (1+2x+3x^2)\, e^{-x}$$

W.K.T

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$= (1+2x+3x^2) \left[1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \right]$$

$$= \left[1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \frac{(-1)^n x^n}{n!} + \cdots \right]$$

$$+ 2x \left[1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \frac{(-1)^n x^n}{n!} + \cdots \right]$$

$$+3x^2\left[1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\frac{(-1)^n x^n}{n!}\right]$$

$$=1-\frac{x}{1!}+\frac{x^2}{2!}+\cdots\frac{(-1)^n x^n}{n!}+\cdots+2x-\frac{2x^2}{1!}+\cdots$$

$$\frac{(-1)^n 2^{n+1}}{n!}+\cdots$$

$$+3x^2-\frac{3x^3}{1!}+\frac{3x^4}{2!}-\frac{3x^5}{3!}+\cdots+\frac{(-1)^n 3x^{n+2}}{n!}+\cdots$$

$$=1-\frac{x}{1!}+\frac{x^2}{2!}+\cdots\frac{(-1)^n x^n}{n!}+\cdots+2x-\frac{2x^2}{1!}+\cdots\frac{(-1)^{n-1}2^n x^n}{(n-1)!}+\frac{(-1)^{n+1}2x}{n!}$$

$$+\cdots+3x^2-\frac{3x^3}{1!}+\frac{3x^4}{2!}+\cdots\frac{(-1)^{n-2}}{(n-2)!}3x^n+\frac{(-1)^{n-1}}{(n-1)!}3x^{n+1}+\cdots$$

$$\therefore \text{Coeff of } x^n=\frac{(-1)^n}{n!}+\frac{2(-1)^{n-1}}{(n-1)!}+\frac{3(-1)^{n-2}}{(n-2)!}$$

$$=(-1)^n\left[\frac{1}{n!}\right]+\frac{2(-1)^n(-1)^{-1}}{(n-1)!}+\frac{3(-1)^n(-1)^{-2}}{(n-2)!}$$

$$=(-1)^n\left[\frac{1}{n!}-\frac{2}{(n-1)!}+\frac{3}{(n-2)!}\right]$$

$$=(-1)^n\left[\frac{1}{n!}-\frac{2n}{n(n-1)!}+\frac{3n(n-1)}{n(n-1)(n-2)!}\right]$$

$$=\frac{(-1)^n}{n!}\left[1-2n+3n(n-1)\right]$$

$$=\frac{(-1)^n}{n!}\left[1-2n+3n^2-3n\right]$$

$$=\frac{(-1)^n}{n!}\left[3n^2-5n+1\right]$$

2 Find the co-eff of x^n in $\frac{2+5x}{e^{2x}}$

Soln:

$$\frac{2+5x}{e^{2x}} = (2+5x)e^{-2x}$$

W.K.T

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

put $x = 2x$

$$e^{-2x} = 1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \cdots + (-1)^n \frac{(2x)^n}{n!} + \cdots$$

$$\frac{2+5x}{e^{2x}} = (2+5x)\left[1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots + (-1)^n \frac{(2x)^n}{n!} + \cdots\right]$$

$$= \left[2 - \frac{2\cdot 2x}{1!} + \frac{2\cdot 2^2 x^2}{2!} + \cdots + 2\cdot\frac{(-1)^n 2^n x^n}{n!} + \cdots\right]$$

$$+ \left[5x - \frac{5\cdot 2x^2}{1!} + \frac{5\cdot 2^2 x^3}{3!} + \cdots + 5\cdot\frac{(-1)^n 2^n x^{n+1}}{n!} + \cdots\right]$$

$$= \left[2 - \frac{2\cdot 2x}{1!} + \frac{2\cdot 2^2 x^2}{2!} + \cdots + 2\cdot\frac{(-1)^n x^n}{n!} + \cdots\right]$$

$$+ \left[5x - \frac{5\cdot 2x^2}{1!} + \cdots + 5\frac{(-1)^{n-1} 2^{n-1} x^n}{(n-1)!} + 5\frac{(-1)^n 2^n x^{n+1}}{n!} + \cdots\right]$$

Co-eff of x^n $= \frac{2^{n+1}(-1)^n}{n!} + 5\frac{(-1)^{n-1}2^{n-1}}{(n-1)!}$

$$= \frac{2\cdot 2^n (-1)^n}{n!} + 5\frac{(-1)^n (-1)^{-1} 2^n \cdot 2^{-1}}{(n-1)!}$$

$$= 2^n (-1)^n \left[\frac{2}{n!} - \frac{5}{2(n-1)!}\right]$$

$$= 2^n(-1)^n\left[\frac{2}{n!} - \frac{5n}{2n(n-1)!}\right]$$

$$= \frac{2^n(-1)^n}{n!}\left[2 - \frac{5n}{2}\right]$$

$$= \frac{2^n(-1)^n}{n!}\left[\frac{4-5n}{2}\right]$$

$$= \frac{2^{n-1}(-1)^n(4-5n)}{n!}$$

3. Find the co-eff of x^n in the series

$$1 + \frac{3-2x}{1!} + \frac{(3-2x)^2}{2!} + \frac{(3-2x)^3}{3!} + \cdots$$

Soln:

$$1 + \frac{3-2x}{1!} + \frac{(3-2x)^2}{2!} + \frac{(3-2x)^3}{3!} + \cdots$$

W.K.T

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Put $x = 3-2x$

$$e^{3-2x} = e^3 \cdot e^{-2x}$$

$$= e^3\left[1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots + (-1)^n\frac{(2x)^n}{n!} + \cdots\right]$$

Co-eff of $x^n = \frac{e^3(-1)^n 2^n}{n!}$

H.W

4. Find the co-eff of x^n in e^{2x-3}

Soln:

$$e^{2x-3} = e^{2x} \cdot e^{-3}$$

$$= e^{-3}\left[1 + \frac{(2x)}{1!} + \frac{(2x)^2}{2!} + \cdots + \frac{(2x)^n}{n!} + \cdots\right]$$

Co-eff of $x^n = e^{-3}\frac{2^n}{n!}$

5. Find the co-eff of x^n in e^{e^x}

Soln.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$e^{e^x} = 1 + \frac{e^x}{1!} + \frac{(e^x)^2}{2!} + \frac{(e^x)^3}{3!} + \cdots$$

$$e^{e^x} = 1 + \frac{e^x}{1!} + \frac{e^{2x}}{2!} + \frac{e^{3x}}{3!} + \cdots$$

$$= 1 + \frac{1}{1!}\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right] + \frac{1}{2!}\left[1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots + \frac{(2x)^n}{n!} + \cdots\right] + \cdots$$

Co-eff of $x^n = \frac{1}{1!}\frac{1}{n!} + \frac{1}{2!}\frac{2^n}{n!} + \frac{1}{3!}\frac{3^n}{n!} + \cdots$

$$= \frac{1}{n!}\left[\frac{1^n}{1!} + \frac{2^n}{2!} + \frac{3^n}{3!} + \cdots\right]$$

6. Find the co-eff of x^n in the expansion of $\frac{1+x+x^2}{e^x}$ in ascending powers of x.

Soln:

$$\frac{(1+x+x^2)}{e^x} = (1+x+x^2)e^{-x}$$

$$(1+x+x^2)e^x = (1+x+x^2)\left(1 - \frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{(-1)^n x^n}{n!} + \cdots\right)$$

$$= \left(1 - \frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{(-1)^n x^n}{n!} + \cdots\right) + x\left(1 - \frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{(-1)^n x^n}{n!} + \cdots\right)$$

$$+ x^2\left[1 - \frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{(-1)^n x^n}{n!} + \cdots\right]$$

$$= 1 - \frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{(-1)^n x^n}{n!} + \cdots + \left(x - \frac{x^2}{1!} + \frac{x^3}{2!} + \cdots \frac{(-1)^n x^{n+1}}{n!} + \cdots\right)$$

$$+ \left(x^2 - \frac{x^3}{1!} + \frac{x^4}{2!} + \cdots + \frac{(-1)^n x^{n+2}}{n!} + \cdots\right)$$

$$= 1 - \frac{x}{1!} + \frac{x^2}{2!} + \cdots \frac{(-1)^n x^n}{n!} + \cdots + x - \frac{x^2}{1!} + \frac{x^3}{2!} + \cdots + \frac{(-1)^{n-1} x^n}{(n-1)!} + \frac{(-1)^n x^{n+1}}{n!} + \cdots$$

$$+ x^2 - \frac{x^3}{1!} + \cdots + \frac{(-1)^{n-2} x^n}{(n-2)!} + \frac{(-1)^{n-1} x^{n+1}}{(n-1)!} + \frac{(-1)^n x^{n+2}}{n!} + \cdots$$

Co-eff of $x^n = \frac{(-1)^n}{n!} + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^{n-2}}{(n-2)!}$

$$= \frac{(-1)^n}{n!} + \frac{(-1)^n (-1)^{-1}}{(n-1)!} + \frac{(-1)^n (-1)^{-2}}{(n-2)!}$$

$$= (-1)^n \left[\frac{1}{n!} - \frac{n}{n(n-1)!} + \frac{n(n-1)}{n(n-1)(n-2)!}\right]$$

$$= \frac{(-1)^n}{n!}\left[1 - n + n(n-1)\right]$$

$$= \frac{(-1)^n}{n!}\left[1 - n + n^2 - n\right]$$

$$= \frac{(-1)^n}{n!}\left[n^2 - 2n + 1\right] = \frac{(-1)^n}{n!}(n-1)^2$$

7. Find the Co-eff of x^3 in $\frac{1+x-x^2}{e^{2x}}$

Soln:

$$\frac{1+x-x^2}{e^{2x}} = (1+x-x^2)\,e^{-2x}$$

$$e^{-2x} = 1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots$$

$$(1+x-x^2)\,e^{-2x} = (1+x-x^2)\left(1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots + \frac{(-1)^n(2x)^n}{n!} + \cdots\right]$$

$$= \left[1 - \frac{2x}{1!} + \frac{2^2x^2}{2!} + \cdots \frac{(-1)^n 2^n x^n}{n!} + \cdots\right] + x\left[1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots \frac{(-1)^n(2x)^n}{n!} + \cdots\right]$$

$$-x^2\left[1 - \frac{2x}{1!} + \frac{2^2x^2}{2!} + \cdots \frac{(-1)^n 2^n x^n}{n!} + \cdots\right]$$

$$= \left[1 - \frac{2x}{1!} + \frac{2^2x^2}{2!} - \frac{2^3x^3}{3!} + \cdots\right] + \left[x - \frac{2x^2}{1!} + \frac{2^2x^3}{2!} + \cdots\right]$$

$$- x^2 + \frac{2x^3}{1!} - \frac{2^2x^4}{2!} + \cdots$$

Co-eff of $x^3 = -\frac{2^3}{3!} + \frac{2^2}{2!} + \frac{2}{1}$

$$= -\frac{8}{6} + \frac{4}{2} + \frac{2}{1}$$

$$= -\frac{4}{3} + 2 + 2$$

$$= -\frac{4}{3} + 4$$

$$= \frac{-4+12}{3}$$

$$= \frac{8}{3}$$

SUMMATION: Exponential series

1. Sum the series $\frac{2^3}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \frac{5^2}{4!} + \cdots$

Soln:

Let T_n be the n^{th} term

$$\therefore T_n = \frac{(n+1)^2}{n!}$$

Degree of numerator is 2.

Let $(n+1)^2 = A + Bn + Cn(n-1)$ ——— (1)

$n^2 + 2n + 1 = A + Bn + Cn(n-1)$

Equate the co-eff of n^2

$$\boxed{1 = C}$$

Equate the co-eff of n

$$2 = B - C$$

$$2 = B - 1$$

$$\boxed{B = 3}$$

Equate the constant term

$$\boxed{1 = A}$$

Sub A, B, C value in (1)

$$(n+1)^2 = 1 + 3n + n(n-1)$$

$$T_n = \frac{(n+1)^2}{n!}$$

$$= \frac{1 + 3n + n(n-1)}{n!}$$

$$= \frac{1}{n!} + \frac{3n}{n!} + \frac{n(n-1)}{n!}$$

$$= \frac{1}{n!} + \frac{3n}{n(n-1)!} + \frac{n(n-1)}{n(n-1)(n-2)!}$$

$$t_n = \frac{1}{n!} + \frac{3}{(n-1)!} + \frac{1}{(n-2)!}$$

Put $n = 1, 2, 3 \ldots$

$$t_1 = \frac{1}{1!} + \frac{3}{0!}$$

$$t_2 = \frac{1}{2!} + \frac{3}{1!} + \frac{1}{0!}$$

$$t_3 = \frac{1}{3!} + \frac{3}{2!} + \frac{1}{1!}$$

$$t_4 = \frac{1}{4!} + \frac{3}{3!} + \frac{1}{2!}$$

. . . .

Adding vertically

$$S = \left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right] + 3\left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right] + \left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right]$$

$$= (e^1 - 1) + 3e^1 + e^1$$

$$= e - 1 + 3e + e$$

$$\boxed{S = 5e - 1}$$

2. Sum the series $\sum_{n=0}^{\infty} \frac{5n+1}{(2n+1)!}$

Soln:

Let $t_n = \frac{5n+1}{(2n+1)!}$, $n = 0, 1, 2, 3 \ldots$

Degree of numerator is 1.

$$5n+1 = A + B(2n+1) \quad \text{— (1)}$$

Equate the co-eff of n

$5 = 2B$

$\boxed{B = 5/2}$

Equate the constant term

$1 = A + B$

$1 = A + 5/2$

$A = 1 - 5/2$

$A = 2 - 5/2 = -3/2$

$\boxed{A = -3/2}$

Sub A, B value in ①

$$5n+1 = \frac{-3}{2} + \frac{5}{2}(2n+1)$$

$$\frac{5n+1}{(2n+1)!} = \frac{-\frac{3}{2} + \frac{5}{2}(2n+1)}{(2n+1)!}$$

$$= \frac{-\frac{3}{2}}{(2n+1)!} + \frac{5/2\,(2n+1)}{(2n+1)!}$$

$$= \frac{-\frac{3}{2}}{(2n+1)!} + \frac{5/2\,(2n+1)}{2(2n+1)n!}$$

$$t_n = \frac{-3/2}{(2n+1)!} + \frac{5/2}{2n!}$$

Put $n = 0, 1, 2, 3 \ldots$

$t_0 = \frac{-3/2}{1!} + \frac{5/2}{0!}$

$t_1 = \frac{-3/2}{3!} + \frac{5/2}{2!}$

$t_2 = \frac{-3/2}{5!} + \frac{5/2}{4!}$

$t_3 = \frac{-3/2}{7!} + \frac{5/2}{6!}$

- - - -

Adding vertically

$$S = \left[\frac{-3}{2} + \frac{-3/2}{3!} + \frac{-3/2}{5!} + \cdots\right] + \left[\frac{5/2}{0!} + \frac{5/2}{2!} + \frac{5/2}{4!} + \cdots\right]$$

$$= -\frac{3}{2}\left[\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right] + \frac{5}{2}\left[\frac{1}{0!} + \frac{1}{2!} + \frac{1}{4!} + \cdots\right]$$

$$= -\frac{3}{2}\left[\frac{e - e^{-1}}{2}\right] + \frac{5}{2}\left[\frac{e + e^{-1}}{2}\right]$$

$$= -\frac{3}{2}\left[\frac{e - 1/e}{2}\right] + \frac{5}{2}\left[\frac{e + 1/e}{2}\right] = -\frac{3}{2}\left[\frac{e^2 - 1}{2e}\right] + \frac{5}{2}\left[\frac{e^2 + 1}{2e}\right]$$

$$= \frac{1}{4e}\left[-3e^2 + 3 + 5e^2 + 5\right]$$

$$= \frac{1}{4e}\left[2e^2 + 8\right] = \frac{2(e^2 + 4)}{4e} = \frac{e^2 + 4}{2e}$$

$$S = \frac{e^1}{2e} + \frac{4^2}{2e}$$

$$\boxed{S = \frac{e}{2} + \frac{2}{e}}$$

3. P.T $\frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + \cdots = 2e - 1$

Soln:

$$t_n = \frac{n+1}{n!} \quad - (1)$$

Degree of numerator is 1

$$n + 1 = A + Bn$$

Put $n = 0$

$$\boxed{1 = A}$$

Equate the co-eff of n

$$\boxed{1 = B}$$

$$\Rightarrow t_n = \frac{n+1}{n!} = \frac{A + Bn}{n!}$$

$$= \frac{1+n}{n!}$$

$$= \frac{1}{n!} + \frac{n(1)}{n(n-1)!}$$

$$t_n = \frac{1}{n!} + \frac{1}{(n-1)!}$$

Put $n = 1, 2, 3 \cdots$

$$t_1 = \frac{1}{1!} + \frac{1}{0!}$$

$$t_2 = \frac{1}{2!} + \frac{1}{1!}$$

$$t_3 = \frac{1}{3!} + \frac{1}{2!}$$

Adding vertically,

$$S = \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) + \left(\frac{1}{1!} + \frac{1}{2!} + \cdots\right)$$

$$= (e^1 - 1) + e^1 = e + e - 1 = 2e - 1$$

$$\boxed{S = 2e - 1}$$

4. Sum the series $\frac{1^2}{3!} + \frac{2^2}{5!} + \frac{3^2}{7!} + \cdots$

Soln:

Let t_n be the n^{th} term

$$t_n = \frac{n^2}{(2n+1)!}$$

Let $n^2 = A + B(2n+1) + C\,2n(2n+1)$

$4n^2 + 2n$

Equate the coeff of n^2

$$1 = 4C$$

$$\boxed{C = 1/4}$$

Equate the coeff of n

$$0 = 2B + 2C$$

$$0 = 2B + 2(1/4)$$

$2B + \frac{1}{2} = 0$

$2B = -\frac{1}{2}$

$\boxed{B = -\frac{1}{4}}$

Equate the coeff of constant

$0 = A + B$

$0 = A - \frac{1}{4}$

$\boxed{A = \frac{1}{4}}$

$$E_n = \frac{n^2}{(2n+1)!}$$

$$= \frac{\frac{1}{4} - \frac{1}{4}(2n+1) + \frac{1}{4}\,2n(2n+1)}{(2n+1)!}$$

$$= \frac{\frac{1}{4}}{(2n+1)!} - \frac{\frac{1}{4}(2n+1)}{(2n+1)2n!} + \frac{\frac{1}{4}\,2n(2n+1)}{(2n+1)\,2n(2n-1)!}$$

$$E_n = \frac{\frac{1}{4}}{(2n+1)!} - \frac{\frac{1}{4}}{2n!} + \frac{\frac{1}{4}}{(2n-1)!}$$

Put $n = 1, 2, 3 \ldots$

$$E_1 = \frac{\frac{1}{4}}{3!} - \frac{\frac{1}{4}}{2!} + \frac{\frac{1}{4}}{0!}$$

$$E_2 = \frac{\frac{1}{4}}{5!} - \frac{\frac{1}{4}}{4!} + \frac{\frac{1}{4}}{3!}$$

$$E_3 = \frac{\frac{1}{4}}{7!} - \frac{\frac{1}{4}}{6!} + \frac{\frac{1}{4}}{5!}$$

$\ldots$

Adding vertically

$$S = \frac{1}{4}\left[\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots\right] - \frac{1}{4}\left[\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\right]$$

$$+\frac{1}{4}\left[\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots\right]$$

$$= \frac{1}{4}\left[\frac{e-e^{-1}}{2} - \frac{1}{1!}\right] - \frac{1}{4}\left[\frac{e+e^{-1}}{2} - 1\right] + \frac{1}{4}\left[\frac{e-e^{-1}}{2}\right]$$

$$= \frac{1}{4}\left\{\frac{e-\frac{1}{e}}{2} - 1 - \left[\left(\frac{e+\frac{1}{e}}{2}\right) - 1\right] + \frac{e-\frac{1}{e}}{2}\right\}$$

$$= \frac{1}{4}\left\{\frac{e^2-1}{2e} - 1 - \left[\left(\frac{e^2+1}{2e}\right) - 1\right] + \frac{e^2-1}{2e}\right\}$$

$$= \frac{1}{8}\left\{\frac{e^2-1}{e} - 1 - \frac{e^2-1}{2e} + 1 + \frac{e^2-1}{e}\right\}$$

$$= \frac{1}{8e}\left(e^2 - 1 - e^2 - 1 + e^2 - 1\right)$$

$$= \frac{1}{8e}\left(e^2 - 3\right)$$

$$= \frac{e^2}{8e} - \frac{3}{8e}$$

$$\boxed{S = \frac{e - 3e^{-1}}{8}}$$

5. Sum the series $\frac{1}{1!} + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots \infty$

Soln:

Let $T_n = \frac{2n-1}{n!}$

Numerator degree is 1

$2n-1 = A + Bn$

Equate the co-eff of n

$$\boxed{2 = B}$$

Equate the constant term

$$\boxed{-1 = A}$$

$$t_n = \frac{2n-1}{n!} = \frac{A+Bn}{n!}$$

$$= \frac{-1+2n}{n!}$$

$$= \frac{-1}{n!} + \frac{2n}{n(n-1)!}$$

$$t_n = \frac{-1}{n!} + \frac{2}{(n-1)!}$$

Put $n = 1, 2, 3 \ldots$

$$t_1 = \frac{-1}{1!} + \frac{2}{0!}$$

$$t_2 = \frac{-1}{2!} + \frac{2}{1!}$$

$$t_3 = \frac{-1}{3!} + \frac{2}{2!}$$

.....

Adding vertically

$$S = \left[\frac{-1}{1!} - \frac{1}{2!} - \frac{1}{3!} - \cdots\right] + \left[\frac{2}{0!} + \frac{2}{1!} + \frac{2}{2!} + \cdots\right]$$

$$= -1\left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right] + 2\left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\right]$$

$$= -1(e-1) + 2e$$

$$= -e + 1 + 2e$$

$$\boxed{S = e+1}$$

6. S.T $\frac{1}{1!} + \frac{1+2}{2!} + \frac{1+2+3}{3!} + \frac{1+2+3+4}{4!} + \cdots = \frac{3e}{2}$

Soln:

Let T_n be the n^{th} term

$$T_n = \frac{1+2+3+\cdots+n}{n!}, \quad n = 1, 2, 3 \ldots$$

$$T_n = \frac{n(n+1)}{2} \cdot \frac{1}{n!}$$

$$= \frac{n(n+1)}{2n(n-1)!} = \frac{n+1}{2(n-1)!}$$

Numerator degree is one

$$n+1 = A + B\underbrace{2(n-1)}_{2Bn-2B}$$

Equate the co-eff of n

$$1 = 2B$$

$$\boxed{B = \frac{1}{2}}$$

Equate the co-eff of constant

$$1 = A - 2B$$

$$1 = A - 2\left(\frac{1}{2}\right)$$

$$1 = A - 1$$

$$\boxed{A = 2}$$

$$T_n = \frac{n+1}{2(n-1)!} = \frac{2 + \frac{1}{2}2(n-1)}{2(n-1)!}$$

$$= \frac{2 + n - 1}{2(n-1)!} = \frac{2}{2(n-1)!} + \frac{n-1}{2(n-1)!}$$

$$= \frac{1}{(n-1)!} + \frac{nA}{2(n-1)(n-2)!}$$

$$T_n = \frac{1}{(n-1)!} + \frac{1}{2(n-2)!}$$

put $n = 1, 2, 3, \ldots$

$$T_1 = \frac{1}{0!}$$

$$T_2 = \frac{1}{1!} + \frac{1}{0!}\left(\tfrac{1}{2}\right)$$

$$T_3 = \frac{1}{2!} + \frac{1}{1!}\left(\tfrac{1}{2}\right)$$

$$T_4 = \frac{1}{3!} + \frac{1}{2!}\cdot\tfrac{1}{2}$$

. . . .

Adding vertically

$$S = \left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right] + \frac{1}{2}\left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right]$$

$$= e + \tfrac{1}{2}e$$

$$= e\left(1 + \tfrac{1}{2}\right)$$

$$= e\left(\frac{2+1}{2}\right) = \frac{3e}{2}$$

$$\boxed{S = \frac{3e}{2}}$$

T.S.T $1 + \dfrac{1+3}{2!} + \dfrac{1+3+3^2}{3!} + \dfrac{1+3+3^2+3^3}{4!} + \cdots = \dfrac{e(e^2-1)}{2}$

Soln:

Let T_n be the n^{th} term

$$T_n = \frac{1+3+3^2+\cdots+3^{n-1}}{n!}$$

$$\boxed{1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x-1}}$$

$$\frac{3^n - 1}{3-1} = \frac{3^n - 1}{2}$$

$$\therefore T_n = \frac{3^n - 1}{(3-1)n!} = \frac{3^n - 1}{2n!}$$

$$T_n = \frac{3^n}{2n!} - \frac{1}{2n!}$$

$$T_n = \frac{1}{2}\frac{3^n}{n!} - \frac{1}{2}\cdot\frac{1}{n!}$$

put $n = 1, 2, 3 \ldots$

$$T_1 = \frac{1}{2}, \frac{3}{1!} - \frac{1}{2}, \frac{1}{1!}$$

$$T_2 = \frac{1}{2}\frac{3^2}{2!} - \frac{1}{2}\frac{1}{2!}$$

$$T_3 = \frac{1}{2}\frac{3^3}{3!} - \frac{1}{2}\frac{1}{3!}$$

. . .

Adding vertically

$$S = \frac{1}{2}\left[\frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \cdots\right] - \frac{1}{2}\left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right]$$

$$= \frac{1}{2}\left[e^3 - 1\right] - \frac{1}{2}\left[e - 1\right]$$

$$= \frac{1}{2}\left[e^3 - e - 1 + 1\right]$$

$$= \frac{1}{2}(e^3 - e)$$

$$S = \frac{e(e^2 - 1)}{2}$$

HW 8. Sum the series $\frac{1}{1!} + \frac{1+5}{2!} + \frac{1+5+5^2}{3!} + \cdots$

Soln.

Let T_n be the n^{th} term

$$T_n = \frac{1 + 5 + 5^2 + \cdots + 5^{n-1}}{n!} = \frac{5^n - 1}{(5-1)n!}$$

$$T_n = \frac{5^n - 1}{4n!}$$

$$T_n = \frac{5^n}{4n!} - \frac{1}{4n!}$$

$$T_n = \frac{1}{4}\frac{5^n}{n!} - \frac{1}{4}\frac{1}{n!}$$

Put $n = 1, 2, 3, \ldots$

$$T_1 = \frac{1}{4}\frac{5}{1!} - \frac{1}{4}\frac{1}{1!}$$

$$T_2 = \frac{1}{4}\frac{5^2}{2!} - \frac{1}{4}\frac{1}{2!}$$

$$T_3 = \frac{1}{4}\frac{5^3}{3!} - \frac{1}{4}\frac{1}{3!}$$

$\ldots \quad \ldots \quad \ldots$

Adding vertically

$$S = \frac{1}{4}\left[\frac{5}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \cdots\right] - \frac{1}{4}\left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right]$$

$$= \frac{1}{4}\left[e^5 - 1\right] - \frac{1}{4}\left[e - 1\right]$$

$$= \frac{1}{4}\left[e^5 - 1 - e + 1\right]$$

$$\boxed{S = \frac{e(e^4 - 1)}{4}}$$

9. Sum the series $\frac{1 \cdot 2}{1!} + \frac{2 \cdot 3}{2!} + \frac{3 \cdot 4}{3!} + \cdots$

Soln:

Let T_n be the n^{th} term

$$T_n = \frac{n(n+1)}{n!} = \frac{n^2 + n}{n!}$$

Degree of numerator is 2

$n^2+n = A + Bn + C\,n(n-1)$

$n^2 - n$

Equate the co-eff of n^2

$\boxed{1 = C}$

Equate the co-eff of n

$1 = B - C$

$1 = B - 1$

$\boxed{B = 2}$

Equate the co-eff of constant term

$\boxed{0 = A}$

$n^2+n = 0 + 2n + n(n-1)$

$$\frac{n^2+n}{n!} = \frac{2n + n(n-1)}{n!}$$

$$= \frac{2n}{n(n-1)!} + \frac{n(n-1)}{n(n-1)(n-2)!}$$

$$t_n = \frac{2}{(n-1)!} + \frac{1}{(n-2)!}$$

$$t_n = 2\frac{1}{(n-1)!} + \frac{1}{(n-2)!}$$

Put $n = 1, 2, 3 \ldots$

$$t_1 = 2\frac{1}{0!}$$

$$t_2 = 2\frac{1}{1!} + \frac{1}{0!}$$

$$t_3 = 2\cdot\frac{1}{2!} + \frac{1}{1!}$$

$$t_4 = 2\cdot\frac{1}{3!} + \frac{1}{2!}$$

$\ldots$

Adding vertically

$$S = 2\left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right] + \left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right]$$

$$S = 2e + e$$

$$\boxed{S = 3e}$$

10. Sum the series $\frac{5}{1!} + \frac{7}{3!} + \frac{9}{5!} + \cdots$

Soln:

Let T_n be the n^{th} term

$$T_n = \frac{2n+3}{(2n-1)!}$$

$$2n+3 = A + B(2n-1)$$

Equate the co-~~eff of~~ n.

$$2 = 2B$$

$$\boxed{B = 1}$$

Equate the co-~~eff of~~ constants

$$3 = A - B$$

$$3 = A - 1$$

$$\boxed{A = 4}$$

$$2n+3 = 4 + (2n-1)$$

$$\frac{2n+3}{(2n-1)!} = \frac{4 + 2n - 1}{(2n-1)!}$$

$$= \frac{4}{(2n-1)!} + \frac{2n-1}{(2n-1)(2n-2)!}$$

$$T_n = \frac{4}{(2n-1)!} + \frac{1}{(2n-2)!}$$

Put $n = 1, 2, 3, \ldots$

$t_1 = 4\left(\frac{1}{11} + \frac{1}{01}\right)$

$t_2 = 4\left(\frac{1}{31} + \frac{1}{21}\right)$

$t_3 = 4\left(\frac{1}{51} + \frac{1}{41}\right)$

...

Adding vertically

$$S = 4\left[\frac{1}{11} + \frac{1}{31} + \frac{1}{51} + \ldots\right] + \left[1 + \frac{1}{3!} + \frac{1}{5!} + \ldots\right]$$

$$= 4\left[\frac{e - e^{-1}}{2e}\right] + \left[\frac{e + e^{-1}}{2}\right]$$

$$= 4\left[\frac{e - \frac{1}{e}}{2}\right] + \left[\frac{e + \frac{1}{e}}{2}\right]$$

$$= 4\left[\frac{e^2 - 1}{2e}\right] + \left[\frac{e^2 + 1}{2e}\right]$$

$$= \frac{4e^2 - 4 + e^2 + 1}{2e} = \frac{5e^2 - 3}{2e}$$

$$\boxed{S = \frac{5e}{2} - \frac{3}{2e}}$$

Put $n = 1, 2, 3 \ldots$

$T_1 = 4\left(\frac{1}{1!} + \frac{1}{0!}\right)$

$T_2 = 4\left(\frac{1}{3!} + \frac{1}{2!}\right)$

$T_3 = 4\left(\frac{1}{5!} + \frac{1}{4!}\right)$

. . .

Adding vertically

$$S = 4\left[\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right] + \left[1 + \frac{1}{2!} + \frac{1}{4!} + \cdots\right]$$

$$= 4\left[\frac{e - e^{-1}}{2e}\right] + \left[\frac{e + e^{-1}}{2}\right]$$

$$= 4\left[\frac{e - \frac{1}{e}}{2}\right] + \left[\frac{e + \frac{1}{e}}{2}\right]$$

$$= 4\left[\frac{e^2 - 1}{2e}\right] + \left[\frac{e^2 + 1}{2e}\right]$$

$$= \frac{4e^2 - 4 + e^2 + 1}{2e} = \frac{5e^2 - 3}{2e}$$

$$\boxed{S = \frac{5e}{2} - \frac{3}{2e}}$$

Logarithmic Series

Def:

The infinite series $x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots$ is called the logarithmic series. If the value of x is such that $-1<x\leq 1$.

The sum is

$$\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

Formulae.

1. $\log(1-x) = -\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right]$
2. $-\log(1-x) = x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$
3. $\log 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$
4. $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots$

Problems

1. If $|x|>1$ then find $\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\cdots$

Soln:

W.K.T

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots$$

$$\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\cdots = \frac{1/x}{1}+\frac{1/x^3}{3}+\frac{1/x^5}{5}+\cdots$$

$$= \frac{1}{2}\log\left(\frac{1+1/x}{1-1/x}\right)$$

$$= \frac{1}{2}\log\left(\frac{\frac{x+1}{x}}{\frac{x-1}{x}}\right)$$

$$= \frac{1}{2}\log\left(\frac{1+x}{x} \times \frac{x}{1-x}\right)$$

$$= \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

2. S.T $\log 10 = 3\log 2 + \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4^2} + \frac{1}{3}\cdot\frac{1}{4^3} + \cdots \infty$

Soln:

Let $x = \frac{1}{4}$

$$R.H.S = 3\log 2 + x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$

$$= 3\log 2 + \left[x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right]$$

$$= 3\log 2 + \log(1+x)$$

$$= 3\log 2 + \log\left(1+\frac{1}{4}\right)$$

$$= 3\log 2 + \log\left(\frac{4+1}{4}\right)$$

$$= 3\log 2 + \log\left(\frac{5}{4}\right) \quad \log a + \log b = \log(ab)$$

$$= \log 2^3 + \log\left(\frac{5}{4}\right)$$

$$= \log\left(2^3 \times \frac{5}{4}\right)$$

$$= \log\left(\overset{2}{\cancel{8}} \times \frac{5}{\cancel{4}}\right)$$

$$= \log 10 = L.H.S$$

$$L.H.S = R.H.S$$

3. If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, S.T

$$x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots$$

Soln.

Given $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$y = \log(1+x)$

Taking exponential on both sides

$e^y = e^{\log(1+x)}$

$e^y = 1+x$

$1+x = e^y$

$x+x = x + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

$x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

4. Sum the series $\log_{10} e - \log_{10^2} e + \log_{10^3} e - \cdots$

Soln:

W.K.T

$\log_{10^n} e = \left[\frac{\log_{10} e}{\log_{10} 10^n}\right]$

$\log_{10} e - \log_{10^2} e + \log_{10^3} e - \cdots$ $\left(\log_{10} 10 = 1\right)$

$= \log_{10} e - \left(\frac{\log_{10} e}{\log_{10} 10^2}\right) + \left(\frac{\log_{10} e}{\log_{10} 10^3}\right) + \cdots$

$= \log_{10} e - \left(\frac{\log_{10} e}{2\log_{10} 10}\right) + \left[\frac{\log_{10} e}{3\log_{10} 10}\right]$

$= \log_{10} e - \frac{\log_{10} e}{2} + \frac{\log_{10} e}{3} - \cdots$

$= \log_{10} e \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right]$

$= \log_{10} e \cdot \log_e 2$ $\quad \left[\log_{10} x = \log_{10} e \cdot \log_e x\right]$

$= \log_{10} 2$

5. S.T $\frac{a-x}{a} + \frac{1}{2}\left(\frac{a-x}{a}\right)^2 + \frac{1}{3}\left(\frac{a-x}{a}\right)^3 + \cdots = \log a - \log x$

Soln:-

Let $y = \frac{a-x}{a}$

L.H.S $= y + \frac{y^2}{2} + \frac{y^3}{3} + \cdots$

$= -\log(1-y)$

$= -\log\left(1 - \frac{a-x}{a}\right) = -\log\left(\frac{a-a+x}{a}\right)$

$= -\log\left(\frac{x}{a}\right)$

$= -\{\log x - \log a\} = -\log x + \log a$

$= \log a - \log x = R.H.S$

L.H.S = R.H.S

6. S.T $\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \cdots = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots$

Soln:

Let $x = \frac{1}{n+1}$

L.H.S $= x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$= -\log(1-x)$

$= -\log\left(1 - \frac{1}{n+1}\right)$

$= -\log\left(\frac{n+1-1}{n+1}\right) = -\log\left(\frac{n}{n+1}\right)$

$= \log\left(\frac{n}{n+1}\right)^{-1}$

$= \log\left(\frac{n+1}{n}\right)$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots$

$= \log\left(\frac{n}{n} + \frac{1}{n}\right) = \log\left(1 + \frac{1}{n}\right)$

$= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \ldots$

L.H.S = R.H.S

7. S.T $\log_3 e - \log_9 e + \log_{27} e - \ldots = \frac{\log e^2}{\log e^3}$

Soln:

W.K.T $\log_a b = \frac{1}{\log_b a}$

$\log_3 e = \frac{1}{\log e^3}$

$\log_9 e = \frac{1}{\log_e 9}$

$= \frac{1}{\log e^{3^2}} = \frac{1}{2\log e^3}$

$\log_{27} e = \frac{1}{\log_e 27} = \frac{1}{\log_e 3^3}$

$= \frac{1}{3\log e^3}$

L.H.S $= \frac{1}{\log_e 3} - \frac{1}{2\log_e 3} + \frac{1}{3\log_e 3} + \ldots$

$= \frac{1}{\log_e 3}\left[1 - \frac{1}{2} + \frac{1}{3}(+\ldots)\right]$

$= \frac{1}{\log_e 3} \log e^2 = \frac{\log e^2}{\log e^3}$

8. P.T $\log\sqrt{12} = 1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\cdots$

Soln:

R.H.S

$= 1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\cdots$

$= 1+\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{4^2}+\frac{1}{5}\cdot\frac{1}{4^2}+\frac{1}{6}\cdot\frac{1}{4^3}+\frac{1}{7}\frac{1}{4^3}+\cdots$

$= \left[1+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{5}\cdot\frac{1}{4^2}+\frac{1}{7}\cdot\frac{1}{4^3}+\cdots\right]+$

$\left[\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{4^2}+\frac{1}{6}\cdot\frac{1}{4^3}+\cdots\right]$

$= \left[1+\frac{1}{3}\left(\frac{1}{2}\right)^2+\frac{1}{5}\left(\frac{1}{2}\right)^4+\frac{1}{7}\left(\frac{1}{2}\right)^6+\cdots\right]+$

$\left[\frac{1}{2}\left\{\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4^2}+\frac{1}{3}\cdot\frac{1}{4^3}+\cdots\right\}\right]$

Multiply and divide by 2 in First term

$= 2\left[\frac{1}{2}+\frac{1}{3}\left(\frac{1}{2}\right)^3+\frac{1}{5}\left(\frac{1}{2}\right)^5+\frac{1}{7}\left(\frac{1}{2}\right)^7+\cdots\right]$

$+\frac{1}{2}\left[-\log\left(1-\frac{1}{4}\right)\right]$

$= 2\left[\frac{1}{2}\log\left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right)\right]+\frac{1}{2}\log\left(\frac{4}{4-1}\right)$

$= 2\left[\frac{1}{2}\log\left(\frac{3/2}{1/2}\right)\right]-\frac{1}{2}\log\left(\frac{3}{4}\right)$

$= \log 3-\frac{1}{2}\left[\log 3-\log 4\right]$

$= \log 3-\frac{1}{2}\log 3+\log 4\cdot\frac{1}{2}$

$= \log 3\left(1-\frac{1}{2}\right)+\log 4\cdot\frac{1}{2}$

$= \log 3\left(\frac{1}{2}\right)+\log 4\cdot\frac{1}{2}$

$= \frac{1}{2}\log 3+\frac{1}{2}\log 4$

$= \frac{1}{2}\log(4\times 3)$

$= \frac{1}{2}\log 12$

$= \log 12^{1/2}$

$= \log\sqrt{12} = L.H.S.$

$R.H.S = L.H.S$

9. S.T $\left(\frac{1}{1}+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)\frac{1}{9}+\left(\frac{1}{5}+\frac{1}{6}\right)\frac{1}{9^2}+\cdots = 9\log 3 - 12\log 2$

Soln.

L.H.S

$\left(\frac{1}{1}+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)\frac{1}{9}+\left(\frac{1}{5}+\frac{1}{6}\right)\frac{1}{9^2}+\cdots$

$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{9}+\frac{1}{4}\cdot\frac{1}{9}+\frac{1}{5}\cdot\frac{1}{9^2}+\frac{1}{6}\cdot\frac{1}{9^2}+\cdots$

$\left[\frac{1}{1}+\frac{1}{3}\cdot\frac{1}{9}+\frac{1}{5}\cdot\frac{1}{9^2}+\cdots\right]+\left[\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{9}+\frac{1}{6}\cdot\frac{1}{9^2}+\cdots\right]$

Multiply and divide by 3 in the first term

Multiply and divide by 9 in second term

$3\left[\frac{1}{3}+\frac{1}{3}\left(\frac{1}{3}\right)^3+\frac{1}{5}\left(\frac{1}{3}\right)^5+\cdots\right]+$

$\frac{9}{2}\left[\frac{1}{9}+\frac{1}{2}\left(\frac{1}{9}\right)^2+\frac{1}{3}\left(\frac{1}{9}\right)^3+\cdots\right]$

$= 3\left[\frac{1}{2}\log\left(\frac{1+1/3}{1-1/3}\right)\right]-\frac{9}{2}\log\left(\frac{1+1/9}{1-1/9}\right)$

$= 3\left[\frac{1}{2}\log\left(\frac{4/3}{2/3}\right)\right]-\frac{9}{2}\log\left(\frac{8}{9}\right)$

$= 3\left[\frac{1}{2}\log 2\right]-\frac{9}{2}\log\frac{8}{9}$

$= 3\log 2^{1/2}-\frac{9}{2}\log 8+\frac{9}{2}\log 9$

$= 3\log\sqrt{2}-\log(2\sqrt{2})^9+\log 3^9$

$= \log(\sqrt{2})^3-\log(2\sqrt{2})^9+\log 3^9$

$8^{1/2} = 2\sqrt{2}$

$9^{1/2} = 3$

$$= \log \frac{(\sqrt{2})^3}{(2\sqrt{2})^9} + 9\log 3$$

$$= \log \frac{2\sqrt{2}}{(2\sqrt{2})^9} + 9\log 3$$

$$= \log \frac{1}{(2\sqrt{2})^8} + 9\log 3$$

$$= \log (2\sqrt{2})^{-8} + 9\log 3$$

$$= -8\log 2\sqrt{2} + 9\log 3$$

$$= -8\log 2^1 \cdot 2^{1/2} + 9\log 3$$

$$= -8\log 2^{3/2} + 9\log 3$$

$$= -8^4 \times \frac{3}{2}\log 2 + 9\log 3$$

$$= -12\log 2 + 9\log 3$$

$$= 9\log 3 - 12\log 2 = R.H.S$$

$L.H.S = R.H.S$

SUMMATION - Logarithmic series

1. S.T $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots = \log 2$

Soln:

Let T_n be the nth term

$$T_n = \frac{1}{(2n-1)(2n)}$$

$$\frac{1}{(2n-1)(2n)} = \frac{A}{2n-1} + \frac{B}{2n}$$

$$1 = A(2n) + B(2n-1)$$

Put $n=0$

$1 = A(0) + B(2(0)-1)$

$1 = -B$

$\boxed{B=-1}$

Equate the coeff of n^{th} term

$0 = 2A + 2B$

$0 = 2A - 2B$

$2A = 2B$

$A = \frac{2}{2}$

$\boxed{A=1}$

$t_n = \frac{1}{(2n-1)(2n)} = \frac{1}{2n-1} - \frac{1}{2n}$

Put $n = 1, 2, 3, \ldots$

$t_1 = \frac{1}{1} - \frac{1}{2}$

$t_2 = \frac{1}{3} - \frac{1}{4}$

$t_3 = \frac{1}{5} - \frac{1}{6}$

$\ldots$

$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

$\boxed{S = \log 2}$

2. $\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \ldots = -\log(1-x) + \log\frac{(1-x)+x}{x}$

Soln:

L.H.S

Let $t_n = \frac{x^n}{n(n+1)}$ → (1)

$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$ → (2)

$1 = A(n+1) + Bn$ — Put $n = 0$

$1 = A(0+1) + B(0)$

$\boxed{1 = A}$

Equate nth term

$0 = A + B$

$0 = 1 + B$

$\boxed{B = -1}$

Sub A & B values in (2)

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

(1) $\Rightarrow t_n = \left[\frac{1}{n} - \frac{1}{n+1}\right] x^n$

$t_n = \frac{x^n}{n} - \frac{x^n}{n+1}$

Put $n = 1, 2, 3 \ldots$

$t_1 = \frac{x}{1} - \frac{x}{2} = x\left(\frac{1}{1} - \frac{1}{2}\right)$

$t_2 = \frac{x^2}{2} - \frac{x^2}{3} = x^2\left(\frac{1}{2} - \frac{1}{3}\right)$

$t_3 = \frac{x^3}{3} - \frac{x^3}{4} = x^3\left(\frac{1}{3} - \frac{1}{4}\right)$

. . . .

Adding these values

$$S = \frac{x}{1} - \frac{x^2}{2} + \frac{x^2}{2} - \frac{x^2}{3} + \frac{x3}{3} - \frac{x^3}{4} + \cdots$$

$$= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) - \left(\frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots\right)$$

multiply & divide by x in second term

$$= -\log(1-x) - \frac{1}{x}\left[\frac{x^2}{2} + \frac{x3}{3} + \frac{x^4}{4} + \cdots\right) - x$$

$$= -\log(1-x) - \frac{1}{x}\left[-\log(1-x) - x\right]$$

$$= -\log(1-x) + \frac{\log(1-x) + x}{x} = R.H.S$$

$$L.H.S = R.H.S$$

3. S.T $\frac{1}{2\cdot3\cdot4} + \frac{1}{4\cdot5\cdot6} + \frac{1}{6\cdot7\cdot8} + \cdots = \frac{3}{4} - \log 2$

Soln:

L.H.S

Let t_n be the nth term

$$t_n = \frac{1}{(2n)(2n+1)(2n+2)}$$

$$\frac{1}{2n(2n+1)(2n+2)} = \frac{A}{2n} + \frac{B}{2n+1} + \frac{C}{2n+2}$$

$$1 = A(2n+1)(2n+2) + B2n(2n+2) + C2n(2n+1)$$

$$4n^2+6n+2 \qquad 4n^2+4n \qquad 4n^2+2n$$

Put $n = 0$

$$1 = A(1)(2) + B(0) + C(0)$$

$$1 = 2A$$

$$\boxed{A = \frac{1}{2}}$$

$2n+1 = 0$

$2n = -1$

$n = -1/2$

Put $n = -1/2$

$$1 = A(2(-1/2)+1)[2(-1/2)+2] + B(2(-1/2))[2(-1/2)+2] + C[2(-1/2)(2(-1/2)+1)]$$

$$1 = A(0) + B(-1)(1) + C(0)$$

$$1 = -B$$

$$\boxed{B = -1}$$

Equate coeff of n^{th} terms

$$0 = 4A + 4B + 2C$$

$$0 = 4(1/2) + 4(-1) + 2C$$

$$= 2 - 4 + 2C$$

$$0 = -1 + 2C$$

$$2C = 1$$

$$\boxed{C = 1/2}$$

Sub A, B, C values

$$\frac{1}{2n(2n+1)(2n+2)} = \frac{1/2}{2n} - \frac{1}{2n+1} + \frac{1/2}{2n+2}$$

Tn

Put $n = 1, 2, 3 \ldots$

$$T_1 = \frac{1/2}{2} - \frac{1}{3} + \frac{1/2}{4}$$

$$T_2 = \frac{1/2}{4} - \frac{1}{5} + \frac{1/2}{6}$$

$$T_3 = \frac{1/2}{6} - \frac{1}{7} + \frac{1/2}{8}$$

$$\ldots \ldots$$

$$S = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} + \frac{1}{8} - \frac{1}{5} + \frac{1}{12} + \frac{1}{12} - \frac{1}{7} + \frac{1}{16} + \ldots$$

$$= \frac{1}{4} - \frac{1}{3} + \frac{2}{8_4} - \frac{1}{5} + \frac{2}{12_6} - \frac{1}{7} + \ldots$$

$= \frac{1}{4} - \left[\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right]$

Add 1/2 and -1/2 in R.H.S

$= \frac{1}{4} - \left\{\left[1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots\right] - \left[1 - \frac{1}{2}\right]\right\}$

$= \frac{1}{4} - \left[\log 2 - \left(\frac{1}{2}\right)\right]$

$= \frac{1}{4} - \log 2 + \frac{1}{2}$

$= \frac{1}{4} + \frac{1}{2} - \log 2$

$= \frac{4+2}{8} - \log 2$

$= \frac{3}{4} - \log 2$

$= \frac{3}{4} - \log 2 = R.H.S$

L.H.S = R.H.S

4. S.T $\frac{1}{1.2.3} + \frac{5}{3.4.5} + \frac{9}{5.6.7} + \cdots = \frac{5}{2} - 3\log 2$

Soln:

L.H.S

Let t_n be the n^{th} term

$t_n = \frac{4n-3}{(2n-1)(2n)(2n+1)}$

$\frac{4n-3}{(2n-1)(2n)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n} + \frac{C}{2n+1}$

$4n-3 = A\frac{2n(2n+1)}{4n^2+2n} + B\frac{(2n-1)(2n+1)}{4n^2-1} + C\frac{2n(2n-1)}{4n^2-2n}$

Put $n = 0$

$-3 = A(0) + B(-1)(1) + C(0)$

$-3 = -B$

$\boxed{B = 3}$

Put $n = \frac{1}{2}$

$4\left(\frac{1}{2}\right)^2 - 3 = A\,2\left(\frac{1}{2}\right)\left(2\left(\frac{1}{2}\right)+1\right) + B\left(2\cdot\frac{1}{2}-1\right)\left(2\cdot\frac{1}{2}+1\right) + C\,2\left(\frac{1}{2}\right)\left(2\left(\frac{1}{2}\right)-1\right)$

$2 - 3 = A(1)(2) + B(0) + C(0)$

$-1 = 2A$

$\boxed{A = -1/2}$

Equate the co-eff of n term

$4 = 2A - 2C$

$4 = 2(-1/2) - 2C$

$4 = -1 - 2C$

$5 = -2C$

$\boxed{C = -5/2}$

Sub A, B, C values

$t_n = \frac{-\frac{1}{2}}{2n-1} + \frac{3}{2n} - \frac{5/2}{2n+1}$

Put $n = 1/2/3 \ldots$

$t_1 = \frac{-\frac{1}{2}}{1} + \frac{3}{2} - \frac{5/2}{3}$

$t_2 = \frac{-1/2}{3} + \frac{3}{4} - \frac{5/2}{5}$

$t_3 = \frac{-1/2}{5} + \frac{3}{6} - \frac{5/2}{7}$

$t_4 = \frac{-1/2}{7} + \frac{3}{8} - \frac{5/2}{9}$

Adding the values $-\frac{5}{2}\left(\frac{1}{3}\right)$

$$= -\frac{1}{2} + \frac{3}{2} - \frac{5}{2}\left(\frac{1}{3}\right) - \frac{1}{2}\left(\frac{1}{3}\right) + \frac{3}{4} - \frac{3}{2}\left(\frac{4}{5}\right)$$

$$= -\frac{1}{2} + \frac{3}{2} - \frac{1}{3}(3) + \frac{3}{4} - \frac{3}{5} + \frac{3}{6} - \frac{3}{7} + \cdots$$

$$= -\frac{1}{2} - 3\left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \cdots\right)$$

Add & sub 1 on R.H.S

$$= -\frac{1}{2} - 3\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots\right] - 1$$

$$= -\frac{1}{2} - 3\log 2 + 3$$

$$= -\frac{1}{2} + 3 - 3\log 2$$

$$= \frac{-1+6}{2} - 3\log 2$$

$$= \frac{5}{2} - 3\log 2 = \text{R.H.S}$$

$\therefore$ L.H.S = R.H.S

5. S.T $\frac{1}{1\cdot 3} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 7} + \cdots = 2 - 2\log 2$

Soln:

$$T_n = \frac{2}{2n(2n+1)} = \frac{A}{2^n} + \frac{B}{2n+1}$$

$$2 = A(2n+1) + B2n$$

Equate the constant term

$A = 2 \Rightarrow A = 2$

Equate the co-eff of n^{th} term

$$0 = 2A + 2B$$

$0 = 2A + 2B$

$-2A = 2B$

$\boxed{B = -2}$

$t_n = \frac{2}{2n} - \frac{2}{2n+1}$

Put $n = 1, 2, 3 \cdots$

$t_1 = \frac{2}{2} - \frac{2}{3}$

$t_2 = \frac{2}{4} - \frac{2}{5}$

$t_3 = \frac{2}{6} - \frac{2}{7}$

Adding these values

$S = \frac{2}{2} - \frac{2}{3} + \frac{2}{4} - \frac{2}{5} + \frac{2}{6} - \frac{2}{7} + \cdots$

$= 2\left[\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \cdots\right]$

$= -2\left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right]$

Add & Sub 1

$= -2\left\{\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right) - 1\right\}$

$= -2\left[\log 2 - 1\right]$

$= -2\log 2 + 2$

$= 2 - 2\log 2$

L.H.S = R.H.S