MARUDHAR KESARI JAIN COLLEGE FOR WOMEN, VANIYAMBADI

PG & RESEARCH DEPARTMENT OF MATHEMATICS

CLASS : I B.Sc CHEMISTRY

SUBJECT CODE : MATHEMATICS I

SUBJECT NAME : 23UEMA10C

SYLLABUS

UNIT- IV

Trigonometry

Expansions of $\sin\theta$, $\cos\theta$ in a series of powers of $\sin\theta$ and $\cos\theta$ - Expansions of $\sin(n\theta)$ and $\cos(n\theta)$ in a series sines and cosines of multiples of "θ" - Expansions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in a series of powers of "θ" – Hyperbolic and inverse hyperbolic functions .

UNIT - 4.

TRIGONOMETRY.

Expansions of $\sin^n\theta, \cos^n\theta, \sin n\theta, \cos n\theta, \tan n\theta$ - Expansions of $\sin\theta, \cos\theta, \tan\theta$ in terms of θ.

Let, $x = \cos\theta + i\sin\theta$ — ①

$\frac{1}{x} = \cos\theta - i\sin\theta$ — ②.

① + ② $\Rightarrow x + \frac{1}{x} = 2\cos\theta$.

① − ② $\Rightarrow x - \frac{1}{x} = 2i\sin\theta$.

Also,

$x^n = \cos n\theta + i\sin n\theta$ — ③

$\frac{1}{x^n} = \cos n\theta - i\sin n\theta$ — ④

③ + ④ $\Rightarrow x^n + \frac{1}{x^n} = 2\cos n\theta$.

③ − ④ $\Rightarrow x^n - \frac{1}{x^n} = 2i\sin n\theta$.

NOTE:

1. $(x+a)^n = nC_0 x^n a^n + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \dots + nC_n x^n a^n$.

2. $(x-a)^n = nC_0 x^n a^n - nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \dots + nC_n x^n a^n$.

3. $(1+x)^n = 1 + \frac{nx}{\lfloor 1} + \frac{n(n-1)x^2}{\lfloor 2} + \frac{n(n-1)(n-2)x^3}{\lfloor 3} + \dots$

4. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

5. $(1+x)^{-1} = 1 - x + x^2 + \dots + (-1)^n x^n + \dots$

6. $e^{x} = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \cdots$

7. $e^{-x} = 1 - \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \cdots$

8. $\frac{e^{x}+e^{-x}}{2} = 1 + \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} + \cdots$

9. $\frac{e^{x}-e^{-x}}{2} = x + \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} + \cdots$

10. $nC_r = \frac{n!}{r!(n-r)!} = \frac{\lfloor n}{\lfloor r \, \lfloor n-r}$.

11. $nC_0 = 1,\ nC_n = 1;\ nC_1 = n;\ nC_{n-1} = n$

12. $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$

Examples:

1. State that $2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$.

Sol:

Let $x = \cos\theta + i\sin\theta$.

$\frac{1}{x} = \cos\theta - i\sin\theta$.

$\Rightarrow x + \frac{1}{x} = 2\cos\theta$ and $x^n + \frac{1}{x^n} = 2\cos n\theta$.

Consider,

$2^5\cos^6\theta = \frac{1}{2}(2^6\cos^6\theta)$.

$= \frac{1}{2}(2\cos\theta)^6$.

$$= \frac{1}{2}\left(x+\frac{1}{x}\right)^6 \Rightarrow (x+a)^n$$

$$2^5\cos^6\theta = \frac{1}{2}\Big[6C_0 x^6\left(\frac{1}{x}\right)^0 + 6C_1 x^{6-1}\left(\frac{1}{x}\right)^1 + 6C_2 x^{6-2}\left(\frac{1}{x}\right)^2 +$$

$$6C_3 x^{6-3}\left(\frac{1}{x}\right)^3 + 6C_4 x^{6-4}\left(\frac{1}{x}\right)^4 + 6C_5 x^{6-5}\left(\frac{1}{x}\right)^5 +$$

$$6C_6 x^{6-6}\left(\frac{1}{x}\right)^6\Big]$$

$\frac{\cancel{6}^3\cdot 5}{1\cdot\cancel{2}_1}$ $\frac{\cancel{6}\cdot 5\cdot\cancel{4}^2}{1\cdot\cancel{2}\cdot\cancel{3}}$

$$= \frac{1}{2}\Big[x^6 + 6x^5\left(\frac{1}{x}\right) + 15x^4\left(\frac{1}{x^2}\right) + 20x^3\left(\frac{1}{x^3}\right) +$$

$\frac{\cancel{6}^3\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot 2}{1\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}}$

$$15x^2\left(\frac{1}{x^4}\right) + 6x\left(\frac{1}{x^5}\right) + \frac{1}{x^6}\Big]$$

$$= \frac{1}{2}\left[x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}\right].$$

$$= \frac{1}{2}\left[\left(x^6+\frac{1}{x^6}\right) + 6\left(x^4+\frac{1}{x^4}\right) + 15\left(x^2+\frac{1}{x^2}\right) + 20\right].$$

$$= \frac{1}{2}\left[2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20\right].$$

$$= \frac{2}{2}\left[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10\right].$$

$$2^5\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10.$$

2. $2^6\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta$.

state that.

Sol:

Let, $x = \cos\theta + i\sin\theta$.

$\frac{1}{x} = \cos\theta - i\sin\theta$.

$\Rightarrow x + \frac{1}{x} = 2\cos\theta$ and $x^n + \frac{1}{x^n} = 2\cos n\theta$.

Consider,

$$2^6\cos^7\theta = \frac{1}{2}\left[2^7\cos^7\theta\right]$$

$$= \frac{1}{2}\left[2\cos\theta\right]^7 \Rightarrow \frac{1}{2}\left[x+\frac{1}{x}\right]^7$$

$$2^6\cos^7\theta = \frac{1}{2}\left[7C_0\, x^7\left(\frac{1}{x}\right)^0 + 7C_1\, x^{7-1}\left(\frac{1}{x}\right)^1 + 7C_2\, x^{7-2}\left(\frac{1}{x}\right)\right.$$

$$+7C_3\, x^{1-3}\left(\frac{1}{2}\right)^3 + 7C_4\, x^{7-4}\left(\frac{1}{x}\right)^4 + 7C_5\, x^{7-5}\left(\frac{1}{x}\right)^5 +$$

$$\left. 7C_6\, x^{7-6}\left(\frac{1}{x}\right)^6 + 7C_7\, x^{7-7}\left(\frac{1}{x}\right)^7\right].$$

$$2^6\cos^7\theta = \frac{1}{2}\left[x^7 + 7x^6\left(\frac{1}{x}\right) + 21x^5\left(\frac{1}{x^2}\right) + 35x^4\left(\frac{1}{x^3}\right)\right.$$

$$\left. + 35x^3\left(\frac{1}{x^4}\right) + 21x^2\left(\frac{1}{x^5}\right) + 7x\left(\frac{1}{x^6}\right) + \frac{1}{x^7}\right]$$

$$= \frac{1}{2}\left[\left(x^7+\frac{1}{x^7}\right) + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5}\right]$$

$$= \frac{1}{2}\left[\left(x^7+\frac{1}{x^7}\right) + 7\left(x^5+\frac{1}{x^5}\right) + 21\left(x^3+\frac{1}{x^3}\right) + 35\left(x+\frac{1}{x}\right)\right].$$

$$= \frac{1}{2}\left[2\cos7\theta + 7(2\cos5\theta) + 21(2\cos3\theta) + 35(2\cos\theta)\right].$$

$$= \frac{2}{2}\left[\cos7\theta + 7\cos5\theta + 21\cos3\theta + 35\cos\theta\right]$$

$$2^6\cos^7\theta = \left[\cos7\theta + 7\cos5\theta + 21\cos3\theta + 35\cos\theta\right].$$

Expansion of $\cos n\theta . \sin n\theta$ in powers of $\sin\theta, \cos\theta$.

If n is a positive integer

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ — (1)

By Binomial Theorem.

$$(x+y)^n = x^n + nC_1 x^{n-1} y^1 + nC_2 x^{n-2} y^2 + \dots + y^n$$

Here $x = \cos\theta$, $y = i\sin\theta$.

$$(\cos\theta + i\sin\theta)^n = \cos^n\theta + nC_1 \cos^{n-1}\theta \, i\sin\theta + nC_2 \cos^{n-2}\theta \, i^2 \sin^2\theta + \dots + i^n \sin^n\theta.$$

$$(\cos\theta + i\sin\theta)^n = \cos^n\theta + i nC_1 \cos^{n-1}\theta \sin\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + \dots + i^n \sin^n\theta.$$

Applying eqn (1).

$$(\cos n\theta + i\sin n\theta) = \cos^n\theta + i nC_1 \cos^{n-1}\theta \sin\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + \dots + i^n \sin^n\theta.$$

Equating real and imaginary Part.

$$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta + \dots$$

$$\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + \dots$$

The above eqation can written as.

$$\cos n\theta = c^n - nC_2 c^{n-2} s^2 + \dots$$

$$\sin n\theta = nC_1 c^{n-1} s - nC_3 c^{n-3} s^3 + \dots$$

(or)

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!} \cos^{n-2}\theta \sin^2\theta + \dots$$

$$\sin n\theta = n\cos^{n-1}\theta \sin\theta - \frac{n(n-1)(n-2)}{3!} \cos^{n-3}\theta \sin^3\theta + \dots$$

4. Expansion of $\tan n\theta$ in powers of $\tan\theta$.

Sol:

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$$\tan n\theta = \frac{nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta}{\cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta}$$

In R.H.S Dividing $\cos^n\theta$ in numerator and Denominator.

$$\tan n\theta = \frac{\frac{nC_1 \cos^{n-1}\theta \sin\theta}{\cos^n\theta} - \frac{nC_3 \cos^{n-3}\theta \sin^3\theta}{\cos^n\theta} + \dots}{\frac{\cos^n\theta}{\cos^n\theta} - \frac{nC_2 \cos^{n-2}\theta \sin^2\theta}{\cos^n\theta} + \frac{{}^nC_4 \cos^{n-4}\theta \sin^4\theta}{\cos^n\theta} + \dots}$$

$$\tan n\theta = \frac{nC_1 \cos^{n-1}\theta \sin\theta \cos^{-n}\theta - nC_3 \cos^{n-3}\theta \sin^3 \cos^{-n}\theta + \dots}{1 - nC_2 \cos^{n-2}\theta \sin^2\theta \cos^{-n}\theta + nC_4 \cos^{n-4}\theta \sin^4\theta \cos^{-n}\theta + \dots}$$

$$\tan n\theta = \frac{nC_1 \cos^{-1}\theta \sin\theta - nC_3 \cos^{-3} \sin^3\theta + \dots}{1 - nC_2 \cos^{-2} \sin^2\theta + nC_4 \cos^{-4} \sin^4\theta + \dots}$$

$$\tan n\theta = \frac{nC_1 \frac{\sin\theta}{\cos\theta} - nC_3 \frac{\sin^3\theta}{\cos^3\theta} + \dots}{1 - nC_2 \frac{\sin^2\theta}{\cos^2\theta} + nC_4 \frac{\sin^4\theta}{\cos^4\theta} + \dots}$$

$$\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + \dots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta + \dots}$$

Formula:

$$^nC_r = \frac{n!}{(n-r)!\,r!}$$

Problems:

1. Expand $\cos 6\theta$ interms of $\sin\theta$.

Sol:

By De-moivre's theorem.

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

put, $n = 6$.

$$(\cos\theta + i\sin\theta)^6 = \cos 6\theta + i\sin 6\theta \quad \text{—— (1)}$$

By Binomial Theorem:

$$(x+y)^n = x^n + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots + y^n$$

$i = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$

$$(\cos\theta + i\sin\theta)^6 = \cos^6\theta + 6C_1 \cos^5\theta\, i\sin\theta + 6C_2 \cos^4\theta\, i^2\sin^2\theta + 6C_3\cos^3\theta\, i^3\sin^3\theta + 6C_4\cos^2\theta\, i^4\sin^4\theta + 6C_5\cos\theta\, i^5\sin^5\theta + 6C_6\cos^0\theta\, i^6\sin^6\theta.$$

$$= \cos^6\theta + i6\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta + i6\cos\theta\sin^5\theta - \sin^6\theta \quad \text{—— (2)}$$

sub eqn (1) in eqn (2).

$$\cos 6\theta + i\sin 6\theta = \cos^6\theta + i6\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta + i6\cos\theta\sin^5\theta - \sin 6\theta.$$

Formula:

$$\cos^2\theta = 1 - \sin^2\theta.$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

$$(a-b)^2 = a^2 + b^2 - 2ab.$$

Equating the real part. $a^3-3a^2b+3ab^2-b^3$

$$\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta.$$

$(a-b)^2 = a^2+b^2-2ab$

$$\cos 6\theta = (\cos^2\theta)^3 - 15(1-\sin^2\theta)^2\sin^2\theta + 15(1-\sin^2\theta)$$
$$15\cos^2\theta\sin^4\theta - \sin^6\theta.$$

$$= (1-\sin^2\theta)^3 - 15(1-\sin^2\theta)^2\sin^2\theta + 15(1-\sin^2\theta)\sin^4\theta - \sin^6\theta.$$

$$= (1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta) - 15\sin^2\theta.(1+\sin^4\theta - 2\sin^2\theta) + 15\sin^4\theta - 15\sin^6\theta - \sin^6\theta.$$

$$= 1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta - 15\sin^2\theta - 15\sin^6\theta + 30\sin^4\theta + 15\sin^4\theta - 15\sin^6\theta - \sin^6\theta.$$

$$\cos 6\theta = 1 - 18\sin^2\theta + 48\sin^4\theta - 32\sin^6\theta.$$

2. Prove that $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.

<u>Sol:</u>

W.K.T.

$$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta - nC_6\cos^{n-6}\theta\sin^6\theta$$

<u>Here $n=6$.</u> $\frac{6\times5}{2\times1}$

$$\cos 6\theta = \cos^6\theta - 6C_2\cos^4\theta\sin^2\theta + 6C_4\cos^2\theta\sin^4\theta - 6C_6\cos^0\theta\sin^6\theta.$$

$$= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta.$$

$= \cos^6\theta - 15\cos^4\theta \ \sin^2\theta + 15\cos^2\theta (\sin^2\theta)^2 - (\sin^2\theta)^3.$

$\cos 6\theta = \cos^6\theta - 15\cos^4\theta(1-\cos^2\theta) + 15\cos^2\theta(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3.$

$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta(1+\cos^4\theta - 2\cos^2\theta) - (1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$

$= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta + 15\cos^6\theta - 30\cos^4\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta.$

$\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.$

Hence Proved.

3. Prove that $\cos 4\theta = 8\sin^4\theta - 8\sin^2\theta + 1$.

Sol:

W.K.T

$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta.$

Here $n=4$.

$\cos 4\theta = \cos^4\theta - 4C_2\cos^2\theta \sin^2\theta + 4C_4 \cos^0\theta \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta \cdot \sin^2\theta + \sin^4\theta.$

$= (1-\sin^2\theta)^2 - 6(1-\sin^2\theta)\sin^2\theta + \sin^4\theta.$

$= 1 + \sin^4\theta - 2\sin^2\theta - 6\sin^2\theta + 6\sin^4\theta + \sin^4\theta.$

$\cos 4\theta = 8\sin^4\theta - 8\sin^2\theta + 1.$

Hence Proved.

4. Prove that $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$.

Sol:

W.K.T

$$\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3\cos^{n-3}\theta\sin^3\theta + nC_5\cos^{n-5}\theta\sin^5\theta - nC_7\cos^{n-7}\theta\sin^7\theta$$

n=7.

$$7\theta = 7C_1\cos^6\theta\sin\theta - 7C_3\cos^4\theta\sin^3\theta + 7C_5\cos^2\theta\sin^5\theta - 7C_7\cos^0\theta\sin^7\theta$$

$$= 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$

$$= 7(\cos^2\theta)^3\sin\theta - 35(\cos^2\theta)^2\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$

$$= 7\sin\theta(1-\sin^2\theta)^3 - 35\sin^3\theta(1-\sin^2\theta)^2 + 21\sin^5\theta(1-\sin^2\theta) - \sin^7\theta.$$

$$= [7\sin\theta(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta)] - [35\sin^3\theta(1-2\sin^2\theta+\sin^4\theta)] + 21\sin^5\theta - 21\sin^7\theta - \sin^7\theta$$

$$= 7\sin\theta - 21\sin^3\theta + 21\sin^5\theta - 7\sin^7\theta - 35\sin^3\theta + 70\sin^5\theta - 35\sin^7\theta + 21\sin^5\theta - 21\sin^7\theta - \sin^7\theta$$

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta.$$

Hence proved

5. Expand $\tan 7\theta$ in terms of $\tan\theta$.

Sol:

W.K.T

$$\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + \cdots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta + \cdots}$$ &

Here, $n = 7$

$$\tan 7\theta = \frac{7C_1 \tan\theta - 7C_3 \tan^3\theta + 7C_5 \tan^5\theta - 7C_7 \tan^7\theta}{1 - 7C_2 \tan^2\theta + 7C_4 \tan^4\theta - 7C_6 \tan^6\theta}$$

$$\tan 7\theta = \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}$$

6. Expand $\frac{\cos 5\theta}{\cos\theta}$ interms of $\cos\theta$.

Sol:

W.K.T.

$$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta$$

$n = 5$,

$$\cos 5\theta = \cos^5\theta - 5C_2 \cos^3\theta \sin^2\theta + 5C_4 \cos\theta \sin^4\theta$$

$$= \cos^5\theta - 10\cos^3\theta \sin^2\theta + 5\cos\theta \sin^4\theta \quad \text{--- (1)}$$

Divide $\cos\theta$ in above eqn.

$$\frac{\cos 5\theta}{\cos\theta} = \cos^4\theta - 10\cos^2\theta \sin^2\theta + 5\sin^4\theta$$

$$= \cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + 5(1-\cos^2\theta)^2$$

$$= \cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 5(1 - 2\cos^2\theta + \cos^4\theta)$$

$$= \cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 5 - 10\cos^2\theta + 5\cos^2\theta$$

$$\frac{\cos 5\theta}{\cos\theta} = 16\cos^4\theta - 20\cos^2\theta + 5$$

7. Expand $\tan 5\theta$.

Sol:

W.K.T.

$$\tan n\theta = \frac{nC_1 \tan\theta - nC_3 \tan^3\theta + \dots}{1 - nC_2 \tan^2\theta + nC_4 \tan^4\theta + \dots}$$

$n=5$

$$\tan 5\theta = \frac{5C_1 \tan\theta - 5C_3 \tan^3\theta + 5C_5 \tan^5\theta}{1 - 5C_2 \tan^2\theta + 5C_4 \tan^4\theta + \dots}$$

$$\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$$

8. Prove that $\cos 8\theta = 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta$.

Sol:

W.K.T.

$$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta - \dots$$

$n=8$,

$$\cos 8\theta = \cos^8\theta - 8C_2 \cos^6\theta \sin^2\theta + 8C_4 \cos^4\theta \sin^4\theta - 8C_6 \cos^2\theta \sin^6\theta + 8C_8 \cos^0\theta \sin^8\theta.$$

$$= \cos^8\theta - 28\cos^6\theta \sin^2\theta + 70\cos^4\theta \sin^4\theta - 28\cos^2\theta \sin^6\theta + \sin^8\theta.$$

$$= (\cos^2\theta)^4 - 28(\cos^2\theta)^3 \sin^2\theta + 70(\cos^2\theta)^2 \sin^4\theta - 28\cos^2\theta \sin^6\theta + \sin^8\theta.$$

$$= [(1-\sin^2\theta)^2]^4 - [28(1-\sin^2\theta)^3 \sin^2\theta] + [70(1-\sin^2\theta)^2 \sin^4\theta] - 28(1-\sin^2\theta) \sin^6\theta + \sin^8\theta.$$

$= [1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta] -$
$28\sin^2\theta(1 - 3\sin^2\theta + 3\sin^4\theta - \sin^6\theta)] + [70\sin^4\theta$
$(1 - 2\sin^2\theta + \sin^4\theta)] - 28\sin^6\theta + 28\sin^8\theta + \sin^8\theta.$

$= 1 - 4\sin^2\theta + 6\sin^4\theta - 4\sin^6\theta + \sin^8\theta - 28\sin^2\theta +$
$84\sin^4\theta - 84\sin^6\theta - 28\sin^8\theta + 70\sin^4\theta -$
$140\sin^6\theta + 70\sin^8\theta - 28\sin^6\theta + 28\sin^8\theta$
$+ \sin^8\theta.$

$\cos 8\theta = 1 - 32\sin^2\theta + 160\sin^4\theta - 256\sin^6\theta + 128\sin^8\theta.$

Hence Proved.

1. Expand $\cos 7\theta$ interms of $\cos\theta$.

Sol: W.K.T.

$\cos n\theta = \cos^n\theta - nC_2\cos^{n-2}\theta\sin^2\theta + nC_4\cos^{n-4}\theta\sin^4\theta \ldots$

Here $n = 7$.

$\cos 7\theta = \cos^7\theta - 7C_2\cos^5\theta\sin^2\theta + 7C_4\cos^3\theta\sin^4\theta$
$- 7C_6\cos\theta\sin^6\theta.$

$= \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$

$= \cos^7\theta - [21\cos^5\theta(1 - \cos^2\theta)] + [35\cos^3\theta(1 - \cos^2\theta)^2]$
$- [7\cos\theta(1 - \cos^2\theta)^3]$

$= \cos^7\theta - 21\cos^5\theta - 21\cos^7\theta + 35\cos^3\theta\,\cancel{\cos^3\theta} -$
$70\cos^5\theta + 35\cos^7\theta - 7\cos\theta + 21\cos^3\theta -$
$21\cos^5\theta + 7\cos^7\theta.$

$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta.$

10. Expand $\cos 9\theta$ as a series in powers of $\cos\theta$. (1)

Sol:

W.K.T.

$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta \ldots$

$n = 9$,

$\cos 9\theta = \cos^9\theta - 9C_2 \cos^7\theta \sin^2\theta + 9C_4 \cos^5\theta \sin^4\theta$

$- 9C_6 \cos^3\theta \sin^6\theta + 9C_8 \cos\theta \sin^8\theta$.

$= \cos^9\theta - 36 \cos^7\theta \sin^2\theta + 126 \cos^5\theta \sin^4\theta$

$- 84 \cos^3\theta \sin^6\theta + 9 \cos\theta \sin^8\theta$.

$= \cos^9\theta - 36\cos^7\theta(1-\cos^2\theta) + 126\cos^5\theta(1-\cos^2\theta)^2$

$- 84\cos^3\theta(1-\cos^2\theta)^3 + 9\cos\theta(1-\cos^2\theta)^4$.

$= \cos^9\theta - 36\cos^7\theta + 36\cos^9\theta + [126\cos^5\theta(1-2\cos^2\theta$

$+ \cos^4\theta)] - [84\cos^3\theta(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)] +$

$9\cos\theta(1-4\cos^2\theta+6\cos^4\theta-4\cos^6\theta+\cos^8\theta)]$.

$= \cos^9\theta - 36\cos^7\theta + 36\cos^9\theta + 126\cos^5\theta -$

$252\cos^7\theta + 126\cos^9\theta - 84\cos^3\theta + 252\cos^5\theta$

$- 252\cos^7\theta + 84\cos^9\theta + 9\cos\theta - 36\cos^3\theta$

$+ 54\cos^5\theta - 36\cos^7\theta + 9\cos^9\theta$.

$\cos 9\theta = 256\cos^9\theta - 576\cos^7\theta + 432\cos^5\theta -$

$120\cos^3\theta + 9\cos\theta$.

Using $\frac{\cos 7\alpha}{\cos\alpha} = 64\cos^6\alpha - 112\cos^4\alpha + 56\cos^2\alpha - 7$

Prove that $\frac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2$ where $x = 2\cos\theta$.

Sol:

$$\frac{1+\cos 7\theta}{1+\cos\theta} = \frac{2\cos^2\frac{7\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

Let $\frac{\theta}{2} = \alpha$.

$$\frac{1+\cos 7\theta}{1+\cos\theta} = \frac{2\cos^2 7\alpha}{2\cos^2\alpha}$$

$$= \frac{\cos^2 7\alpha}{\cos^2\alpha} \Rightarrow \left[\frac{\cos 7\alpha}{\cos\alpha}\right]^2 \text{ --- } (1)$$

$\because x = 2\cos\theta$.

$$x = 2\left[2\cos^2\frac{\theta}{2} - 1\right]$$

$$x = 4\cos^2\frac{\theta}{2} - 2.$$

$$x+2 = 4\cos^2\alpha \text{ --- } (2)$$

$$\left[\frac{\cos 7\alpha}{\cos\alpha}\right]^2 = (64\cos^6\alpha - 112\cos^4\alpha + 56\cos^2\alpha - 7)^2.$$

$$= [(4\cos^2\alpha)^3 - 7(4\cos^2\alpha)^2 + 14(4\cos^2\alpha) - 7]^2.$$

$$(1) \Rightarrow \frac{1+\cos 7\theta}{1+\cos\theta} = \left[\frac{\cos 7\alpha}{\cos\alpha}\right]^2$$

$=(64\cos^6\alpha - 112\cos^4\alpha + 56\cos^2\alpha - 7)^2.$

$= [(x+2)^3 - 7(x+2)^2 + 14(x+2) - 7]^2.$

$= [x^3 + 6x^2 + 12x + 8 - 7x^2 - 28x - 28 + 14x + 28 - 7]^2.$

$= [x^3 - x^2 - 2x + 1]^2$

$\therefore \dfrac{1+\cos 7\theta}{1+\cos\theta} = (x^3 - x^2 - 2x + 1)^2.$

---

2. Expansion for $\tan(\theta_1 + \theta_2 + \dots + \theta_n)$.

W.K.T.

$\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i\sin(\theta_1 + \theta_2 + \dots + \theta_n).$

$= \cos\theta_1 + \cos\theta_2 + \dots + \cos\theta_n + i\sin\theta_1 + i\sin\theta_2 + \dots + i\sin\theta_n.$

$= (\cos\theta_1 + i\sin\theta_1) + (\cos\theta_2 + i\sin\theta_2) + \dots + (\cos\theta_n + i\sin\theta_n).$

$= \cos\theta_1\left[1 + \dfrac{i\sin\theta_1}{\cos\theta_1}\right] + \cos\theta_2\left[\dfrac{1 + i\sin\theta_2}{\cos\theta_2}\right] + \dots + \cos\theta_n\left[\dfrac{1 + i\sin\theta_n}{\cos\theta_n}\right]$

$= \cos\theta_1(1 + i\tan\theta_1) + \cos\theta_2(1 + i\tan\theta_2) + \dots + \cos\theta_n(1 + i\tan\theta_n).$

$= \cos\theta_1 \cos\theta \dots \cos\theta_n [1 + i\tan\theta_1][1 + i\tan\theta_2] \dots [1 + i\tan\theta_n].$

$= \cos\theta_1 \cos\theta_2 \dots \cos\theta_n [1 + i\Sigma\tan\theta_1 + i^2\Sigma\tan\theta_1\tan\theta_2 + i^3\Sigma\tan\theta_1\tan\theta_2\tan\theta_3]$

Let,

$S_1 = \tan\theta_1 + \tan\theta_2 + \dots + \tan\theta_n = \Sigma\tan\theta_1.$

$S_2 = \tan\theta_1\tan\theta_2 + \tan\theta_2\tan\theta_3 + \dots = \Sigma_1 \tan\theta_1\tan\theta_2.$

$S_3 = \tan\theta_1\tan\theta_2\tan\theta_3 + \tan\theta_3\tan\theta_4\tan\theta_5 + \dots =$

$\Sigma_1 \tan\theta_1\tan\theta_2\tan\theta_3.$

$\cos(\theta_1+\theta_2+\theta_n) + i\sin(\theta_1+\theta_2+\dots+\theta_n) = \cos\theta_1 \cos\theta_2$

$\cos\theta_3 [1+is_1+i^2 s_2+i^3 s_3+\dots]$

$= \cos\theta_1 \cos\theta_2 \cos\theta_3 \dots \cos\theta_n [1+is_1 - s_2 - is_3+\dots]$

Equating real and imaginary part.

$\cos(\theta_1+\theta_2+\dots+\theta_n) = \cos\theta_1 . \cos\theta_2 , \cos\theta_3 \dots \cos\theta_n$

$[1-s_2+s_4\dots] \rightarrow ①$

$\sin(\theta_1+\theta_2+\dots+\theta_n) = \cos\theta_1 \cos\theta_2 , \cos\theta_3 \dots \cos\theta_n$

$[s_1 - s_2 + s_3 - \dots] \longrightarrow ②$

divide $\frac{②}{①}$

$$\tan(\theta_1+\theta_2+\dots+\theta_n) = \frac{s_1 - s_3 + s_5 - s_7 + \dots}{1 - s_2 + s_4 - s_6 + \dots}$$

Result

If the give eqn is $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$

$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$

$(\div a_0)$ $x^n + \frac{a_1}{a_0} x^{n-1} + \frac{a_2}{a_0} x^{n-2} + \dots \frac{a_{n-1}}{a_0} x + \frac{a_n}{a_0} = 0$

$s_1 = \frac{a_1}{a_0}$, $s_2 = \frac{a_2}{a_0}$, ..., $s_n = \frac{a_n}{a_0}$

2. If α, β are The roots of eqn $ax^2+bx+c=0$

Then, $s_1 = \alpha+\beta = \frac{-b}{a}$

$s_2 = \alpha\beta = \frac{c}{a}$

3. If α, β, γ are The roots of eqn $ax^3+bx^2+cx+d=0$

Then

$s_1 = \alpha+\beta+\gamma = \frac{-b}{a}$

$s_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$s_3 = \alpha\beta\gamma = \frac{-d}{a}$

4. If $\alpha, \beta, \gamma, \delta$ are the roots of the eqn $ax^4+bx^3+cx^2+dx+e=0$ then

$$S_1 = \alpha+\beta+\gamma+\delta = \frac{-b}{a}$$

$$S_2 = \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a}$$

$$S_3 = \alpha\beta\gamma+\beta\gamma\delta+\alpha\beta\delta+\alpha\gamma\delta = \frac{-d}{a}$$

$$S_4 = \alpha\beta\gamma\delta = \frac{e}{a}$$

5. If α, β, γ, s are the roots of eqn $x^4+px^3+qx^3+qx^2+rx+dx$

Then
$$S_1 = \alpha+\beta+\gamma \text{ (or) } \Sigma\alpha = -p$$

$$S_2 = \Sigma\alpha\beta = q$$

$$S_3 = \Sigma\alpha\beta\gamma = -r$$

$$S_4 = \alpha\beta\gamma s = s$$

Problems

1. If α, β, γ are the roots of eqn $x^3+px^2+qx+p=0$ P.T: $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ (radian) except when $q=1$

Sol

G.T α, β, γ are the roots of eqn

$$x^3+px^2+qx+p=0$$

$[a=1,\ b=p,\ c=q,\ D=p]$

$$S_1 = \alpha+\beta+\gamma = -p$$

$$S_2 = \alpha\beta+\beta\gamma+\gamma\alpha = q$$

$$S_3 = \alpha\beta\gamma = -p$$

let $\tan^{-1}\alpha = \theta_1$ | $\tan^{-1}\beta = \theta_2$ | $\tan^{-1}\gamma = \theta_3$

$\alpha = \tan\theta_1$ | $\beta = \tan\theta_2$ | $\gamma = \tan\theta_3$

W.K.T

$$\tan(\theta_1+\theta_2+\dots+\theta_n) = \frac{s_1+s_3+s_5\dots}{1-s_2+s_4\dots}$$

$$= \frac{-p+p}{1-q} = 0$$

$$\theta_1+\theta_2+\dots+\theta_n = \tan^{-1}(0)$$

$$\Rightarrow \theta_1+\theta_2+\theta_3 = \tan^{-1}(0)$$

$$\boxed{\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi}$$

---

2. If $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$, $\tan\theta_4$ are the root of eqn $x^4+px^3+qx^2+rx+s=0$ show that

$$\tan(\theta_1+\theta_2+\theta_3+\theta_4) = \frac{r-p}{1-q+s}$$

sol

$$ax^4+bx^3+cx^2+Dx+E=0$$

$a=1$, $b=p$, $c=q$, $D=r$, $E=s$

$S_1 = \alpha+\beta+\gamma+\delta = -p$

$S_2 = \alpha\beta+\beta\gamma+\gamma\delta+\delta\alpha = q$

$S_3 = \alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha = -r$

$S_4 = \alpha\beta\gamma\delta = s$

W.K.T

$$\tan(\theta_1+\theta_2+\theta_3+\dots+\theta_n) = \frac{S_1-S_3}{1-S_2+S_4}$$

$$= \frac{-p+r}{1-q+s}$$

$$\boxed{\tan(\theta_1+\theta_2+\theta_3+\theta_4) = \frac{r-p}{1+s-q}}$$

---

3. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z = \pi/2$ ST $xy+yz+zx=1$

soll

$$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z = \pi/2 \rightarrow (1)$$

let ~~$\tan^{-1}x=\theta_1$~~ $\Rightarrow x=\tan\theta_1$

$\tan^{-1} y \Rightarrow \theta_2 \Rightarrow y = \tan\theta_2$

$\tan^{-1} z = \theta_3 \Rightarrow z = \tan\theta_3$

$\theta \Rightarrow \theta_1 + \theta_2 + \theta_3 = \pi/2$

Multply by tan θ on b.s

$\tan(\theta_1 + \theta_2 + \theta_3) = \tan \pi/2$

$\tan(\theta_1 + \theta_2 + \theta_3) = \infty$

$$\frac{S_1 - S_3}{1 - S_2} = \infty$$

$$\frac{S_1 - S_3}{1 - S_2} = \frac{1}{0}$$

$0 = 1 - S_2$

$1 - S_2 = 0$

$S_2 = 1$

$\tan\theta_1 \tan\theta_2 + \tan\theta_2 \tan\theta_3 + \tan\theta_3 \tan\theta_1 = 1$

$\boxed{xy + yz + zx = 1}$

---

4. Proove the eqn $\frac{ah}{\cos\theta} - \frac{bk}{\sin\theta} = a^2 - b^2$ has four roots and that sum of four Value of θ with satisfied its equal to odd multiplies of π (radians)

given//

$$\frac{ah}{\cos\theta} - \frac{bk}{\sin\theta} = a^2 - b^2 \quad \text{--- (1)}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= \frac{\frac{2\tan\theta/2}{1+\tan^2\theta/2}}{\frac{1-\tan^2\theta/2}{}}$$

$$\tan\theta = \frac{2\tan\theta/2}{1-\tan^2\theta/2}$$

put $t = \tan\theta/2$

$$\sin\theta = \frac{2t}{1-t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$① - \frac{ah}{\frac{1-t^2}{1+t^2}} - \frac{bk}{\frac{2t}{1+t^2}} = a^2-b^2$$

$$\frac{ah(1+t^2)}{1-t^2} - \frac{bk(1+t^2)}{2t} = a^2-b^2$$

$$2aht(1+t^2) - \frac{bk(1+t^2)}{2t} = a^2-b^2$$

$$2aht + 2aht^3 - bk + bkt^4 = (a^2-b^2)(2t-2t^3)$$

$$= 2a^2t - 2a^2t^3 - 2t^2 - t + 2b^2t^3$$

$$2aht + 2aht^3 - bk + bkt^4 = 2a^2t - 2a^2t^3 + 2b^2 + 2b^2t$$

$$bkt^4 + 2aht^3 + 2aht - bk - 2a^2t + 2a^2t^3 - 2b^2t^3 = 0$$

$$bkt^4 + t^3(2ah + 2a^2 - 2b^2) + 0t^2 + t(2ah - 2a^2 + 2b^2) - bk = 0$$

Compare this eqn to

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$a = bk$, $b = (2ah + 2a^2 - 2b^2)$, $c = 0$

$d = (2ah - 2a^2 + 2b^2)$, $e = -bk$

$$S_1 = \frac{-b}{a} = -\frac{(2ah + 2a^2 - 2b^2)}{bk}$$

$$S_2 = \frac{c}{a} = 0$$

$$S_3 = \frac{-d}{a} = \frac{-(2ah - 2a^2 + 2b^2)}{bk}$$

$$S_4 = \frac{e}{a} = \frac{-bk}{bk} = -1$$

$$\tan\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2}\right) = \frac{S_1 - S_3}{1 - S_2 + S_4}$$

$$= \frac{-\frac{(2ah + 2a^2 - 2b^2)}{bk} + \frac{(2ah - 2a^2 + 2b^2)}{bk}}{}$$

$$\tan \frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2} \; {}^{1-0-1} = \infty$$

$$\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2} = \tan^{-1}\infty$$

$$\frac{\theta_1 + \theta_2 + \theta_3 + \theta_4}{2} = n\pi + \pi/2.$$

$$\boxed{\theta_1 + \theta_2 + \theta_3 + \theta_4 = \pi(2\pi + 1)}$$

Sum of the Value of θ is equal to the add Multiples of π (radians)

Powers of sines and cosines of θ in terms of $\cos$ of θ

Formulae:

1. $(a+b)^n = a^n + nC_1 a^{n-1} b + nC_2 a^{n-2} b^2 + \dots + b^n$.
2. $(x+a)^n = x^n + nC_1 x^{n-1} a + nC_2 x^{n-2} a^2 + \dots + a^n$.
3. $(x-a)^n = x^n - nC_1 x^{n-1} a + nC_2 x^{n-2} a^2 + \dots + (-1)^n a^n$
4. $\left(x+\frac{1}{x}\right)^n = x^n + nC_1 x^{n-1}\left(\frac{1}{x}\right) + nC_2 x^{n-2}\left(\frac{1}{x}\right)^2 + \dots + \left(\frac{1}{x}\right)^n$.
5. $\left(x-\frac{1}{x}\right)^n = x^n - nC_1 x^{n-1}\left(\frac{1}{x}\right) + nC_2 x^{n-2}\left(\frac{1}{x}\right)^2 \dots + (-1)^n\left(\frac{1}{x}\right)^n$.
6. Let $x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = \cos\theta - i\sin\theta$$

$$\left(x+\frac{1}{x}\right) = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta.$$

$$\left(x-\frac{1}{x}\right) = \cos\theta + i\sin\theta - \cos\theta + i\sin\theta.$$

$$= 2i\sin\theta$$

7. Let $x^n = (\cos\theta + i\sin\theta)^n$

$$x^n = \cos n\theta + i\sin n\theta.$$

$$\frac{1}{x^n} = \cos n\theta - i\sin n\theta.$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta. \quad x^n - \frac{1}{x^n} = 2i\sin n\theta.$$

8. 1 Radian $= 57^\circ 17' 44.8''$.

Problems:

1. Expand $\sin^7\theta$ interms of $\sin\theta$ (or) Show that $-64\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$.

Sol:

$$x - \frac{1}{x} = 2i\sin\theta, \quad x^n - \frac{1}{x^n} = 2i\sin n\theta.$$

$$\left(x-\frac{1}{x}\right)^n = x^n - nC_1 x^{n-1}\left(\frac{1}{x}\right) + nC_2 x^{n-2}\left(\frac{1}{x^2}\right) + \cdots + (-1)^n\left(\frac{1}{x}\right)^n.$$

Here $n=7$.

$$\left(x-\frac{1}{x}\right)^7 = x^7 - 7C_1 x^{7-1}\left(\frac{1}{x}\right) + 7C_2 x^{7-2}\left(\frac{1}{x^2}\right) - 7C_3 x^{7-3}\left(\frac{1}{x^3}\right)$$

$$+ 7C_4 x^{7-4}\left(\frac{1}{x^4}\right) - 7C_5 x^{7-5}\left(\frac{1}{x^5}\right) + 7C_6 x^{7-6}\left(\frac{1}{x^6}\right)$$

$$- 7C_7 x^{7-7}\left(\frac{1}{x^7}\right)$$

$$= x^7 - 7x^6\left(\frac{1}{x}\right) + 21x^5\left(\frac{1}{x^2}\right) - 35x^4\left(\frac{1}{x^3}\right) + 35x^3\left(\frac{1}{x^4}\right) - 21x^2\left(\frac{1}{x^5}\right) + 7x\left(\frac{1}{x^6}\right) - \left(\frac{1}{x^7}\right).$$

$$= x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}.$$

$$\left(x - \frac{1}{x}\right)^7 = \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right).$$

$$(2i\sin\theta)^7 = 2i\sin 7\theta - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta)$$

$$(2i)^7(\sin\theta)^7 = 2i(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta)$$

$$-64\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta.$$

2. Prove that $2^8\cos^9\theta = \cos 9\theta + 9\cos 7\theta + 36\cos 5\theta + 84\cos 3\theta + 126\cos\theta$.

$5C_3$ $\quad \frac{5\times 4\times 3}{1\times 2\times 3}$

Sol:

W.K.T.

$$x + \frac{1}{x} = 2\cos\theta, \quad x^n + \frac{1}{x^n} = 2\cos n\theta.$$

$$\left(x + \frac{1}{x}\right)^n = x^n + nC_1 x^{n-1}\left(\frac{1}{x}\right) + nC_2 x^{n-2}\left(\frac{1}{x}\right)^2 + \cdots$$

Put $n = 9$,

$$\left(x + \frac{1}{x}\right)^9 = x^9 + 9C_1 x^8\left(\frac{1}{x}\right) + 9C_2 x^7\left(\frac{1}{x^2}\right) + 9C_3 x^6\left(\frac{1}{x}\right)^3 + 9C_4 x^5\left(\frac{1}{x}\right)^4 + 9C_5 x^4\left(\frac{1}{x}\right)^5 + 9C_6 x^3\left(\frac{1}{x}\right)^6 + 9C_7 x^2\left(\frac{1}{x}\right)^7 + 9C_8 x\left(\frac{1}{x}\right)^8 + 9C_9 x^0\left(\frac{1}{x}\right)^9.$$

$$= x^9 + 9x^8\left(\frac{1}{x}\right) + 36x^7\left(\frac{1}{x^2}\right) + 84x^6\left(\frac{1}{x^3}\right) + 126x^5\left(\frac{1}{x^4}\right) + 126x^4\left(\frac{1}{x^5}\right) + 84x^3\left(\frac{1}{x^6}\right) + 36x^2\left(\frac{1}{x^7}\right) + 9x\left(\frac{1}{x^8}\right) + \frac{1}{x^9}.$$

$$= x^9 + 9x^7 + 36x^5 + 84x^3 + 126x + 126\frac{1}{x} + 84\frac{1}{x^3} + 36\frac{1}{x^5} + 9\frac{1}{x^7} + \frac{1}{x^9}$$

$$\left(x+\frac{1}{x}\right)^9 = \left(x^9+\frac{1}{x^9}\right) + 9\left(x^7+\frac{1}{x^7}\right) + 36\left(x^5+\frac{1}{x^5}\right) + 84\left(x^3+\frac{1}{x^3}\right) + 126\left(x+\frac{1}{x}\right)$$

$$2^9\cos^9\theta = 2\cos 9\theta + 9(2\cos 7\theta) + 36(2\cos 5\theta) + 84(2\cos 3\theta) + 126(2\cos\theta)$$

$$2^9\cos^9\theta = 2[\cos 9\theta + 9\cos 7\theta + 36\cos 5\theta + 84\cos 3\theta + 126\cos\theta]$$

$$2^8\cos^9\theta = \cos 9\theta + 9\cos 7\theta + 36\cos 5\theta + 84\cos 3\theta + 126\cos\theta.$$

5. Prove that $16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

Sol:

W.K.T

$$x - \frac{1}{x} = 2i\sin\theta, \quad x^n - \frac{1}{x^n} = 2i\sin n\theta.$$

$$\left(x-\frac{1}{x}\right)^n = x^n - nC_1 x^{n-1}\left(\frac{1}{x}\right) + nC_2 x^{n-2}\left(\frac{1}{x}\right)^2 + \cdots$$

Put $n=5$.

$$\left(x-\frac{1}{x}\right)^5 = x^5 - 5C_1 x^4\left(\frac{1}{x}\right) + 5C_2 x^3\left(\frac{1}{x}\right)^2 - 5C_3 x^2\left(\frac{1}{x}\right)^3 + 5C_4 x^1\left(\frac{1}{x}\right)^4 + 5C_5 x^0\left(\frac{1}{x}\right)^5$$

$$= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$$

$$\left(x-\frac{1}{x}\right)^5 = \left(x^5-\frac{1}{x^5}\right) - 5\left(x^3-\frac{1}{x^3}\right) + 10\left(x-\frac{1}{x}\right).$$

$$(2i\sin 5)^5 = (2i\sin 5\theta) - 5(2i\sin 3\theta) + 10(2i\sin\theta)$$

$(2i)^5 \sin^5\theta = 2i\,[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$.

$16 i^4 \sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$.

Hence Proved.

4. Expand $\cos^7\theta \sin^3\theta$ interms of $\sin\theta$.

Sol:

W.K.T.

$$x+\frac{1}{x} = 2\cos\theta, \quad x-\frac{1}{x} = 2i\sin\theta.$$

$$\left(x+\frac{1}{x}\right)^7\left(x-\frac{1}{x}\right)^3 = \left(x+\frac{1}{x}\right)^3\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3.$$

$$= \left(x^2-\frac{1}{x^2}\right)^3\left(x+\frac{1}{x}\right)^4.$$

$$= \left[(x^2)^3 - 3C_1(x^2)^2\left(\frac{1}{x^2}\right) + 3C_2(x^2)\left(\frac{1}{x^2}\right)^2 - 3C_3(x^3)^0\left(\frac{1}{x^2}\right)^3\right]\left(x+\frac{1}{x}\right)^4.$$

$$= \left[x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right]\left[x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4}\right]$$

$$= x^{10} + 4x^8 + 6x^6 + 4x^4 + x^2 - 3x^6 - 12x^4 - 18x^2$$
$$-12 - \frac{3}{x^2} + 3x^2 + 12 + \frac{24}{x^2} + \frac{12}{x^2} + \frac{3}{x^6} - \frac{1}{x^2}$$
$$- \frac{4}{x^4} - \frac{6}{x^6} - \frac{4}{x^8} - \frac{1}{x^{10}}.$$

$$= \left(x^{10}-\frac{1}{x^{10}}\right) + 4\left(x^8-\frac{1}{x^8}\right) + 3\left(x^6-\frac{1}{x^6}\right) - 8\left(x^4-\frac{1}{x^4}\right)$$
$$-14\left(x^2-\frac{1}{x^2}\right).$$

$2^7 \cos^7\theta \, 2^3 i^3 \sin^3\theta = 2i\sin 10\theta + 4(2i\sin 8\theta) + 3(2i\sin 6\theta)$
$- 8(2i\sin 4\theta) - 14(2i\sin 2\theta)$.

$2^{10} i^3 \cos^7\theta \sin^3\theta = 2i[\sin 10\theta + 4\sin 8\theta + 3\sin 6\theta - 8\sin 4\theta$
$- 14\sin 2\theta]$

$\cos^7\theta \sin^3\theta = \frac{-1}{2^9}[\sin 10\theta + 4\sin 8\theta + 3\sin 6\theta -$
$8\sin 4\theta - 14\sin 2\theta]$.

5. Prove that $\sin^3\theta \cos^4\theta = \frac{1}{2^6}[\sin 7\theta + \sin 5\theta -$
$3\sin 3\theta - 3\sin\theta]$.

Sol:

$x + \frac{1}{x} = 2\cos\theta$; $x - \frac{1}{x} = 2i\sin\theta$.

$\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3 = \left(x+\frac{1}{x}\right)^1\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^3$.

$= \left(x^2 - \frac{1}{x^2}\right)^3\left(x+\frac{1}{x}\right)$.

$= \left[(x^2)^3 - 3C_1(x^2)^2\left(\frac{1}{x^2}\right) + 3C_2(x^2)\left(\frac{1}{x^2}\right)^2 -\right.$
$\left. 3C_3(x^2)^0\left(\frac{1}{x^2}\right)^3\right]\left(x+\frac{1}{x}\right)$.

$= \left[x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right]\left[x + \frac{1}{x}\right]$.

$= x^7 + x^5 - 3x^3 - 3x + \frac{3}{x} + \frac{3}{x^3} - \frac{1}{x^5} - \frac{1}{x^7}$.

$\left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3 = \left(x^7 - \frac{1}{x^7}\right) + \left(x^5 - \frac{1}{x^5}\right) - 3\left(x^3 - \frac{1}{x^3}\right)$
$- 3\left(x - \frac{1}{x}\right)$.

$2^4\cos^4\theta . 2^3 i^3 \sin^3\theta = 2i\sin 7\theta + 2i\sin 5\theta -$
$3(2i\sin 3\theta) - 3(2i\sin\theta)$.

$2^{7}i^{3} \sin^3\theta \cos^4\theta = 2i\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right].$

$-\sin^3\theta \cos^4\theta = \frac{1}{2^6}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right].$

Hence proved.

Formula.

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots$$

1. Evaluate $\lim_{x\to 0} \frac{\tan 2x - 2\tan x}{x^3}$

Sol:

$\theta = 2x.$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$$

$$\frac{\tan 2x - 2\tan x}{x^3} = \frac{\left[2x + \frac{(2x)^3}{3} + \frac{2(2x)^5}{15} + \cdots\right] - 2\left[x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\right]}{x^3}$$

$$= \frac{\left[2x + \frac{8x^3}{3} + \frac{64}{15}x^5 + \cdots\right] - \left[2x + \frac{2x^3}{3} + \frac{4x^5}{15} + \cdots\right]}{x^3}.$$

$$\frac{\frac{x^3}{3} + \frac{64x^5}{15} + \cdots - 2x - \frac{2x^3}{3} - \frac{4x^5}{15} + \cdots}{x^3}$$

$$= \frac{\frac{6x^3}{3} + \frac{60x^5}{15}}{x^3} \Rightarrow \frac{x^3\left(\frac{6}{x} + \frac{60x^2}{15}\right)}{x^3}$$

$$\frac{\tan 2x - 2\tan x}{x^3} = \frac{6}{3} + \frac{60x^2}{15}.$$

$$\lim_{x \to 0} \frac{\tan 2x - 2\tan x}{x^3} = \frac{6}{3} + \frac{60(0)}{15} = 2.$$

$$\lim_{x \to 0} \frac{\tan 2x - 2\tan x}{x^3} = 2.$$

2. Evaluate $\lim_{x \to 0} \frac{\sin x - x\cos x}{x^3}$.

Sol

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \frac{\theta^6}{6!} + \cdots$$

$$\frac{\sin x - x\cos x}{x^3} = \frac{\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right] - x\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right]}{x^3}$$

$$= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + x + \frac{x^3}{2!} - \frac{x^5}{4!}}{x^3}$$

$$= \frac{-\frac{x^3}{6} + \frac{x^5}{120} - \cdots + \frac{x^3}{2} - \frac{x^5}{24}}{x^3}$$

$$= \frac{x^3}{x^3}\left(\frac{-1}{6} + \frac{x^2}{120} - \cdots + \frac{1}{2} - \frac{x^2}{24}\right).$$

$$= \lim_{x \to 0}\left(\frac{-1}{6} + \frac{x^2}{120} \cdots\cdots + \frac{1}{2} - \frac{x^2}{24}\right).$$

$$= \frac{-1}{6} + \frac{1}{2} = \frac{1}{3}.$$

3. Evaluate $\lim_{x \to 0} \dfrac{\sin x - x + \frac{x^6}{5}}{x^5}$.

Sol:

W.K.T

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\frac{\sin x - x + \frac{x^6}{5}}{x^5} = \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)}{x^5}.$$

$$= \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots x + \frac{x^6}{5}}{x^5}$$

$$= \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - x + \frac{x^6}{5}}{x^5} \Rightarrow \frac{x^5\left(\frac{-1}{6x^2} + \frac{1}{120} + \frac{x}{5}\right)}{x^5}.$$

$$= \frac{\frac{-1}{6x^2} + \frac{1}{120} + \frac{x}{5}}{x^5}.$$

$$\lim_{x \to 0} \frac{\sin x - x + \frac{x^6}{5}}{x^5} = \frac{-1}{6(0)} + \frac{1}{120} + \frac{0}{5}$$

Ans $= \dfrac{1}{120}$.

4. Evaluate $\lim_{x \to 0} \dfrac{\sin 2x - 2\sin x}{x^3}$.

Sol:
W.K.T. $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + $

$$\frac{\sin 2x - 2\sin x}{x^3} = \frac{\left[2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \cdots\right] - 2\left[x - \frac{x^3}{6} + \frac{x^5}{120}\right]}{x^3}$$

$$= \frac{2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \cdots - 2x + \frac{2x^3}{6} - \frac{2x^5}{120}}{x^3}$$

$$= \frac{x^3\left(-\frac{8}{6} + \frac{32x^2}{120} + \frac{2}{6} - \frac{2x^2}{120}\right)}{x^3} = -\frac{8}{6} + \frac{32x^2}{120} + \frac{2}{6} - \frac{2x^2}{120}.$$

$$\lim_{x\to 0} \frac{\sin 2x - 2\sin x}{x^3} = \frac{-8}{6} + \frac{32(0)}{120} + \frac{2}{6} - \frac{2(0)}{120}$$

Ans :- -1

5. Evaluate $\lim_{x\to 0} \frac{x - \sin x}{x^3}$

Sol
W.K.T $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$

$$\frac{x - \sin x}{x^3} = \frac{x - x + \frac{x^3}{6} - \frac{x^5}{120} + \cdots}{x^3} = \frac{x^3\left(\frac{1}{6} - \frac{x^2}{120}\right)}{x^3}.$$

$$\lim_{x\to 0} \frac{x - \sin x}{x^3} = \frac{1}{6} - \frac{0}{120}.$$

Ans :- $\frac{1}{6}$

6. Evaluate $\lim_{x\to 0} \frac{\sin x - \tan x}{x^2}$

Sol:
W.K.T. $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$

$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots$

$$\frac{\sin x - \tan x}{x^2} = \frac{\left(x - \frac{x^3}{6} + \frac{x^5}{120} \cdots\right) - \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\right)}{x^2}$$

$$= \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \dots - x - \frac{x^3}{3} - \frac{2x^5}{15} - \dots}{x^2}$$

$$= \frac{x^2\left(\frac{-x}{6} + \frac{x^3}{120} - \frac{x}{3} - \frac{2x^3}{15}\right)}{x^2}$$

$$\lim_{x \to 0} \frac{\sin x - \tan x}{x^2} = \frac{-0}{6} + \frac{0}{120} - \frac{0}{3} - \frac{2(0)}{15} = 0.$$

7. If $\frac{\tan\theta}{\theta} = \frac{2524}{2523}$ show that value of θ is $1^\circ 58'$

Sol:

$$\frac{\tan\theta}{\theta} = \frac{2524}{2523}$$

$$\frac{\theta + \frac{\theta^3}{3} + \frac{\theta^5}{15} + \dots}{\theta} = \frac{2524}{2523} \Rightarrow \frac{\theta\left[1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \dots\right]}{\theta} = \frac{2524}{2523}$$

$$\Rightarrow 1 + \frac{\theta^2}{3} = \frac{2524}{2523} \Rightarrow \frac{\theta^2}{3} = \frac{2524}{2523} - 1 \Rightarrow \frac{\theta^2}{3} = \frac{1}{2523}.$$

$$\theta^3 = \frac{3}{2523} \Rightarrow \theta^2 = 0.00118 \Rightarrow \theta = \sqrt{0.00118}.$$

$$\theta = 0.03435 \times 57^\circ 17' 448''$$

$$\theta = 1^\circ 58'$$

8. Find the approximate value of θ in radians if $\frac{\sin\theta}{\theta} = \frac{863}{864}$

Sol:

$$\frac{\sin\theta}{\theta} = \frac{863}{864} \Rightarrow \frac{\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right]}{\theta} = \frac{863}{864}.$$

$$\frac{\theta\left[1 - \frac{\theta^3}{6} + \frac{\theta^5}{120}\right]}{\theta} = \frac{863}{864} \Rightarrow 1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} + \dots = \frac{863}{864}.$$

Neglect higher power.

$$1 - \frac{\theta^2}{6} = \frac{863}{864} \Rightarrow \frac{-\theta^2}{6} = \frac{863}{864} - 1$$

$$\frac{-\theta^2}{6} = \frac{-1}{864} \Rightarrow \theta^2 = \frac{6}{864}$$

$$\theta^2 = 0.006944.$$

$$\theta = 0.08333.$$

UNIT-1.

1. Express $\frac{\sin 6\theta}{\sin\theta\cos\theta}$ interms of $\cos\theta$ and $\sin\theta$.

Sol: W.K.T

$\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + \ldots$

$n=6$

$\sin 6\theta = 6C_1 \cos^5\theta \sin\theta - 6C_3 \cos^3\theta \sin^3\theta + 6C_5 \cos\theta \sin^5\theta$

$= 6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta$.

Divide by $\sin\theta\cos\theta$.

$\frac{\sin 6\theta}{\sin\theta\cos\theta} = \frac{6\cos^5\theta \sin\theta}{\sin\theta\cos\theta} - \frac{20\cos^3\theta \sin^3\theta}{\sin\theta\cos\theta} + \frac{6\cos\theta \sin^5\theta}{\sin\theta\cos\theta}$

$= 6\cos^4\theta - 20\cos^2\theta \sin^2\theta + 6\sin^4\theta$ —— (1).

Interms of $\cos\theta$.

$\frac{\sin 6\theta}{\sin\theta\cos\theta} = 6\cos^4\theta - 20\cos^2\theta \cancel{(s-}\sin^2\theta) + 6(1-\cos^2\theta)^2$.

$= 6\cos^4\theta - 20\cos^2\theta(1-\cos^2\theta) + 6(1-2\cos^2\theta+\cos^4\theta)$.

$= 6\cos^4\theta - 20\cos^2\theta + 20\cos^4\theta + 6 - 12\cos^2\theta + 6\cos^4\theta$.

$\frac{\sin 6\theta}{\sin\theta\cos\theta} = 32\cos^4\theta - 32\cos^2\theta + 6$.

Interms of $\sin\theta$.

$\frac{\sin 6\theta}{\sin\theta\cos\theta} = 6(1-2\sin^2+\sin^4\theta) - 20\sin^2\theta(1-\sin^2\theta) + 6\sin^4\theta$.

$= 6 - 12\sin^2\theta + 6\sin^4\theta - 20\sin^2\theta + 20\sin^4\theta + 6\sin^4\theta$.

$\frac{\sin 6\theta}{\sin\theta\cos\theta} = 6 - 32\sin^2\theta + 32\sin^4\theta$.

2. Express $\frac{\cos 5\theta}{\cos\theta}$ interms of $\sin\theta$ and $\cos\theta$.

Sol:

W.K.T.

$\cos n\theta = \cos^n\theta - nC_2 \cos^{n-2}\theta \sin^2\theta + nC_4 \cos^{n-4}\theta \sin^4\theta \ldots$

$n=5$.

$\cos 5\theta = \cos^5\theta - 5C_2 \cos^3\theta \sin^2\theta + 5C_4 \cos\theta \sin^4\theta$.

$\frac{\cos 5\theta}{\cos\theta} = \frac{\cos^5\theta}{\cos\theta} - \frac{10\cos^3\theta \sin^2\theta}{\cos\theta} + \frac{5\cos\theta \sin^4\theta}{\cos\theta}$

$= \cos^4\theta - 10\cos^2\theta \sin^2\theta + 5\sin^4\theta$.

In terms of $\sin\theta$.

$\frac{\cos 5\theta}{\cos\theta} = (1-\sin^2\theta)^2 - 10(1-\sin^2\theta)\sin^2\theta + 5\sin^4\theta$.

$= 1 - 2\sin^2\theta + \sin^4\theta - 10\sin^2\theta + 10\sin^4\theta + 5\sin^4\theta$

$\frac{\cos 5\theta}{\cos\theta} = 16\sin^4\theta - 12\sin^2\theta + 1$

In terms of $\cos\theta$.

$\frac{\cos 5\theta}{\cos\theta} = \cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + 5(1-\cos^2\theta)^2$

$= \cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 5 - 10\cos^2\theta - 5\cos^4\theta$.

$\frac{\cos 5\theta}{\cos\theta} = 16\cos^4\theta - 20\cos^2\theta + 5$.

3. Express $\frac{\sin 7\theta}{\sin\theta}$ interms of $\sin\theta$ and $\cos\theta$

Sol:

W.K.T $\sin n\theta = nC_1\cos^{n-1}\theta\sin\theta - nC_3\cos^{n-3}\theta\sin^3\theta + \cdots$

$n = 7$.

$\sin 7\theta = 7C_1\cos^6\theta\sin\theta - 7C_3\cos^4\theta\sin^3\theta + 7C_5\cos^2\theta\sin^5\theta - 7C_7\cos^0\theta\sin^7\theta$.

$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$.

$\frac{\sin 7\theta}{\sin\theta} = \frac{7\cos^6\theta\sin\theta}{\sin\theta} - \frac{35\cos^4\theta\sin^3\theta}{\sin\theta} + \frac{21\cos^2\theta\sin^5\theta}{\sin\theta} - \frac{\sin^7\theta}{\sin\theta}$

$= 7\cos^6\theta - 35\cos^4\theta\sin^2\theta + 21\cos^2\theta\sin^4\theta - \sin^6\theta$.

Interms of $\sin\theta$.

$\frac{\sin 7\theta}{\sin\theta} = 7(1-\sin^2\theta)^3 - 35(1-\sin^2\theta)^2\sin^2\theta + 21(1-\sin^2\theta)\sin^4\theta - \sin^6\theta$.

$= 7 - 21\sin^2\theta + 21\sin^4\theta - 7\sin^6\theta - 35\sin^2\theta + 70\sin^4\theta - 35\sin^6\theta + 21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$.

$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta + 64\sin^6\theta$.

Interms of $\cos\theta$.

$\frac{\cos 7\theta}{\cos\theta} = 7\cos^6\theta - 35\cos^4\theta(1-\cos^2\theta) + 21\cos^2\theta(1-\sin^2\theta)^2 - (1-\sin^2\theta)^3$

$= 7\cos^6\theta - 35\cos^4\theta + 35\cos^6\theta + 21\cos^2\theta(1 - 2\cos^2\theta + \cos^4\theta) - (1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$.

$= 7\cos^6\theta - 35\cos^4\theta + 35\cos^6\theta + 21\cos^2\theta - 42\cos^4\theta + 21\cos^6\theta - 1 + 3\cos^2\theta - 3\cos^4\theta + \cos^6\theta$.

$\frac{\cos 7\theta}{\cos\theta} = 64\cos^6\theta - 80\cos^4\theta + 24\cos^2\theta - 1$.