MARUDHAR KESARI JAIN COLLEGE FOR WOMEN (AUTONOMOUS)

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1st B.Sc. Statistics – Semester - I

E-Notes (Study Material)

Core Course -2:MATHEMATICS FOR STATISTICS	**Code:24UMAA17**
Unit: 2 - Series: Summation and approximations related to Binomial, Exponential and Logarithmic series -Taylor‘s series. **(12 Hours)**	
Learning Objectives: To gain the knowledge of Series	
Course Outcome: Demonstrate the knowledge to determine the sums, expansion and approximation of series including binomial, exponential, logarithmic.	

Overview:

In mathematics, we can describe a series as adding infinitely many numbers or quantities to a given starting number or amount. We use series in many areas of mathematics, even for studying finite structures, for example, combinatory for forming functions. The knowledge of the series is a significant part of calculus and its generalization as well as mathematical analysis. Apart from these applications in mathematics, infinite series are also extensively used in different quantitative disciplines such as statistics, physics, computer science, finance, etc.

1. Summation and approximations related to Binomial Series.
2. Summation and approximations related to Exponential series.
3. Summation and approximations related to Logarithmic series.
4. Taylor‘s series.

15/07/2024

Binomial Series.

When n is a rational number $(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \dots + x$. such that $-1 < x < 1$

Result:-

1. $(1-x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \dots$

2. $(1-x)^{-n} = 1 - \frac{(-n)}{1}x + \frac{(-n)(-n)^{-1}}{1\cdot 2}x^2$

When $-1 < x < 1$ and n is a positive integer.

3. $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

4. $\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

5. $\frac{1}{(1-x)^3} = (1-x)^{-3} = \frac{1}{1\cdot 2}[1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + 4\cdot 5x^3 + \dots]$

6. $\frac{1}{(1-x)4} = (1-x)^{-4} = \frac{1}{1\cdot 2\cdot 3}[1\cdot 2\cdot 3 + 2\cdot 3\cdot 4x + 3\cdot 4\cdot 5x^2 + \dots]$

7. $\frac{1}{(1-x)^n} = (1-x)^{-n} = \frac{1}{1\cdot 2\cdot 3\dots(n-1)}[1\cdot 2\dots(n-1) + 2\cdot 3\cdot 4 \dots nx + \dots]$

8. $\dfrac{1}{(1+x)} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

9. $\dfrac{1}{(1+x)^2} = (1+x)^{-2} = \dfrac{1}{1\cdot 2}\left[1\cdot 2 - 2\cdot 3x + 3\cdot 4x^2 - \cdots\right]$

10. $\dfrac{1}{(1+x)^3} = (1+x)^{-3} = \dfrac{1}{1\cdot 2}\left[1\cdot 2 - 2\cdot 3x + 3\cdot 4x^2 \cdots\right]$

11. $\dfrac{1}{(1+x)^4} = (1+x)^{-4} = \dfrac{1}{1\cdot 2\cdot 3}\left[1\cdot 2\cdot 3 - 2\cdot 3\cdot 4x + 3\cdot 4\cdot 5x^2 \cdots\right]$

12. $\dfrac{1}{(1+x)^n} = (1+x)^{-n} = \dfrac{1}{1\cdot 2\cdot 3\cdots n}\left[1\cdot 2\cdot 3\cdots(n-1) - 2\cdot 3\cdots nx + \cdots\right]$

When n is a positive number.

1. Find the Coefficient of x^n in the expansion of $\frac{1}{(1-x^2)}$

Soln:- $\frac{1}{(1-x^2)} = (1-x^2)^{-1}$

$[\because (1-x^2)^{-1} = 1+x+x^2+x^3+\cdots x^n.]$

$(1-x^2)^{-1} = 1+x^2+(x^2)^2+(x^2)^3+\cdots(x^2)^n.$

$= 1+x^2+x^4+x^6+\cdots+x^{2n}.$

Coefficient of $x^n = \begin{cases} 1 \text{ if } n \text{ is even} \\ 0 \text{ if } n \text{ is odd.} \end{cases}$

2. Find the Coefficient of x^{2n} in the expansion of

$(1-x^2)^{-1}$

Soln:- $(1-x)^{-1} = 1+x+x^2+\cdots+x^n.$

$x = x^2$

$(1-x^2)^{-1} = 1+x^2+(x^2)^2+(x^2)^3+\cdots+(x^2)^n$

$= 1+x^2+x^4+x^6+\cdots+x^{2n}$

$\therefore$ Coefficient of $x^{2n} = 1$.

3. Find the Coefficient of x^2 in the expansion of $(1+x)^{-3}$.

Soln:- $(1+x)^{-3} = \frac{1}{1\cdot 2}[1\cdot 2 - 2\cdot 3x + 3\cdot 4x^2 + \cdots]$

Coefficient of $x^2 = \frac{1}{1\cdot 2}[3\cdot \overset{2}{\cancel{4}}]$

Coefficient of $x^2 = 6$..

4. Find the coefficient of x^n in $\frac{1}{1-2x} + \frac{1}{1-3x}$.

Sol: $(1-x)^{-1} = 1+x+x^2+\cdots+x^n$.

$(x=2x), (x=3x)$

$\frac{1}{1-2x} + \frac{1}{1-3x} = (1-2x)^{-1} + (1-3x)^{-1}$

$= (1+2x)+(2x)^2+\cdots(2x)^n + (1+3x+(3x)^2+\cdots (3x)^n$

$= ($

Coefficient of $x^n = 2^n + 3^n$

Coefficient of x^n in binomial expansion.

5. (10) Find the sum of the following series.

i) $1+2(1/2)+3(1/2)^2+\cdots+\infty$

ii) $1+2(1/3)+3(1/3)+4(1/3)+\cdots+\infty$

Sol: (i) $(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots$

$x = 1/2$

$1+2(1/2)+3(1/2)^2+4(1/2)^3+\cdots = (1-1/2)^{-2}$

$= \left(\frac{2-1}{2}\right)^{-2}$

$= \left(\frac{1}{2}\right)^{-2}$

$= (2)^2$

$= 4.$

(ii) $x = 1/2$

$1+2(1/3)+3\left(\frac{1}{3}\right)^2+4(1/3)^3 = (1-1/3)^{-2}$

$= \left(\frac{3-1}{3}\right)^{-2}$

$= \left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2$

$= \frac{9}{4}$

2. Find the coefficient of x^n in expansion of

$[1+2x+3x^2+4x^3+\dots]^2$

Soln: $1+2x+3x^2+4x^3+\dots = (1-x)^{-2}$

$[1+2x+3x^2+4x^3+\dots]^2 = [(1-x)^{-2}]^2$

$= (1-x)^{-4}$

$(1-x)^{-4} = \frac{1}{1\cdot2\cdot3}[1\cdot2\cdot3 + 2\cdot3\cdot4\,x + 3\cdot4\cdot5\,x^2 + \dots + x^n]$

$= \frac{1}{1\cdot2\cdot3}[(n+1)(n+2)(n+3)x^n]$

Coefficient of $x^n = \frac{1}{1\cdot2\cdot3}[(n+1)(n+2)(n+3)x^n]$

3. Write the $(n+1)$ termed in the expansion of

$(3-2x)^{-2}$ where x is small

Soln: $(3-2x)^{-2} = 3^{-2}\left[1-\frac{2x}{3}\right]^{-2}$

$= \frac{1}{3^2}\left(1-\frac{2x}{3}\right)^{-2}$

$= \frac{1}{9}\left(1-\frac{2x}{3}\right)^{-2}$

$\because (1-x)^{-2} = 1+2x+3x^2+4x^3+\dots$ —— (1)

Put $x = 2x/3$ in (1)

$\frac{1}{9}\left(1-\frac{2x}{3}\right)^{-2} = \frac{1}{9}\left[1+2\left(\frac{2x}{3}\right)+3\left(\frac{2}{3}x\right)^2+4\left(\frac{2}{3}x\right)^3+\dots + (n+1)\left(\frac{2}{3}x\right)^n\right]$

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$$= \frac{1}{9}\left[1+2\left(\frac{2}{3}x\right)+3\left(\frac{2^2}{3^2}x^2\right)+4\left(\frac{2^3}{3^3}x^3\right)\right.$$

$$\left.+\dots(n+1)\left(\frac{2^n}{3^n}x^n\right)\right]$$

$$(n+1)^{th} \text{ term} = \frac{1}{9}\left[\frac{2^n}{3^n}x^n\right]$$

$$= \frac{1}{3^2}\left[\frac{2^n}{3^n}x^n\right]$$

$$= \left(\frac{2^n}{3^{n+2}}\right)x^n.$$

4. Find the coefficient of x^2 in the expansion of

$$\left(1+\frac{2}{3x}\right)^{3/2}$$

Soln: $\because (1+x)^n = 1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\dots$ —— ①

$$\left(1+\frac{2}{3}x\right)^n = 1+\frac{3/2}{1}\left(\frac{2}{3^2}\right)+\frac{3/2(3/2-1)}{1\cdot 2}\left(\frac{2}{3}x\right)^2+\frac{3/2(3/2-1)(3/2-2)}{1\cdot 2\cdot 3}\left(\frac{2}{3}\right.$$

$$\text{Coefficient of } x^2 = \frac{3/2\,(3/2-1)}{1\cdot 2}\left(\frac{2}{3}x\right)^2$$

$$= \frac{3/2\,(1/2)}{2}\left(\frac{4}{9}\right)$$

$$= \frac{\left(\frac{3}{4}\right)}{2}\left(\frac{4}{9}\right)$$

$$= \frac{\cancel{3}^{1}}{\cancel{8}_{2}}\times\frac{\cancel{4}^{1}}{\cancel{9}_{3}}$$

$\therefore$ Coefficient of $x^2 = \frac{1}{6}$ //

5. Find the coefficient of x^6 in the expansion of

$$\frac{1}{(1-x^2)^{+3}}$$

Soln: We know that :-

$$\because (1-x)^{-3} = \frac{1}{1\cdot2}\left[1\cdot2 + 2\cdot3x + 3\cdot4x^2 + 4\cdot5x^3 + \cdots\right] \text{ --- } (1)$$

Put $x = x^2$ in (1)

$$(1-x^2)^{-3} = \frac{1}{1\cdot2}\left[1\cdot2 + 2\cdot3x^2 + 3\cdot4\,x^4 + 4\cdot5x^6 + 5\cdot6x^8 + \cdots\right]$$

$$\text{Coefficient of } x^6 = \frac{1}{1\cdot\cancel{2}}\left[\overset{2}{\cancel{4}}\cdot5\right]$$

Coefficient of $x^6 = 10$

6. If x is small, what is the value of a if $\sqrt{x^2+4} - \sqrt{x^2+1} = 1 - ax^2$ nearly.

Soln: $\sqrt{x^2+4} - \sqrt{x^2+1} = (x^2+4)^{1/2} - (x^2+1)^{1/2}$

$$= \underset{2}{\cancel{4^{1/2}}}\left(1+\frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

$$= 2\left(1+\frac{x^2}{4}\right)^{1/2} - (1-x^2)^{1/2}$$

We know that

$$\because (1+x)^n = \left[1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot2}x^2 + \cdots\right] \text{ --- } (1)$$

Put $x = x^2/4$, $n = 1/2$ in (1)

$$\left(1+x^2/4\right)^{1/2} = 2\left[1 + \frac{1/2}{1}\left(\frac{x^2}{4}\right) + \frac{1/2\,(1/2-1)}{1\cdot2}\cdot\left(\frac{x^2}{4}\right)^2 + \cdots\right] -$$

$$\left(1 + \frac{\frac{1}{2}}{1}x^2 + \frac{1/2\,(1/2-1)}{1\cdot2}(x^2)^2 + \cdots\right]$$

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$$= 2\left(\left[1+\frac{1}{2}\left(\frac{x^2}{4}\right)\right] - \left[1+\frac{1}{2}(x^2)\right]\right) \text{ nearly}$$

$$= 2\left(\left[1+\frac{x^2}{8}\right] - 1 - \tfrac{1}{2}x^2\right)$$

$$= 2 + \frac{\cancel{2}^{1}x^2}{\cancel{8}_4} - 1 - \frac{1}{\cancel{2}}x^2$$

$$= 1 + \frac{x^2}{4} - \frac{x^2}{2}$$

$$= 1 + \frac{2x^2 - 4x^2}{8}$$

$$= 1 + \frac{-\cancel{4}x^2}{\cancel{8}}$$

$$= 1 - \frac{x^2}{4}$$

$$\therefore \sqrt{x^2+4} - \sqrt{x^2+1} = 1 - \frac{x^2}{4}$$

$$\boxed{a = 1/4}$$

7. When x is small prove that $(1-x)^{-1/2} + (1-x)^{1/2} = 2 + x + \frac{x^2}{4}$ nearly.

Soln: We know that :-

$$(1-x)^n + (1+x)^n = 1 - \frac{(n)x}{1} + \frac{(n)(n-1)x^2}{1\cdot 2} + \cdots \quad \text{--- (1)}$$

$$+ \left[1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \cdots\right.$$

$$(1-x)^{-1/2} + (1+x)^{1/2} = \frac{1(-1/2)x}{1} + \frac{(-1/2)(-1/2-1)x^2}{1\cdot 2} + \cdots$$

$$+ 1 + \frac{1/2}{1}x + \frac{1/2(1/2-1)}{1\cdot 2}x^2 + \cdots$$

$$= 1 + \frac{1}{2}x + \frac{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)x^2}{2} + \cdots$$

$$+ 1 + \frac{1}{2}x + \frac{1/2(-1/2)}{2}x^2 + \cdots$$

$= 2+\frac{1}{2}x+\frac{1}{8}x^2+1+\frac{1}{2}x+\frac{1}{8}x^2$

$= 2+x-\frac{\cancel{2}x^2}{\cancel{8}\,4}$

$= 2+x+\frac{x^2}{4}$ nearly

(8.) When x is small prove that

$\sqrt{x^2+4}-\sqrt{x^2+1} = 1-\frac{1}{4}x^2+\frac{7}{64}x^4$ nearly.

Sol: $\sqrt{x^2+4}-\sqrt{x^2+1} = (x^2+4)^{1/2}-(x^2+1)^{1/2}$

$= \cancel{4}^2\left(1+\frac{x^2}{4}\right)^{1/2}-(1+x^2)^{1/2}$

$= 2\left(1+\frac{x^2}{4}\right)^{1/2}-\left(1+x^2\right)^{1/2}$

We know that :-

$(1+x)^n = 1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 2}x^3+\cdots\cdots$

Put $x=\frac{x^2}{4}$ and $x=x^2$, $n=\frac{1}{2}$ in ①

$\left(1+\frac{x^2}{4}\right)-(1-x^2)^{1/2} = 2\left[1+\frac{\frac{1}{2}}{1}\left(\frac{x^2}{4}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}\left(\frac{x^2}{4}\right)^2+\cdots\right]$

$-\left[1+\frac{\frac{1}{2}}{1}x^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}(x^2)^2+\cdots\right]$

$= 2\left[1+\frac{x^2}{8}+\frac{\frac{1}{2}(-1/2)}{1\cdot 2}\,\frac{x^4}{16}+\cdots\right]$

$-\left[1+\frac{x^2}{2}+\frac{\frac{1}{2}(-1/2)}{1\cdot 2}x^4+\cdots\right]$

$= 2\left[1+\frac{x^2}{8}-\frac{1}{8}\cdot\frac{x^4}{16}\cdots\right]-\left[1+\frac{x^2}{2}-\frac{1}{8}x^4+\cdots\right]$

$= 2+\frac{\cancel{2}x^2}{\cancel{8}\,4}-\frac{\cancel{2}}{\cancel{8}\,4}\cdot\frac{x^4}{16}-1-\frac{x^2}{2}-\frac{1}{8}x^4$

$$= 2 + \frac{x^2}{4} - \frac{x^4}{64} - 1 - \frac{x^2}{2} - \frac{x^4}{8}$$

$$= 1 + x^2\left[\frac{1}{4} - \frac{1}{2}\right] - x^4\left[\frac{1}{64} - \frac{1}{8}\right]$$

$$= 1 + x^2\left[\frac{2-4}{8}\right] - x^4\left[\frac{1-8}{64}\right]$$

$$= 1 + x^2\left[\frac{-\cancel{2}^{1}}{\cancel{8}_{4}}\right] - x^4\left[\frac{-7}{64}\right]$$

$$= 1 - \frac{1}{4}x^2 + \frac{7}{64}x^4 \text{ nearly.}$$

9. Find the coefficient of x^n in the expansion of $(1+x+x^2+x^3+\ldots)^{-n}$.

Soln:

$$(1+x+x^2+x^3+\cdots)^{-n} = ((1-x)^{-1})^{-n}$$

$$= (1-x)^n.$$

$$(1-x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 \ldots$$

$$+ \frac{(-1)n(n-1)(n-2)\cdots 2}{1\cdot 2\cdot 3 \quad (n-2)(n-1)n} x^n$$

$$= 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \cdots\cdots (-1)x^n$$

Coefficient of $x^n = (-1)^n$.

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Summation.

2M

Binomial Series.

The formula for finding the sum of binomial series.

$$\therefore (1-x)^{-p/q} = 1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2+\frac{p(p+q)(p+2q)}{1\cdot 2\cdot 3}\left(\frac{x}{q}\right)^3+\cdots$$

Working rule to obtain the sum of a binomial series :-

1. The series should commence with 1 as the first, term.

2. The second term should be of the form $\frac{p}{1}\left(\frac{x}{q}\right)$

   Where in the power of $\left(\frac{x}{q}\right)$ is 1.

3. In the succeeding terms, the powers of (x/q) should be 2, 3, 4, ...

4. The Coefficients should be such that the numerator and denominator have same number of factors as the power of x/q. If the power is n, then

   (i) The numerator of the coefficient is a product of n terms of an A.P whose initial term is p and Common difference is q and

   (ii) The denominator is the product of the first n integers $1, 2, 3, \ldots n$.

5. Find p, q, x and use $(1-x)^{-p/q}$ to get the sum.

NOTE :-

1. It is not necessary to write the general term, i, the $(n+1)^{th}$ term.

2. It should be ensured that x lies between -1 & 1.

3. For the series $1+\frac{n}{1}x+\frac{n(n+1)}{1\cdot2}x^2+\frac{n(n+1)(n+2)}{1\cdot2\cdot3}x^3+\dots$

We have $p=n$, $q=1$, $x=x$ and so the sum is

$$S = (1-x)^{-n/1}$$
$$= (1-x)^{-n}$$

1) Sum the series $1+\frac{1}{3}+\frac{1}{3}\cdot\frac{3}{6}+\frac{1}{3}\cdot\frac{3}{6}\cdot\frac{5}{9}+\dots$

Soln:- $S = 1+\frac{1}{3}+\frac{1}{3}\cdot\frac{3}{6}+\frac{1}{3}\cdot\frac{3}{6}\cdot\frac{5}{9}+\dots$

$$\because (1-x)^{-p/q} = 1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2+\dots$$

$$S = 1+\frac{1}{1}\left(\frac{1}{3}\right)+\frac{1\cdot3}{1\cdot2}\left(\frac{1}{3}\right)^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\left(\frac{1}{3}\right)^3+\dots \quad \text{---} (1)$$

$p=1$

$p+q=3$

$1+q=3$

$q=3-1$

$\boxed{q=2}$

$x/q=\frac{1}{3}$

$x/2=\frac{1}{3}$

$\boxed{x=\frac{2}{3}}$

$\therefore S = (1-x)^{-p/q} = (1-2/3)^{-1/2}$

$$= \left(\frac{3-2}{3}\right)^{-1/2}$$

$$= \left(\frac{1}{3}\right)^{-1/2}$$

$$= (3^{-1})^{-1/2}$$

$$= (3)^{1/2}$$

$S = (1-x)^{-p/q} = \sqrt{3}$

2. Sum the series $\frac{1}{10} + \frac{1}{10} \cdot \frac{4}{20} + \frac{1 \cdot 4 \cdot 7}{10 \cdot 20 \cdot 30} + \cdots$

Soln:- $S = \frac{1}{10} + \frac{1 \cdot 4}{10 \cdot 20} + \frac{1 \cdot 4 \cdot 7}{10 \cdot 20 \cdot 30} + \cdots$

$p = 1$ $\qquad 1+q = 4$

$p+q = 4$ $\qquad q = 4-1$

$\boxed{q = 3}$

Add and sub ①

$S = 1 + \frac{1}{1}\left(\frac{1}{10}\right) + \frac{1 \cdot 4}{1 \cdot 2}\left(\frac{1}{10}\right)^2 + \frac{1 \cdot 4 \cdot 7}{1 \cdot 2 \cdot 3}\left(\frac{1}{10}\right)^3 - 1$

$\frac{x}{q} = \frac{1}{10}$

$\frac{x}{3} = \frac{1}{10}$

$\boxed{x = \frac{3}{10}}$

$S = (1-x)^{-p/q} = (1-3/10)^{-1/3} - 1$

$$= \left(\frac{10-3}{10}\right)^{-1/3} - 1$$

$$= \left(\frac{7}{10}\right)^{-1/3} - 1$$

$$= \left(\frac{1}{7/10}\right)^{1/3} - 1$$

$$S=(1-x)^{-p/q}=\left(\frac{10}{7}\right)^{1/3}-1$$

(3) Sum the series $\frac{1}{3\cdot 6}+\frac{1\cdot 3}{3\cdot 6\cdot 9}+\frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9\cdot 12}+\cdots$

Soln:

$$S=\frac{1}{3\cdot 6}+\frac{1\cdot 3}{3\cdot 6\cdot 9}+\frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9\cdot 12}+\cdots$$

Multiple by (-1) on both sides.

$$(-1)S=\frac{-1\cdot 1}{3\cdot 6}+\frac{-1\cdot 1\cdot 3}{3\cdot 6\cdot 9}+\frac{-1\cdot 1\cdot 3\cdot 5}{3\cdot 6\cdot 9\cdot 12}+\cdots$$

$$=\frac{-1\cdot 1}{1\cdot 2}\left(\frac{1}{3}\right)^2+\frac{-1\cdot 1\cdot 3}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^3+\cdots$$

Add and sub $1+\frac{-1}{1}\left(\frac{1}{3}\right)$ on both side.

$$-S=\left[1+\frac{-1}{1}\left(\frac{1}{3}\right)+\frac{-1\cdot 1}{1\cdot 2}\left(\frac{1}{3}\right)^2+\frac{-1\cdot 1\cdot 3}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^3\right]$$

$$-\left[1+\frac{-1}{1}\left(\frac{1}{3}\right)\right]$$

$p=-1$

$p+q=1$

$-1+q=1$

$q=1+1$

$\boxed{q=2}$

$x/q=\frac{1}{3}$

$x/2=1/3$

$\boxed{x=\frac{2}{3}}$

$$-S=(1-x)^{-p/q}=\left(1-2/3\right)^{1/2}-\left[1+\frac{(-1)}{1}\left(1/3\right)\right]$$

$$=\left(\frac{3-2}{3}\right)^{1/2}-\left[1-\frac{1}{3}\right]$$

$$=\left(\frac{1}{3}\right)^{1/2}-\left[\frac{3-1}{3}\right]$$

$$=\left(\frac{1}{3}\right)^{1/2}-\left(\frac{2}{3}\right)$$

$$-S = \frac{1}{\sqrt{3}} - \frac{2}{3}$$

$$S = \frac{2}{3} - \frac{1}{\sqrt{3}}.$$

H.W.

4. Sum the Infinity the series $\frac{7}{9} + \frac{7 \cdot 9}{9 \cdot 12} + \frac{7 \cdot 9 \cdot 11}{9 \cdot 12 \cdot 15} + \cdots$

Soln:

$$S = \frac{7}{9} + \frac{7 \cdot 9}{9 \cdot 12} + \frac{7 \cdot 9 \cdot 11}{9 \cdot 12 \cdot 15} + \cdots$$

$$S = \frac{7}{3}\left(1/3\right) + \frac{7 \cdot 9}{3 \cdot 4}\left(1/3\right)^2 + \frac{7 \cdot 9 \cdot 11}{3 \cdot 4 \cdot 5}\left(1/3\right)^3 + \cdots$$

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Multiplying by both sides $\frac{3 \cdot 5}{1 \cdot 2}$

$$\frac{3 \cdot 5}{1 \cdot 2} S = \frac{3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3}\left(\frac{1}{3}\right) + \frac{3 \cdot 5 \cdot 7 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{1}{3}\right)^2 + \cdots$$

Multiplying by $\left(\frac{1}{3}\right)^2$ on both sides

$$\frac{15}{2} S\left(\frac{1}{3}\right)^2 = \frac{3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3}\left(\frac{1}{3}\right)^3 + \frac{3 \cdot 5 \cdot 7 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{1}{3}\right)^4 + \cdots$$

Add & sub $1 + \frac{3}{1}\left(\frac{1}{3}\right) + \frac{3 \cdot 5}{1 \cdot 2}\left(\frac{1}{3}\right)^2$ in R.H.S.

$$\frac{\cancel{15}^{5} s}{2}\left(\frac{1}{\cancel{9}_{3}}\right) = 1 + \frac{3}{1}\left(\frac{1}{3}\right) + \frac{3 \cdot 5}{1 \cdot 2}\left(\frac{1}{3}\right)^2 + \frac{3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3}\left(\frac{1}{3}\right)^3 + \cdots$$

$$- \left[1 + \frac{3}{1}\left(\frac{1}{3}\right) + \frac{3 \cdot 5}{1 \cdot 2}\left(\frac{1}{3}\right)^2\right]$$

$$\frac{5s}{6} = 1 + \frac{3}{1}\left(\frac{1}{3}\right) + \frac{3 \cdot 5}{1 \cdot 2}\left(\frac{1}{3}\right)^2 + \cdots - \left[1 + \frac{3}{1}\left(\frac{1}{3}\right) + \frac{\cancel{15}^{5}}{2}\left(\frac{1}{\cancel{9}_{3}}\right)\right]$$

$p = 3$

$p + q = 5$

$3 + q = 5$

$\boxed{q = 2}$

$x/q = 1/3$

$x/2 = 1/3$

$\boxed{x = 2/3}$

$$\frac{5s}{6} = \left(1-\frac{2}{3}\right)^{-3/2} - \left[2+\frac{5}{6}\right]$$

$$= \left(\frac{1}{3}\right)^{-3/2} - \left[\frac{12+5}{6}\right]$$

$$= \left[\frac{1}{3}\right]^{-3/2} - \left[\frac{17}{6}\right]$$

$$= (3)^{3/2} - \frac{17}{6}$$

$$\left\{ \because 3^{3/2} = 3(3)^{1/2} = 3\sqrt{3} \right.$$

$$\frac{5}{6}s = 3\sqrt{3} - \frac{17}{6}$$

$$s = \frac{6}{5}\left[3\sqrt{3} - \frac{17}{6}\right]$$

$$= \frac{18}{5}\sqrt{3} - \frac{6}{5} \times \frac{17}{6}$$

$$s = \frac{18}{5}\sqrt{3} - \frac{17}{5}.$$

5. Find

7. Show that $\sqrt{8} = 1 + \frac{3}{4} + \frac{3\cdot 5}{2\cdot 4^2} + \frac{3\cdot 5\cdot 7}{2\cdot 3\cdot 4^3} + \cdots$

Sol:

$$S = 1 + \frac{3}{4} + \frac{3\cdot 5}{2\cdot 4^2} + \frac{3\cdot 5\cdot 7}{2\cdot 3\cdot 4^3} + \cdots$$

$$S = 1 + \frac{3}{1}\left(\frac{1}{4}\right) + \frac{3\cdot 5}{2}\left(\frac{1}{4}\right)^2 + \frac{3\cdot 5\cdot 7}{2\cdot 3}\left(\frac{1}{4}\right)^3$$

$p = 3$

$p + q = 5$

$3 + q = 5$

$q = 5 - 3$

$\boxed{q = 2}$

$\frac{x}{q} = \frac{1}{4}$

$\frac{x}{2} = \frac{1}{4}$

$x = \frac{1}{\cancel{4}_2} \times \cancel{2}$

$\boxed{x = \frac{1}{2}}$

$$S = (1-x)^{-p/q} = \left[1-\tfrac{1}{2}\right]^{-3/2}$$

$$= \left[\frac{2-1}{2}\right]^{-3/2}$$

$$= \left[\frac{1}{2}\right]^{-3/2}$$

$$= (2)^{3/2}$$

$$= (2^3)^{1/2}$$

$$= (8)^{1/2}$$

$$\boxed{S = \sqrt{8}}$$

8. Sum the Series $\frac{2}{6} + \frac{2\cdot5}{6\cdot12} + \frac{2\cdot5\cdot8}{6\cdot12\cdot18} + \cdots$

Soln:

$$S = \frac{2}{6} + \frac{2\cdot5}{6\cdot12} + \frac{2\cdot5\cdot8}{6\cdot12\cdot18} + \cdots$$

$$S = \frac{2}{1}\left(\frac{1}{6}\right) + \frac{2\cdot5}{1\cdot2}\left(\frac{1}{6}\right)^2 + \frac{2\cdot5\cdot8}{1\cdot2\cdot3}\left(\frac{1}{6}\right)^3 + \cdots$$

Add & sub (−1) in R.H.S.

$$S = 1 + \frac{2}{1}\left(\frac{1}{6}\right) + \frac{2\cdot5}{1\cdot2}\left(\frac{1}{6}\right)^2 + \cdots - 1$$

$p = 2$

$p+q = 5$

$2+q = 5$

$q = 5-2$

$\boxed{q = 3}$

$\frac{x}{q} = \frac{1}{6}$

$\frac{x}{3} = \frac{1}{6}$

$x = \frac{3}{6} = \frac{1}{2}$

$\boxed{x = \frac{1}{2}}$

$$S = (1-x)^{-p/q} - 1$$

$$= \left[1-\frac{1}{2}\right]^{-2/3} - 1 = \left(\frac{2-1}{2}\right)^{-2/3} - 1 = \left(\frac{1}{2}\right)^{-2/3} - 1$$

$$\boxed{S = (2)^{2/3} - 1}$$

9. Sum to infinity of the series.

$$\frac{5}{3\cdot 6}\cdot\frac{1}{4^2}+\frac{5\cdot 8}{3\cdot 6\cdot 9}\cdot\frac{1}{4^3}+\frac{5\cdot 8\cdot 11}{3\cdot 6\cdot 9\cdot 12}\cdot\frac{1}{4^2}+\cdots$$

Soln:-

$$S=\frac{5}{3\cdot 6}\cdot\frac{1}{4^2}+\frac{5\cdot 8}{3\cdot 6\cdot 9}\cdot\frac{1}{4^3}+\frac{5\cdot 8\cdot 11}{3\cdot 6\cdot 9\cdot 12}\cdot\frac{1}{4^4}+\cdots$$

$$S=\frac{5}{1\cdot 2}\left(\frac{1}{12}\right)^2+\frac{5\cdot 8}{1\cdot 2\cdot 3}\left(\frac{1}{12}\right)^3+\cdots$$

$$2S=\frac{2\cdot 5}{1\cdot 2}\left(\frac{1}{12}\right)^2+\frac{2\cdot 5\cdot 8}{1\cdot 2\cdot 3}\left(\frac{1}{12}\right)^3+\cdots$$

Add & sub in $1+\frac{2}{1}\left(\frac{1}{12}\right)$ in R.H.S

$$2S=1+\frac{2}{1}\left(\frac{1}{12}\right)+\frac{2\cdot 5}{1\cdot 2}\left(\frac{1}{12}\right)^2+\cdots$$

$p=2$

$p+q=5$

$2+q=5$

$q=5-2$

$\boxed{q=3}$

$\frac{x}{q}=\frac{1}{12}$

$\frac{x}{3}=\frac{1}{12}$

$x=\frac{1}{\cancel{12}_4}\times\cancel{3}^1$

$\boxed{x=\frac{1}{4}}$

$$2S=(1-x)^{-p/q}-\left[1+\frac{x}{1}\left(\frac{1}{1}\right)\right]$$

$$=\left(1-\frac{1}{4}\right)^{-2/3}-\left[1+\frac{1}{6}\right]$$

$$=\left(\frac{4-1}{4}\right)^{-2/3}-\left[\frac{6+1}{6}\right]$$

$$=\left(\frac{3}{4}\right)^{-2/3}-\left[\frac{7}{6}\right]$$

$$2S=\left(\frac{4}{3}\right)^{2/3}-\frac{7}{6}$$

$$S=2\left(\frac{4}{3}\right)^{2/3}-{}^{1}\cancel{2}\times\frac{7}{\cancel{6}3}$$

$$S=2\left(\frac{4}{3}\right)^{2/3}-\frac{7}{3}.$$

10. If a, b and $n>0$, find the value of

$$1+\frac{na}{a+b}+\frac{n(n+1)}{2!}\cdot\left[\frac{a}{a+b}\right]^2+\cdots$$

Soln:

$$S=1+\frac{na}{a+b}+\frac{n(n+1)}{2!}\left[\frac{a}{a+b}\right]^2+\cdots$$

$p=n$, $q=1$, $\frac{x}{q}=\frac{a}{a+b}$ (or) $x=\frac{a}{a+b}$.

$$S=\left(1-\frac{a}{a+b}\right)^{-n/1} \quad (1-x)^{-p/q}$$

$$=\left(\frac{\cancel{a}+b-\cancel{a}}{a+b}\right)^{-n}$$

$$=\left(\frac{b}{a+b}\right)^{-n}$$

$$S=\left(\frac{a+b}{b}\right)^{n}$$

11. Prove that

$$x^n=1+n\left(1-\frac{1}{x}\right)+\frac{n(n+1)}{1\cdot 2}\left(1-\frac{1}{x}\right)^2+\cdots$$

Soln: Given

$$1+n\underbrace{\left(1-\frac{1}{x}\right)}_{y}+\frac{n(n+1)}{1\cdot 2}\underbrace{\left(1-\frac{1}{x}\right)^2}_{y}+\cdots$$

Let

$y=1-\frac{1}{x}$.

$$1+ny+\frac{n(n+1)}{1\cdot 2}y^2+\cdots=(1-x)^{-p/q}$$

Where $p=n$, $q=1$, $\frac{x}{q}=y$ or $x=y$

$$=(1-y)^{-n}$$

$$=(1-(1-1/x))^{-n}$$

$$=(\cancel{1}-\cancel{1}+\frac{1}{x})^{-n}$$

$$=\left(\frac{1}{x}\right)^{-n}$$

$$=x^n$$

$\therefore 1+n(1-\frac{1}{x})+\frac{n(n+1)}{1\cdot 2}(1-\frac{1}{x})^2+\cdots=x^n$.

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Remark:

When the terms of the series are alternately +ve and -ve, then also the same formula.

$$(1-x)^{-p/q} = 1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+1)}{1\cdot 2}\left(\frac{x}{q}\right)^2+\cdots$$

1. Sum the series $1-\frac{3}{4}+\frac{3\cdot5}{4\cdot8}-\frac{3\cdot5\cdot7}{4\cdot8\cdot12}+\cdots$

Sol: $S = 1-\frac{3}{4}+\frac{3\cdot5}{4\cdot8}-\frac{3\cdot5\cdot7}{4\cdot8\cdot12}+\cdots$

$$= 1-\frac{3}{1}\left(\frac{1}{4}\right)+\frac{3\cdot5}{1\cdot2}\left(\frac{1}{4}\right)^2-\frac{3\cdot5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{4}\right)^3+\cdots$$

$$S = 1+\frac{3}{1}\left(-\frac{1}{4}\right)+\frac{3\cdot5}{1\cdot2}\left(-\frac{1}{4}\right)^2+\frac{3\cdot5\cdot7}{1\cdot2\cdot3}\left(-\frac{1}{4}\right)^3+\cdots$$

$p=3$ $\qquad \frac{x}{q} = -\frac{1}{4}$

$p+q=5$ $\qquad \frac{x}{\cancel{2}_1} = \frac{-1}{\cancel{4}_2}$

$3+q=5$ $\qquad \boxed{x=-\frac{1}{2}}$

$q=5-3$

$\boxed{q=2}$

$$\therefore S = (1-x)^{-p/q} = \left(1+\frac{1}{2}\right)^{-3/2}$$

$$= \left(\frac{2+1}{2}\right)^{-3/2}$$

$$= \left(\frac{3}{2}\right)^{-3/2}$$

$$= \left(\frac{2}{3}\right)^{3/2}$$

2. Find the sum of the series $\frac{1\cdot4}{5\cdot10}-\frac{1\cdot4\cdot7}{5\cdot10\cdot15}+\frac{1\cdot4\cdot7\cdot10}{5\cdot10\cdot15\cdot20}+\cdots$

Sol: $S = \frac{1\cdot4}{5\cdot10}-\frac{1\cdot4\cdot7}{5\cdot10\cdot15}+\frac{1\cdot4\cdot7\cdot10}{5\cdot10\cdot15\cdot20}+\cdots$

$$= \frac{1\cdot4}{1\cdot2}\left(\frac{1}{5}\right)-\frac{1\cdot4\cdot7}{1\cdot2\cdot3}\left(\frac{1}{5}\right)^2+\frac{1\cdot4\cdot7\cdot10}{1\cdot2\cdot3\cdot4}\left(\frac{1}{5}\right)^3+\cdots$$

$$S = \frac{1\cdot4}{1\cdot2}\left(-\frac{1}{5}\right)^2 + \frac{1\cdot4\cdot7}{1\cdot2\cdot3}\left(-\frac{1}{5}\right)^3 + \frac{1\cdot4\cdot7\cdot10}{1\cdot2\cdot3\cdot4}\left(-\frac{1}{5}\right)^4 + \cdots$$

Add & Sub $1+\frac{1}{1}\left(\frac{-1}{5}\right)$ in R.H.S

$$S = 1+\frac{1}{1}\left(-\frac{1}{5}\right) + \frac{1\cdot4}{1\cdot2}\left(-\frac{1}{5}\right)^2 + \frac{1\cdot4\cdot7}{1\cdot2\cdot3}\left(-\frac{1}{5}\right)^3 + \cdots - \left[1+\frac{1}{1}\left[\frac{-1}{5}\right]\right].$$

$p = 1$ $\qquad \frac{x}{q} = -\frac{1}{5}$

$p+q = 4$

$1+q = 4$ $\qquad \frac{x}{3} = -\frac{1}{5}$

$q = 4-1$ $\qquad \boxed{x = \frac{-3}{5}}$

$\boxed{q=3}$

$$\therefore\ S = (1-x)^{-p/q} - \left[1+\frac{1}{1}\left(-\frac{1}{5}\right)\right]$$

$$= \left(1+\frac{3}{5}\right)^{-1/3} - \left[1-\frac{1}{5}\right]$$

$$= \left[\frac{5+3}{5}\right]^{-1/3} - \left[\frac{5-1}{5}\right]$$

$$= \left[\frac{8}{5}\right]^{-1/3} - \left[\frac{4}{5}\right]$$

$$S = \left[\frac{5}{8}\right]^{1/3} - \left[\frac{4}{5}\right]$$

Exponential Series :-

The limit of $\left(1+\frac{1}{n}\right)^n$, as n tends to the exists. Its Value is an irrational number lying between 2 and 3. It is denoted by e. Its Value to five decimal places is 2.71828.

Maclaurin's series for e^x.

For all real values of x.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \longrightarrow (1)$$

Replacing x into $-x$ in (1), we get

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \cdots - (2)$$

Adding (1) & (2) and dividing by 2 and also.

subtracting (2) from (1) and dividing by 2.

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots (3)$$

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots (4)$$

$x = 1$ in (1), (2), (3) & (4)

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots (5)$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots (6)$$

$$\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots (7)$$

$$\frac{e - e^{-1}}{2} = \frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots (8)$$

1. Sum the series $\frac{1+3x}{1!}+\frac{(1+3x)^2}{2!}+\frac{(1+3x)^3}{3!}+\cdots$

Soln: $S = 1+\frac{3x}{1!}+\frac{(1+3x)^2}{2!}+\frac{(1+3x)^3}{3!}+\cdots$

Denoting $1+3x$ by α,

$$S = \frac{\alpha}{1!}+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\cdots$$

$$= \left(1+\frac{\alpha}{1!}+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}\cdots\right)-1$$

$$= e^{\alpha}-1$$

$$= e^{1+3x}-1$$

$$S = e \cdot e^{3x}-1.$$

2. Sum the series $1-\log_e 2+\frac{(\log_e 2)^2}{2!}-\frac{(\log_e 2)^3}{3!}+\cdots$

Soln: $S = 1-\log_e 2+\frac{(\log_e 2)^2}{2!}-\frac{(\log_e 2)^3}{3!}+\cdots$

We know that $e^{\log f(x)} = f(x)$

Now denoting $\log_e 2$ by x,

$$S = 1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots$$

$$= e^{-x}$$

$$= e^{-\log_e 2}$$

$$= e^{\log_e 2^{-1}}$$

$$= 2^{-1}$$

$$\boxed{S = \frac{1}{2}}$$

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3. Show that $\frac{1}{2}\left(e-\frac{1}{e}\right) = 1+\frac{1}{3!}+\frac{1}{5!}+\cdots$

Solu:- $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$

$e^{-1} = 1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots$

Subtracting ① & ② and dividing by 2.

We get

$$\frac{1}{2}(e-e^{-1}) = \frac{1}{2}\left[1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots-1+\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}+\cdots\right]$$

$$= \frac{1}{2}\left[2\left(\frac{1}{1!}\right)+2\left(\frac{1}{3!}\right)+2\left(\frac{1}{5!}\right)+\cdots\right]$$

$$= \frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots$$

$$\frac{1}{2}\left(e-\frac{1}{e}\right) = 1+\frac{1}{3!}+\frac{1}{5!}+\cdots$$

4. Sum the series $1+\frac{3^2}{2!}+\frac{3^4}{4!}+\frac{3^6}{6!}+\cdots$

Solu:- $1+\frac{3^2}{2!}+\frac{3^4}{4!}+\frac{3^6}{6!}+\cdots$

$$\frac{e^x+e^{-x}}{2} = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots \rightarrow ①$$

Put $x=3$ in ①

$$\frac{e^3+e^{-3}}{2} = 1+\frac{3^2}{2!}+\frac{3^4}{4!}+\frac{3^6}{6!}+\cdots$$

5. Show that $\dfrac{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots}$

Solu:- We know that

$$\frac{e+e^{-1}}{2} = 1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots$$

$$\frac{e-e^{-1}}{2} = \frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots$$

$$\frac{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots} = \frac{\cancel{\frac{1}{2}}(e+e^{-1})}{\cancel{\frac{1}{2}}(e-e^{-1})}$$

$$= \frac{e^2+1}{e^2-1}$$

6. Sum of the series $\dfrac{\frac{1}{2!}+\frac{1}{4!}+\cdots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots} = \dfrac{e-1}{e+1}$.

Soln: Add & sub 1

$$\frac{\left[1+\frac{1}{2!}+\frac{1}{4!}+\cdots\right]-1}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots} = \frac{\frac{e+e^{-1}}{2}-1}{\frac{e-e^{-1}}{2}}$$

$$= \frac{\frac{e+e^{-1}-2}{2}}{\frac{e-e^{-1}}{2}}$$

$$= \frac{e+e^{-1}-2}{\cancel{2}} \times \frac{\cancel{2}}{e-e^{-1}}$$

$$= \frac{e+e^{-1}-2}{e-e^{-1}}$$

$$= \frac{\frac{e^2+1-2e}{e}}{\frac{e^2-1}{e}}$$

$$= \frac{e^2+1-2e}{e^2-1} \times \frac{\cancel{e}}{\cancel{e}}$$

$$= \frac{e^2+1-2e}{e^2-1}$$

$$= \frac{(e+1)^2}{(e+1)(e-1)}$$

$$= \frac{(e+1)}{(e-1)} \;//$$

7. Show that $\dfrac{1+\frac{1}{2!}+\frac{2}{3!}+\frac{2^2}{4!}+\cdots}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}=\dfrac{e}{2}.$

Soln:
$$\frac{1+\frac{1}{2!}+\frac{2}{3!}+\frac{2^2}{4!}+\cdots}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}$$

Multiple & divide by 2^2 on the numerator only.

$$\frac{\frac{1}{2^2}\left[\frac{2^2}{1}+\frac{2^2}{2!}+\frac{2^3}{3!}+\cdots\right]}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}$$

Add in $\frac{2^2}{?!}=\left(1+\frac{2}{1!}\right)+1$

$$=\frac{\frac{1}{2^2}\left[\left(1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}\right)+1\right]}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}$$

$$=\frac{\frac{1}{2^2}(e^2+1)}{\frac{1}{2}(e+e^{-1})}$$

$$=\frac{\cancel{2}}{2^{\cancel{2}}}\left(\frac{e^2+1}{e+\frac{1}{e}}\right)$$

$$=\frac{1}{2}\left(\frac{e^2+1}{\frac{e^2+1}{e}}\right)$$

$$=\frac{1}{2}\times e$$

$$=\frac{e}{2}.$$

8. Show that $1+\frac{x\log e^4}{1!}+\frac{(x\log e^4)^2}{2!}+\frac{(x\log_e a)^2}{3!}+\cdots$

Sol: $1+\frac{x\log_e^a}{1!}+\frac{(x\log_e^a)^2}{2!}+\frac{(x\log_e^a)^3}{3!}+\cdots$

$x = x\log_e^a$

$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = e^x$

$= e^{x\log_e^a}$

$= e^{\log_e a^x} \cdot e^{\log_e f(x)} = f(x)$

$= a^x$

9. Show that $\log_e^2+\frac{(\log_e^2)^2}{2!}+\frac{(\log_e^2)^3}{3!}+\cdots = 1$

Sol: $\log_e^2+\frac{(\log_e^2)^2}{2!}+\frac{(\log_e^2)^3}{3!}+\cdots$

$\left[1+\log_e^2+\frac{(\log_e^2)^2}{2!}+\cdots\right]-1$

$x = \log_e^2$

$= 1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots-1$

$= e^x-1$

$= e^{\log_e^2}-1$

$= 2-1$

$= 1.$

10. Find the coefficient of x^n in $\frac{1+2x+3x^2}{\cancel{e^x}\, e^x}$

Sol: $\frac{1+2x+3x^2}{e^x} = (1+2x+3x^2)e^{-x}$

$= (1+2x+3x^2)\left[1-\frac{x}{1!}+\cdots\frac{(-1)^{n-2}}{(n-2)!}x^{n-2}\right.$

$\left.+\frac{(-1)^{n-1}}{(n-1)!}x^{n-1}+\frac{(-1)^n}{n!}x^n+\cdots\right.$

By multiplication, we get the coefficient of x^n as

$$1\times\frac{(-1)^n}{n!}+2\times\frac{(-1)^{n-1}}{(n-1)!}+3\times\frac{(-1)^{n-2}}{(n-2)!}$$

$$=\frac{(-1)^n}{n!}\left[1-2n+3n(n-1)\right]$$

$$=\frac{(-1)^n}{n!}\left[1-5n+3n^2\right]$$

11. Find the coefficient of x^n in e^{e^x}.

Sol:

$$e^{e^x}=1+\frac{e^x}{1!}+\frac{(e^x)^2}{2!}+\frac{(e^x)^3}{3!}+\cdots$$

$$=1+\frac{e^x}{1!}+\frac{e^{2x}}{2!}+\frac{e^{3x}}{3!}+\cdots$$

$$=1+\frac{1}{1!}\left[1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots\right]$$

$$+\frac{1}{2!}\left[1+\frac{2x}{1!}+\frac{2^2x^2}{2!}+\cdots+\frac{2^nx^n}{n!}+\cdots\right]$$

$$+\frac{1}{3!}\left[1+\frac{3x}{1!}+\frac{3^2x^2}{2!}+\cdots+\frac{3^nx^n}{n!}\cdots\right]+\cdots$$

Coefficient of $x^n=\frac{1}{1!}\cdot\frac{1}{n!}+\frac{1}{2!}\cdot\frac{2^n}{n!}+\frac{1}{3!}\cdot\frac{3^n}{n!}\cdots$

$$=\frac{1}{n!}\left[\frac{1^n}{1!}+\frac{2^n}{2!}+\frac{3^n}{3!}+\cdots\right]$$

Summation (Exponential series).

In summing an exponential series, we should know how to write the n^{th} terms.

We shall denote the n^{th} term by t_n. As an illustration, we shall sum the series.

$$\frac{1^3}{2!}+\frac{2^3}{3!}+\frac{3^3}{4!}+\frac{4^3}{5!}+\cdots$$

Now the numerators of the terms are cubes of

$1, 2, 3, \ldots n, \ldots$

& the denominators of are the factorials of

$2, 3, 4, \ldots n+1, \ldots$

which exceed the numbers in (1) by 1.

Thus $t_n = \frac{n^3}{(n+1)!}$

Next the nth term should be rewritten as sum of suitable terms to use exponential series.

As an illustration, we shall consider

$$t_n = \frac{n^3}{(n+1)!}$$

Working rule :-

step 1: Consider the numerator n^3 and find its degree. (Now the degree is 3).

step 2: Rewrite n^3 as the sum of

(i) a constant term A

(ii) a first degree term $B(n+1)$

(iii) a second degree term $C(n+1)n$

(iv) a third degree of the sum also is 3.

$$n^3 = A + B(n+1) + C(n+1)n + D(n+1)n(n-1).$$

step3: Obtain the values of A, B, C, D others by alloting particular values for n or by equating the coefficients as follows.

$n=-1 \Rightarrow (-1)^3 = A+0+0+0 \cdots A = -1$

$n=0 \Rightarrow 0^3 = A+B+0+0 \cdots B = -A$

$n=1 \Rightarrow 1^3 = A+2B+2C \cdots C = 0.$

Comparing the coefficients of n^3, $1 = D$.

$\therefore n^3 = -1 + (n+1) + 0 + (n+1)n(n-1)$

$$t_n = -\frac{1}{(n+1)!} + \frac{n+1}{(n+1)!} + \frac{(n+1)(n)(n-1)}{(n+1)!}$$

$$= -\frac{1}{(n+1)!} + \frac{1}{n!} + \frac{1}{(n-2)!}$$

because $(n+1)! = 1\cdot2\cdot3 \cdots (n-2)(n-1)n(n+1)$

Setting $1, 2, 3, \ldots$ for n & remembering that $0! = 1$.

we get.

$$t_1 = -\frac{1}{2!} + \frac{1}{1!}$$

$$t_2 = -\frac{1}{3!} + \frac{1}{2!} + \frac{1}{0!}$$

$$t_3 = -\frac{1}{4!} + \frac{1}{3!} + \frac{1}{1!}$$

$$t_4 = -\frac{1}{5!} + \frac{1}{4!} + \frac{1}{2!}.$$

Adding the terms Vertically.

$$S = -\left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots\right)$$

$$+ \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right)$$

$$+ \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right)$$

$$= -\left[e - \left(1 + \frac{1}{1!}\right)\right] + [e-1] + e$$

$$= -[e-(2)] + e - 1 + e$$

$$= -\cancel{e} + 2 + \cancel{e} - 1 + e$$

$$= e + 1$$

1. Sum the Series $\frac{2^2}{1!}+\frac{3^2}{2!}+\frac{4^2}{3!}+\frac{5^2}{4!}+\dots$

Let t_n be the n^{th} term.

$$t_n = \frac{(n+1)^2}{n!}$$

Degree of numerator is 2.

Let $(n+1)^2 = A+Bn+cn(n-1)$

$n^2+2n+1 = A+Bn+cn(n-1)$ — (1)

Equate Coefficient of n^2

$1 = c$

$\boxed{c=1}$.

Equate coefficient of n.

$2 = B-c$

$2 = B-1$

$\boxed{B=3}$

Equate the constant term.

$\boxed{A=1}$.

Sub A, B & c value in (1)

$(n+1)^2 = 1+3n+n(n-1)$

$$t_n = \frac{(n+1)^2}{n!}$$

$$= \frac{1+3n+n(n-1)}{n!}$$

$$= \frac{1}{n!}+\frac{3n}{n!}+\frac{n(n-1)}{n!}$$

$$= \frac{1}{n!}+\frac{3\cancel{n}}{\cancel{n}(n-1)!}+\frac{\cancel{n(n-1)}}{\cancel{n(n-1)}(n-2)!}$$

$$t_n = \frac{1}{n!}+\frac{3}{(n-1)!}+\frac{1}{(n-2)!}$$

Put $n = 1, 2, 3, \ldots$

$$t_1 = \frac{1}{1!} + \frac{3}{0!}$$

$$t_2 = \frac{1}{2!} + \frac{3}{1!} + \frac{1}{0!}$$

$$t_3 = \frac{1}{3!} + \frac{3}{2!} + \frac{1}{1!}$$

$$t_4 = \frac{1}{4!} + \frac{3}{3!} + \frac{1}{2!}$$

Adding Vertically.

$$S = \left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right] + 3\left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right] + \left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right]$$

$$S = (e^1 - 1) + 3e + e^1$$

$$= e^1 - 1 + 3e^1 + e^1$$

$$S = 5e - 1.$$

2. Sum the series $\sum_{n=0}^{\infty} \frac{5n+1}{(2n+1)!}$

Soln: $t_n = \frac{5n+1}{(2n+1)!}, \; n = 0, 1, 2, \ldots$

Degree of numerator is 1.

$$5n+1 = A + B(2n+1) \longrightarrow (1)$$

Equate the coefficient of n.

$$5 = 2B$$

$$\boxed{B = 5/2}$$

Equate the Constant.

$$1 = A + B$$

$$1 = A + 5/2$$

$$A = 1 - 5/2$$

$$= \frac{2-5}{2}$$

$$\boxed{A = 3/2}$$

Sub A, B Value in (1)

$$(1) \Rightarrow \quad 5n+1 = A + B(2n+1)$$

$$5n+1 = -\frac{3}{2} + \frac{5}{2}(2n+1)$$

$$\frac{5n+1}{(2n+1)!} = \frac{-3/2 + 5/2(2n+1)}{(2n+1)!}$$

$$= \frac{-3/2}{(2n+1)!} + \frac{5/2(2n+1)}{(2n+1)!}$$

$$= \frac{-3/2}{(2n+1)!} + \frac{5/2\cancel{(2n+1)}}{\cancel{(2n+1)}2n!}$$

$$t_n = \frac{-3/2}{(2n+1)!} + \frac{5/2}{2n!} \longrightarrow (2)$$

Put $n = 0, 1, 2, 3 \cdots$ in (2)

$$t_0 = \left[\frac{-3/2}{1!} + \frac{5/2}{2!} + \cdots\right]$$

$$t_1 = \frac{-3/2}{3!} + \frac{5/2}{2!} + \cdots$$

$$t_2 = \frac{-3/2}{5!} + \frac{5/2}{4!} + \cdots$$

$$t_3 = \frac{-3/2}{7!} + \frac{5/2}{6!}$$

Adding Vertically:-

$$S = \left[\frac{-3/2}{1!} + \frac{-3/2}{3!} + \frac{-3/2}{5!} + \cdots\right] + \left[\frac{5/2}{0!} + \frac{5/2}{2!} + \frac{5/2}{4!} + \cdots\right]$$

$$= -\frac{3}{2}\left[\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right] + \frac{5}{2}\left[\frac{1}{0!} + \frac{1}{2!} + \frac{1}{4!} + \cdots\right] \quad (1)$$

$$= -\frac{3}{2}\left[\frac{e-e^{-1}}{2}\right] + \frac{5}{2}\left[\frac{e+e^{-1}}{2}\right]$$

$$= -\frac{3}{2}\left[\frac{e-\frac{1}{e}}{2}\right] + \frac{5}{2}\left[\frac{e+\frac{1}{e}}{2}\right]$$

$$= \frac{-3}{2}\left[\frac{\frac{e^2-1}{e}}{2}\right] + \frac{5}{2}\left[\frac{\frac{e^2+1}{e}}{2}\right]$$

$$= \frac{-3}{2}\left[\frac{e^2-1}{2e}\right] + \frac{5}{2}\left[\frac{e^2+1}{2e}\right]$$

$$= \frac{-3}{4e}[e^2-1] + \frac{5}{4e}[e^2+1]$$

$$= \frac{1}{4e}\left[-3e^2+3+5e^2+5\right]$$

$$= \frac{1}{4e}\left[2e^2+8\right]$$

$$= \frac{\cancel{2}(e^2+4)}{\cancel{4}_2 e}$$

$$= \frac{e^2+4}{2e}$$

$$= \frac{e^{\cancel{2}}}{2\cancel{e}} + \frac{\cancel{4}^2}{\cancel{2}e}$$

$$S = \frac{e}{2} + \frac{2}{e}.$$

3. Prove that $\frac{2}{1!}+\frac{3}{2!}+\frac{4}{3!}+\cdots = 2e^{-1}$

$t_n = \frac{n+1}{n!} \longrightarrow (1)$

Degree of numerator is 1.

$n+1 = A+Bn \longrightarrow (2)$

Put $n=0$ in (2)

$\boxed{A=1}$

Equate the Coefficient of n.

$\boxed{B=1}$

Sub A, B Value in (2)

$n+1 = 1+n \longrightarrow (3)$

Sub (3) in (1)

$(1) \Rightarrow t_n = \frac{n+1}{n!}$

$t_n = \frac{1+n}{n!}$

Put $t_n = \frac{1}{n!} + \frac{n}{n!}$

$= \frac{1}{n!} + \frac{n(1)}{n(n-1)!}$

$t_n = \frac{1}{n!} + \frac{1}{n-1!} \longrightarrow (4)$

Put $n=1,2,3$ in (4)

$t_1 = \frac{1}{1!} + \frac{1}{0!}$

$t_2 = \frac{1}{2!} + \frac{1}{1!}$

$t_3 = \frac{1}{3!} + \frac{1}{2!}$

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Adding Vertically.

$$S=\left[\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\right]+\left[\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots\right]$$

$$= e-1+e$$

$$S = 2e-1.$$

4. Sum the Series $\frac{1^2}{3!}+\frac{2^2}{5!}+\frac{3^2}{7!}+\cdots$

Soln: Let t_n be the n^{th} term.

$$t_n = \frac{-n^2}{(2n+1)'} \longrightarrow ①$$

Degree of numerator 2.

Let $n^2 = A+B(2n+1)+c\ 2n(2n+1) \longrightarrow ②$

Equate the coefficient of n^2 in ②

$$1=4c$$

$$\boxed{c=1/4}$$

Equate coefficient of n in ②

$$0=2B+2c$$
$$=2B+\cancel{2}\left(1/\cancel{4}_2\right)$$

$$0=2B+\frac{1}{2}$$

$$2B=-1/2$$

$$\boxed{B=-1/4}.$$

Equate coefficient of constant in ②

$$0=A+B$$

$$0=A+-\frac{1}{4}$$

$$\boxed{A=\frac{1}{4}}$$

Sub A, B & C values in ②

$$n^2 = \frac{1}{4} - \frac{1}{4}(2n+1) + \frac{1}{4}2n(2n+1) \longrightarrow ③$$

Sub ③ in ①

$$t_n = \frac{n^2}{(2n+1)!} = \frac{\frac{1}{4} - \frac{1}{4}(2n+1) + \frac{1}{4}((2n)(2n+1))}{(2n+1)!}$$

$$= \frac{1/4}{(2n+1)!} - \frac{1/4(2n+1)}{(2n+1)!} + \frac{1/4\,2n(2n+1)}{(2n+1)!}$$

$$t_n = \frac{1/4}{(2n+1)!} - \frac{1/4}{2n!} + \frac{1/4}{(2n-1)!} \longrightarrow ④$$

Sub $n = 1, 2, 3$ in ④.

$$\left.\begin{aligned} t_1 &= \frac{1/4}{5!} - \frac{1/4}{2!} + \frac{1/4}{1!} \\ t_2 &= \frac{1/4}{5!} - \frac{1/4}{4!} + \frac{1/4}{3!} \\ t_3 &= \frac{1/4}{7!} - \frac{1/4}{6!} + \frac{1/4}{5!} \end{aligned}\right\} \longrightarrow ⑤$$

Adding eqn. ⑤ in Vertically.

$$S = \frac{1}{4}\left[\frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots\right] - \frac{1}{4}\left[\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots\right] + \frac{1}{4}\left[\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right]$$

$$= \frac{1}{4}\left[\frac{e-e^{-1}}{2} - \frac{1}{1!}\right] - \frac{1}{4}\left[\frac{e+e^{-1}}{2} - 1\right] + \frac{1}{4}\left[\frac{e-e^{-1}}{2}\right]$$

$$= \frac{1}{4}\left[\left(\frac{e-\frac{1}{e}}{2} - 1\right) - \left(\frac{e^2+1}{2e} - 1\right) + \left(\frac{e^2-1}{2e}\right)\right]$$

$$= \frac{1}{8e}\left[e^2 - 1 - \frac{e^2+1}{2e} + e^2 - 1\right]$$

$$= \frac{1}{8e}(3e^2 - 3)$$

$$= \frac{3e^{\cancel{2}}}{8\cancel{e}} - \frac{3}{8e}$$

$$= \frac{3e}{8} - \frac{3}{8e}$$

$$S = \frac{e - 3e^{-1}}{8}.$$

5. Sum the series $\frac{1}{1!} + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots \infty$

Soln: Given $\frac{1}{1!} + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots \infty$

Let $t_n = \frac{2n-1}{n!} \longrightarrow (1)$

numerator degree is 1.

$$2n - 1 = A + Bn \longrightarrow (2)$$

Equate coefficient of n in (2)

$$\boxed{B = 2}$$

Equate coefficient of constant in (2)

$$\boxed{A = -1}.$$

Sub A, B values in (2)

$$2n - 1 = -1 + 2n \longrightarrow (3)$$

Sub (3) in (1)

$$(1) \Rightarrow t_n = \frac{2n-1}{n!} = \frac{-1+2n}{n!}$$

$$= \frac{-1}{n!} + \frac{2n}{n!}$$

$$= \frac{-1}{n!} + \frac{2\cancel{n}}{\cancel{n}(n-1)!}$$

$$t_n = \frac{-1}{n!} + \frac{2}{(n-1)!} \longrightarrow (4)$$

Put $n = 1, 2, 3$ in (4)

$$\left.\begin{array}{l} t_1 = \frac{-1}{1!} + \frac{2}{0!} \cdots \\ t_2 = -1/2! + 2/1! \cdots \\ t_3 = -1/3! + 2/2! \end{array}\right\} \longrightarrow (5)$$

$\frac{2n-1}{n!}$

Adding eqn. ⑤ in vertically

$$S = \left[\frac{-1}{1!} - \frac{1}{2!} - \frac{1}{3!}\right] + \left[\frac{2}{0!} + \frac{2}{1!} + \frac{2}{2!} + \cdots\right]$$

$$= -1\left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right] + 2\left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right]$$

$$= -1(e-1) + 2e$$

$$= -e + 1 + 2e$$

$$\boxed{S = e+1}$$

6. Show that $1 + \frac{1+3}{2!} + \frac{1+3+3^2}{3!} + \frac{1+3+3^2+3^3}{4!} + \cdots = \frac{e(e^2-1)}{2}$.

Sol. $1 + \frac{1+3}{2!} + \frac{1+3+3^2}{3!} + \frac{1+3+3^2+3^3}{4!} + \cdots$

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x-1}$$

$$1 + 3 + 3^2 + \cdots + 3^{n-1} = \frac{3^n - 1}{3-1} = \frac{3^n - 1}{2}$$

Let $t_n = \frac{1+3+3^2+\cdots+3^{n-1}}{n!} = \frac{3^n-1}{2} \cdot \frac{1}{n!}$

$$t_n = \frac{1}{2} \cdot \frac{3^n}{n!} - \frac{1}{2} \cdot \frac{1}{n!} \longrightarrow ①$$

Put $n = 1, 2, 3, 4 \ldots$

$$\left.\begin{aligned} t_1 &= \frac{1}{2} \cdot \frac{3}{1!} - \frac{1}{2} \cdot \frac{1}{1!} \\ t_2 &= \frac{1}{2} \cdot \frac{3^2}{2!} - \frac{1}{2} \cdot \frac{1}{2!} \\ t_3 &= \frac{1}{2} \cdot \frac{3^3}{3!} - \frac{1}{2} \cdot \frac{1}{3!} \text{ etc} \end{aligned}\right\} \longrightarrow ②$$

Adding eqn. ⑤ in Vertically

$$S = \frac{1}{2}\left(\frac{3}{1!}+\frac{3^2}{2!}+\frac{3^3}{3!}+\dots\right) - \frac{1}{2}\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots\right)$$

$$= \frac{1}{2}(e^3-1) - \frac{1}{2}(e-1)$$

$$= \frac{e^3}{2} - \cancel{\frac{1}{2}} - \frac{e}{2} + \cancel{\frac{1}{2}}$$

$$= \frac{e^3-e}{2}$$

$$\boxed{S = \frac{e(e^2-1)}{2}}$$

Logarithmic Series.

Definition.

The infinite series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

is called the logarithmic series. Its sum is ~~called the logarithmic~~ $\log_e(1+x)$ if the Value of x is such that $-1 < x \le 1$.

Thus, when $-1 < x \le 1$

We have

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad \text{①}$$

Formula.

1) $\log(1-x) = -\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right]$

2) $-\log(1-x) = x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots$

3) $\log 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

4) $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x+\frac{x^2}{3}+\frac{x^5}{5}+\cdots$ odd term.

1) If $|x|>1$, then find $\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\cdots$

Sol: We know that

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots \quad |x|<1.$$

Given :-

$$\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5X5}+\cdots \text{ --- } (1)$$

Replace x by $1/x$ in (1)

We get

$$\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\cdots = \frac{1/x}{1}+\frac{1/x^3}{3}+\frac{1/x^5}{5}+\cdots$$

$$= \frac{1}{2}\log\left(\frac{1+1/x}{1-1/x}\right)$$

$$= \frac{1}{2}\log\left(\frac{\frac{x+1}{\cancel{x}}}{\frac{x-1}{\cancel{x}}}\right)$$

$$= \frac{1}{2}\log\left(\frac{x+1}{x-1}\right)$$

2. Show that $\log 10 = 3\log 2 + \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4^2} + \frac{1}{3}\cdot\frac{1}{4^3} - \dots \infty$

Soln: Let $x = 1/4$

$$3\log 2 + \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4^2} + \frac{1}{3}\cdot\frac{1}{4^3} + \dots = 3\log 2 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$$

$$= 3\log 2 + \left[x - \frac{x^2}{2} + \frac{x^3}{3} + \dots\right]$$

$$= 3\log 2 + \log(1+x)$$

$$= 3\log 2 + \log(1 + 1/4)$$

$$= 3\log 2 + \log\left(\frac{5}{4}\right)$$

$$= \log 2^3 + \log(5/4)$$

$$= \log(2^3 \times 5/4)$$

$$= \log(\overset{2}{8} \times 5/\cancel{4})$$

$$3\log 2 + \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4^2} + \frac{1}{3}\cdot\frac{1}{4^3} + \dots = \log 10.$$

3. If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$ show that

$$x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \dots$$

Soln Given

$$y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$y = \log(1+x)$$

Taking exponential on both sides.

$$e^y = e^{\log(1+x)} \qquad e^{\log f(x)} = f(x)$$

$$e^y = 1+x$$

$$1+x = e^y$$

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$$1+x = 1+\frac{y}{1!}+\frac{y^2}{2!}+\frac{y^3}{3!}+\cdots$$

$$x = \frac{y}{1!}+\frac{y^2}{2!}+\frac{y^3}{3!}+\cdots$$

4. Show that $\frac{a-x}{a}+\frac{1}{2}\left(\frac{a-x}{a}\right)^2+\frac{1}{3}\left(\frac{a-x}{a}\right)^3+\cdots = \log a - \log b.$

Ans: Let $y = \frac{a-x}{a}$.

$$\frac{a-x}{a}+\frac{1}{2}\left(\frac{a-x}{a}\right)^2+\frac{1}{3}\left(\frac{a-x}{a}\right)^3+\cdots = y+\frac{y^2}{2}+\frac{y^3}{3}+\cdots$$

$$= -\log(1-y)$$

$$= -\log\left(1-\frac{a+x}{a}\right)$$

$$= -\log\left(\frac{a-\cancel{a}+x}{a}\right)$$

$$= -\log\left(\frac{x}{a}\right)$$

$$= -[\log x - \log a]$$

$$= -\log x + \log a$$

$$= \log a - \log x.$$

5. Show that $\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\cdots = \frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots$

Ans: Let

$$x = \frac{1}{n+1}.$$

$$\frac{1}{n+1}+\frac{1}{2(n+1)}+\frac{1}{3(n+1)}+\cdots = x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots$$

$= -\log(1-x)$

$= -\log\left(1-\frac{1}{n+1}\right)$

$= -\log\left(\frac{n+1-1}{n+1}\right)$

$= -\log\left(\frac{n}{n+1}\right)$

$= \log\left(\frac{n}{n+1}\right)^{-1}$

$= \log\left(\frac{n+1}{n}\right)$

$= \log\left(\frac{n}{n}+\frac{1}{n}\right)$

$= \log\left(1+\frac{1}{n}\right)$

$= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots$

6. If $x>0$, show that $\log x = \frac{x-1}{x+1} + \frac{1}{2}\,\frac{x^2-1}{(x+1)^2} + \frac{1}{3}\,\frac{x^3-1}{(x+1)^3} + \cdots \infty$

Soln: Rearranging the terms of R.H.S

$$\frac{x-1}{x+1} + \frac{1}{2}\cdot\frac{x^2-1}{(x+1)^2} + \frac{1}{3}\cdot\frac{x^3-1}{(x+1)^3} + \cdots = \left[\left(\frac{x}{x+1}\right) + \frac{1}{2}\left(\frac{x}{x+1}\right)^2 + \frac{1}{3}\left(\frac{x}{x+1}\right)^3 + \cdots\right]$$

$$= \left[\left(\frac{1}{x+1}\right) + \frac{1}{2}\left(\frac{1}{x+1}\right)^2 + \frac{1}{3}\left(\frac{1}{x+1}\right)^3 + \cdots\right]$$

$$= \left[-\log\left(1-\frac{x}{x+1}\right)\right] - \left[-\log\left(1-\frac{1}{x+1}\right)\right]$$

$$= -\log\frac{1}{x+1} + \log\frac{x}{x+1}$$

$$= \log\left(\frac{x}{\cancel{x+1}} \times \frac{\cancel{x+1}}{1}\right)$$

$$= \log x$$

7. $$\log\sqrt{12} = 1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\cdot\frac{1}{4^3}+\cdots\infty$$

Sln:- $$1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\cdot\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\cdots$$

$$= 1+\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{4^2}+\frac{1}{5}\cdot\frac{1}{4^2}+\frac{1}{6}\cdot\frac{1}{4^3}+\frac{1}{7}\cdot\frac{1}{4^4}+\cdots$$

$$= \left[1+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{5}\cdot\frac{1}{4^2}+\frac{1}{7}\cdot\frac{1}{4^3}+\cdots\right]$$

$$+\left[\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{4^2}+\frac{1}{6}\cdot\frac{1}{4^3}+\cdots\right]$$

$$= \left[1+\frac{1}{3}\left(\frac{1}{2}\right)^2+\frac{1}{5}\left(\frac{1}{2}\right)^4+\frac{1}{7}\left(\frac{1}{2}\right)^6+\cdots\right]$$

$$+\frac{1}{2}\left[\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4^2}+\frac{1}{3}\cdot\frac{1}{4^3}+\cdots\right]$$

Multiply & divide by 2 in 1st term.

$$= 2\left[\frac{1}{2}+\frac{1}{3}\left(\frac{1}{2}\right)^3+\frac{1}{5}\left(\frac{1}{2}\right)^5+\frac{1}{7}\left(\frac{1}{2}\right)^7+\cdots\right]$$

$$+\frac{1}{2}\left[-\log(1-1/4)\right]$$

$$= 2\left[\frac{1}{2}\log\left[\frac{1+1/2}{1-1/2}\right]\right]-1/2\cdot\log\left(\frac{4-1}{4}\right)$$

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)=x+\frac{x^3}{3}+\frac{x^5}{5}$$

$$= \cancel{2}\left[\frac{1}{\cancel{x}}\cdot\log\left(\frac{3/x}{1/x}\right)\right]-\frac{1}{2}\cdot\log\left(\frac{3}{4}\right)$$

$$= \log 3-\frac{1}{2}\log 3+\log 4\cdot 1/2$$

$$= \frac{1}{2}\log 3\left[1-1/2\right]+\frac{1}{2}\log 4$$

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$$= \log 3 \left(\frac{1}{2}\right) \mp \frac{1}{2} \log 4$$

$$= \frac{1}{2} \cdot \log 3 \mp \frac{1}{2} \log 4$$

$$= \frac{1}{2}\left[\log 3 \mp \log 4\right]$$

$$= \frac{1}{2} \log (3 \times 4)$$

$$= \frac{1}{2} \log 12$$

$$= \log 12^{1/2}$$

$$= \log \sqrt{12}.$$

Coefficient of x^n in logarithmic expansion.

1. Find the coefficient of x^n in the expansion of $\log(1+2x-3x^2)$ in ascending powers of x. For what values of x is the expansion valid?

(i) $-2/n$ if n is a multiple of 3.

(ii) $1/n$ if n is ~~real~~ not a multiple of 3.

Soln:

$$\log(1+x+x^2) = \log \frac{1-x^3}{1-x}$$

$$= \log(1-x^3) - \log(1-x)$$

$$= -\left[x^3 + \frac{(x^3)^2}{2} + \frac{(x^3)^3}{3} + \dots\right] + \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \dots\right]$$

Series:-1. In the first series the powers of x are multiple of three, namely $3, 6, 9, \dots$

When n is a multiple of 3, $n/3$ is an integer and in this series, the term containing x^n is

$$\frac{-(x^3)^{n/3}}{n/3} \text{ or } -\frac{3}{n}\cdot x^n.$$

So the coefficient of x^n is $-3/n$.

If n is not a multiple of 3, then this series will not contain the x^n term and the coefficient of x^n is 0.

Series 2:- In the second series the powers of x are continuously increasing.

They are $1, 2, 3, \ldots n$

In this series, the coefficient of x^n is $1/n$ immaterial of whether n is a multiple of 3 or not.

Thus, in the expansion of $\log(1+x+x^2)$, we get the following results:

(i) If n is a multiple of 3, the required coefficient of x^n is $-\frac{3}{n}+\frac{1}{n}$ (or) $-\frac{2}{n}$.

(ii) If n is not a multiple of 3, then it is $1/n$.

06/8/24

Summation (logarithmic series).

There are certain series whose sums can be obtained by identifying them as logarithmic series after splitting their nth terms into partial fractions.

The following results are useful in aspect.

1. Show that $\frac{1}{1\cdot 2}+\frac{1}{3\cdot 4}+\frac{1}{5\cdot 6}+\cdots=\log 2$

Soln: In the denominators, the first factors are odd numbers whose nth term is $(2n-1)$.

$$t_n=\frac{1}{(2n-1)(2n)} \longrightarrow ①$$

$$\frac{1}{(2n-1)(2n)}=\frac{A}{2n-1}+\frac{B}{2n} \rightarrow ②$$

$$1=A(2n)+B(2n-1) \rightarrow ③$$

Put $n=0$ in ③

$$1=0-B$$

$$\boxed{B=-1}$$

$$t_n=\frac{1}{(2n-1)(2n)}$$

Put $n = 1/2$ in ③

$$1 = A(\cancel{2} \cdot \tfrac{1}{\cancel{2}}) + B(\cancel{2} \cdot \tfrac{1}{2}) - 1$$

$$1 = A + 0$$

$$\boxed{A = 1}$$

Sub A & B Values in ②

$$t_n = \frac{1}{2n-1} + \frac{-1}{2n}.$$

Put $n = 1, 2, 3, \ldots$

$$\left.\begin{aligned} t_1 &= \frac{1}{1} - \frac{1}{2} \\ t_2 &= \frac{1}{3} - \frac{1}{4} \\ t_3 &= \frac{1}{5} - \frac{1}{6} \end{aligned}\right\} — ④$$

Adding ④

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$\therefore \boxed{S = \log 2}$.

∴ Hence, it is proved.

. Show that $\frac{1}{1\cdot 3} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 7} + \cdots = 2 - 2\log 2.$

$$t_n = \frac{2}{(2n)(2n+1)} — ①$$

$$\frac{2}{(2n)(2n+1)} = \frac{A}{(2n+1)} + \frac{B}{2n} — ②$$

$$1 = A(2n) + B(2n+1) — ③$$

Put $n = 0$ in ③

$$2 = A$$

$$\boxed{A = 2}$$

Put $n = -1/2$ in ③

$$2 = A(\cancel{2} \times -1/2) + A + B(2 - 1/2)$$

$$2 = -\cancel{A} + \cancel{A} - B$$

$$\boxed{B = -2}$$

sub A & B values in ②

$$\frac{2}{2n(2n+1)} = \frac{2}{2n} - \frac{2}{2n+1} \quad \text{— ④}$$

sub ④ in ①

$$t_n = \frac{2}{2n} - \frac{2}{2n+1}$$

$$t_n = 2\left[\frac{1}{2n} - \frac{1}{2n+1}\right] \quad \text{— ⑤}$$

Put $n = 1$ in ⑤

$$\left.\begin{aligned} t_1 &= 2\left[\frac{1}{2} - \frac{1}{3}\right] \\ t_2 &= 2\left[\frac{1}{4} - \frac{1}{5}\right] \\ t_3 &= 2\left[\frac{1}{6} - \frac{1}{7}\right] \end{aligned}\right\} \quad \text{— ⑥}$$

Adding ⑥.

$$S = 2\left[\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \cdots\right]$$

$$= -2\left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \cdots\right]$$

$$= -2\left[(1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \cdots) - 1\right]$$

$$= -2\left[\log 2 - 1\right]$$

$$= -2\log 2 + 2$$

$$\therefore \boxed{S = 2 - 2\log 2}$$

∴ Hence, it is proved.

Show that $\frac{1}{1\cdot 2} - \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} - \cdots = 2\log 2 - 1$

Sol:

$$t_n = (-1)^{n+1}\left[\frac{1}{n} - \frac{1}{n+1}\right] \quad \text{—} ①$$

Put $n = 1, 2, 3$ in ①

$$\left.\begin{aligned} t_1 &= \frac{1}{1} - \frac{1}{2} \\ t_2 &= -\frac{1}{2} + \frac{1}{3} \\ t_3 &= \frac{1}{3} - \frac{1}{4} \\ t_4 &= -\frac{1}{4} + \frac{1}{5} \end{aligned}\right\} \quad \text{—} ②$$

Adding ② Vertically.

$$S = \frac{1}{1} + \left[-\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{3} - \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{5} + \cdots\right]$$

$$= 1 + 2\left[-\tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots\right]$$

$$= 1 + 2\left[\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots\right) - 1\right]$$

$$= 1 + 2\left[\log 2 - 1\right]$$

$$= 1 + 2\log 2 - 2$$

$$\boxed{\therefore S = 2\log 2 - 1}$$

$t_n = (-1)^{n+1}\left[\frac{1}{n} + \frac{1}{n+1}\right]$

$\therefore$ Hence, it is proved.

5. $\log_3 e - \log_9 e + \log_{27} e \cdots = \dfrac{\log_e 2}{\log_e 3}$.

Sol: Changing the bases to e in each term.

$$\log_3 e = \frac{1}{\log_e 3}$$

$$-\log_9 e = \frac{-1}{\log_e 9} = \frac{-1}{\log_e 3^2}$$

$$= \frac{1}{2\log 3}$$

$$\log_{27} e = \frac{1}{\log_e 27} = \frac{1}{\log_e 3^3}$$

$$= \frac{1}{3\log 3}$$

$$S = \frac{1}{\log_e 3} \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots\right]$$

$$= \frac{1}{\log_e 3} \left[\log_e 2\right]$$

$$\therefore \boxed{S = \frac{\log_e 2}{\log_e 3}}$$

23/8/24 Taylor Series.

To find the numeric solution of the equation.

$$\frac{dy}{dx} = f(x, y) \longrightarrow (1)$$

Given the initial condition $y(x_0) = y_0 \longrightarrow (2)$

Now, we expand $y(x)$ about the point $x = x_0$ in a Taylor's series in powers of $(x - x_0)$.

i.e) $$y(x) = y(x_0) + \frac{(x-x_0)}{1!} y'(x_0) + \frac{(x-x_0)^2}{2!} y^2(x_0) + \cdots \longrightarrow (3)$$

Where $y^{(r)} = (x_0) = \left(\frac{d^r y}{dx^r}\right) . x = x_0$

i.e) $$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \cdots$$

$$y_1 = y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \longrightarrow (4)$$

Where $h = x_1 - x_0$ or $x_1 = x_0 + h$.

To find $y_0', y_0'', \dots$ we use (1) and its derivatives at $x = x_0$.

Now, having got y_1, we can calculate $y_1', y_1'', y_1''' \dots$ etc by using $y' = f(x, y)$ about the point $x = x_1$, we get.

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots \longrightarrow (5)$$

Proceeding in the same way, we get

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \cdots \longrightarrow (6)$$

Where $y_n^r = \left(\frac{d^r y}{dx^r}\right)_{(x_n, y_n)}$

The terms upto and including h^n and neglect terms involving h^{n+1} and higher powers of h, the taylor algorithm used is said to be of nth order.

If h is small and the terms after n terms are neglected, the error is $\frac{h^n}{n!} f^n(Q)$ where $x_0 < Q < x$. if $x_1 - x_0 = h$.

1. Solve $\frac{dy}{dx} = x+y$ given $y(1)=0$ and get $y(1\cdot1)$ $y(1\cdot2)$ by taylor series method Compare your result with the explicit.

Soln: Here $x_0 = 1$, $y_0 = 0$, $h = 0\cdot1$

$y' = x+y \qquad y_0 = y(x=1) = 0$

$y'' = 1+y' \qquad y_0' = x_0 + y_0 = 1+0 = 1$

$y''' = y'' \qquad y_0'' = 1+y_0' = 2$

$y^{IV} = y''' \qquad y_0''' = y_0'' = 2$

$y_0^{IV} = 2$

$y_0' = 1+0$, $y_0' = 1$, $y_0'' = 1+y_0'$, $= 1+$, $y_0'' = 2$, $y_0''' = y_0''$, $y''' = 2$

$x_0 = 1$, $y_0 = 0$, $h = 0\cdot1$

By Taylor series, we have

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{IV} + \cdots$$

$$\therefore y_1 = y(1\cdot1) = 0 + \frac{0\cdot1}{1}(1) + \frac{(0\cdot1)^2}{2}(2) + \frac{(0\cdot1)^3}{6}(2) + \frac{(0\cdot1)^4}{24}(2) + \frac{(0\cdot1)^5}{120}(2) + \cdots \quad (2)$$

$$= 0\cdot1 + 0\cdot01 + 0\cdot00033 + 0\cdot00000833 + 0\cdot000000166 + \cdots$$

$y(1\cdot1) = 0\cdot11033847$

Now, take $x_0 = 1\cdot1$, $h = 0\cdot1$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{IV} + \cdots \longrightarrow (3)$$

We calculate y_1', y_1'', y_1'''.....

$x_1 = 1\cdot1$; $y_1 = 0\cdot11033847$

$y_1' = x_1 + y_1$

$= 1\cdot1 + 0\cdot11033847$

$y_1' = 1\cdot21033847$

$y_1'' = 1 + y_1'$

$= 1 + 1\cdot21033847$

$y_1'' = 2\cdot21033847$

$y_1''' = y_1'' = y_1^{IV} = y_1^{V} = \cdots = 2.21033847$

$y_2 = y(1.2) = 0.11033847 + \frac{0.1}{1}(1.21033847) + \frac{(0.1)^2}{2}(2.21033847) +$

$\frac{(0.1)^3}{6}(2.21033847) + \frac{(0.1)^4}{24}(2.21033847) + \cdots$

$= 0.11033847 + 0.121033847 + 2.21033847(0.005 + 0.0016666 + \cdots)$

$y_2 = 0.2461077$

To exact solution of $\frac{dy}{dx} = x+y$ is

$y = -x-1+2e^{x-1}$

$y(1.1) = -1.1-1+2e^{0.1} = -2.1+2e^{0.1}$
$= 0.11034 \quad = -2.1+2.2103$

$y(1.2) = -1.2-1+2e^{0.2}$
$= -1.2-$

$y(1.2) = 0.2428$

$y(1.1) = 0.11033847$

$y(1.2) = 0.2461077.$

Exact Values $y(1.1) = 0.110341836.$
$y(1.2) = 0.24280552.$

2) Using Taylor series method, find correct to four decimal places, the Values of $y(0.1)$, given $\frac{dy}{dx} = x^2+y^2$ and $y(0)=1$.

$y(0) = 1$
$y(x_0) = y_0$
$x_0 = 0, y_0 = 1$

Sol: We have, $y' = x^2+y^2$

$y'' = 2x+2yy'$

$y''' = 2+2yy''+2(y')^2$

$y^{IV} = 2yy''+2y'y''+4y'y''$
$= 2yy'''+6y'y''$

$y_0' = x_0^2+y_0^2$
$= 0+1$

$y_0' = 1$

$y_0'' = 2x_0+2y_0y_0'$
$= 2$

$y_0''' = 2+2y_0y_0''+2(y_0')^2$
$= 2+2(1)(2)+2(1)^2$

$= 8$

$y_0^{IV} = 2y_0y_0'''+6y_0'y_0''$
$= 2\times1\times8+6(1)(2)$

$y_0^{IV} = 28$

By Taylor series method,

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots$$

$$y(0.1) = y_1 = 1 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(8) + \frac{(0.1)^4}{24}(28 + \ldots$$

$$= 1 + 0.1 + 0.01 + 0.001333333 + 0.000116666$$

$$= 1.11144999$$

$$\approx 1.11145$$

3. Using Taylor series method, find $y(1.1)$ & $y(1.2)$, correct to four decimal places given. $\frac{dy}{dx} = xy^{1/3}$ and $y(1) = 1$

Soln: Take $x_0 = 1, y_0 = 1, h = 0.1$

$$\frac{dy}{dx} = x \cdot y^{1/3}$$

$$y' = xy^{1/3}$$

$$y'' = \frac{1}{3}xy^{-2/3}y' + y^{1/3}.$$

$$y''' = \left[\frac{x^2}{3}\left(-\frac{1}{3}\right)y^{-4/3}y' + \frac{2x}{3}y^{-1/3} + \frac{1}{3}y^{-2/3}y'\right]$$

$$y_0' = 1(1)^{1/3} = 1$$

$$y_0'' = \frac{1}{3}x_0 y_0^{-2/3}y_0' + y_0^{1/3}$$

$$= \frac{1}{3} + 1$$

$$y_0'' = 4/3.$$

$$y_0''' = \left[\frac{x_0^2}{3}\left(-\frac{1}{3}\right)y_0^{-4/3} + \frac{2x_0}{3}y_0^{-1/3} + \frac{1}{3}y_0^{-2/3}y_0'\right]$$

$$= \frac{1}{3}\left(-\frac{1}{3}\right)(1)^{-4/3} + \frac{2}{3}(1)^{-1/3} + \frac{1}{3}(1)^{-2/3}]$$

$$y_0''' = \frac{8}{9}$$

By Taylor series.

$$y_1 = y_0 + hy_0' + \frac{h^2}{y_0''} + \cdots$$

$$y_1 = y(1.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{2}\left(\frac{4}{3}\right) + \frac{(0.1)^3}{6}\left(\frac{8}{9}\right) + \cdots$$

$$= 1 + 0.1 + 0.00666 + 0.000148 + \cdots$$

$$y_1 = 1.10681$$

We start with (x_1, y_1) as the starting value

$$y_1 = 1.10681$$

$$y_1' = x_1 y_1^{1/3}$$

$$= (1.1)(1.10681)^{1/3}$$

$$\boxed{y_1' = 1.13785}$$

$$y_1'' = \frac{1}{3}x_1 y_1^{-2/3}y_1' + y_1^{1/3}$$

$$= \frac{1}{3}(1.1)(1.10681)^{-2/3}(1.13785) + (1.10681)^{1/3}$$

$$= 0.38992 + 1.03441$$

$$= 1.42433$$

$$\boxed{y_1''' = 0.929787}$$

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \cdots$$

$$y(1.2) = 1.10681 + (0.1)(1.13785) + \frac{0.01}{2}(1.42\ldots$$

$$+ \frac{0.001}{6}(0.929787) + \cdots$$

$$y(1.2) = 1.227722$$

Additional Resource :

https://r.search.yahoo.com/_ylt=AwrPpnka8DZn8AEAH6S7HAx.;_ylu=Y29sbwNzZzMEcG9zAzEEdnRpZAMEc2VjA3Ny/RV=2/RE=1732863259/RO=10/RU=https%3a%2f%2ftutorial.math.lamar.edu%2fClasses%2fCalcII%2fBinomialSeries.aspx/RK=2/RS=KvK6diaYJEyBmJQSJRPpQQgmGA0-

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Practice Questions:

Question Bank

Section – A

1. Find the coefficient of x^n in the expansion of $\frac{1}{(1-x^2)}$.
2. Find the coefficient of x^{2n} in the expansion of$(1-x^2)^{-1}$.
3. Find the coefficient of x^2 in the expansion of$(1+x)^{-3}$.
4. Find the coefficient of x^n in $\frac{1}{1-2x}+\frac{1}{1-3x}$.
5. Find the sum of the series $1+2\left(\frac{1}{2}\right)+3\left(\frac{1}{2}\right)^2+\cdots\infty$
6. Find the sum of the series $1+2\left(\frac{1}{3}\right)+3\left(\frac{1}{3}\right)^2+\cdots\infty$
7. Find the coefficient ofx^n in expansion of $(2+3x)^{-1}$ in ascending power of x
8. Find the coefficient of x^n in expansion$[1+2x+3x^2+4x^3+\cdots]^2$
9. Find the coefficient of x^2 in the expansion of $\left(1+\frac{2}{3x}\right)^{\frac{3}{2}}$
10. Define binomial series.
11. If a, b and $n>0$, find the values of $1+\frac{na}{a+b}+\frac{n(n+1)}{2!}\left[\frac{a}{a+b}\right]^2+\cdots$
12. Define exponential series.
13. Show that $\frac{1}{2}\left(e-\frac{1}{e}\right)=1+\frac{1}{3!}+\frac{1}{5!}+\cdots$
14. Sum the series $1+\frac{3^2}{2!}+\frac{3^4}{4!}+\frac{3^6}{6!}+\cdots$
15. Define logarithmic series

Section – B

1. If x is small, what is the value of a if prove that$\sqrt{x^2+4}-\sqrt{x^2+1}=1-\frac{1}{4}x^2+\frac{7}{64}x^4$ nearly.
2. When x is small prove that$\sqrt{x^2+4}-\sqrt{x^2+1}=1-\frac{1}{4}x^2+\frac{7}{64}x^4$ nearly.
3. . Sum the series $1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\cdots$

4. Sum the series $\frac{1}{3.6}+\frac{1.3}{3.6.9}+\frac{1.3.5}{3.6.9.12}+\cdots$
5. Sum the infinity the series $\frac{7}{9}+\frac{7.9}{9.12}+\frac{7.9.11}{9.12.15}+\cdots$
6. Show that $\sqrt{8}=1+\frac{3}{4}+\frac{3.5}{2.4^2}+\frac{3.5.7}{2.3.4^2}+\cdots$
7. Sum the series $\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\cdots$
8. Sum to infinity of the series $\frac{5}{3.6}\frac{1}{4^2}+\frac{5.8}{3.6.9}\frac{1}{4^3}+\frac{5.8.11}{3.6.9.12}\frac{1}{4^4}\ldots$
9. If a, b and $n>0$, find the values of $1+\frac{na}{a+b}+\frac{n(n+1)}{2!}\left[\frac{a}{a+b}\right]^2+\cdots$
10. Prove that $x^n=1+n\left(1-\frac{1}{x}\right)+\frac{n(n+1)}{1.2}\left(1-\frac{1}{x}\right)^2+\cdots$
11. Sum the series $1-\frac{3}{4}+\frac{3.5}{4.8}-\frac{3.5.7}{4.8.12}+\cdots$
12. Find the sum of the series $\frac{1.4}{5.10}-\frac{1.4.7}{5.10.15}+\frac{1.4.7.10}{5.10.15.20}-\cdots$
13. Sum the series $\frac{1+3x}{1!}+\frac{(1+3x)^2}{2!}+\frac{(1+3x)^3}{3!}+\cdots$
14. Show that $\frac{1}{2}\left(e-\frac{1}{e}\right)=1+\frac{1}{3!}+\frac{1}{5!}+\cdots$
15. Sum the series $1+\frac{3^2}{2!}+\frac{3^4}{4!}+\frac{3^6}{6!}+\cdots$

Section – C

1. Sum the series$\frac{2^2}{1!}+\frac{3^2}{2!}+\frac{4^2}{3!}+\ldots$
2. Show that $\frac{5}{1.2.3}+\frac{7}{3.4.5}+\frac{9}{5.6.7}+\cdots=3\log 2-1$
3. Sum to infinity of the series $1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\cdots$
4. Sum the series$\frac{5}{1.2}(\frac{1}{3})+\frac{5.7}{1.2.3}\left(\frac{1}{3}\right)^2+\frac{5.7.9}{1.2.3.4}(\frac{1}{3})^3+\ldots$
5. Sum the series $\frac{1.2}{1!}+\frac{2.3}{2!}+\frac{3.4}{3!}+\cdots$
6. Find the sum to infinity of series $\frac{1.4}{5.10}+\frac{1.4.7}{5.10.15}+\frac{1.4.7.10}{510.15.20}+\cdots$
7. Prove that $\sum_{n=0}^{\infty}\frac{5n+1}{(2n+1)!}=\frac{e}{2}+\frac{2}{e}$

Text Book: Duraipandian, P. and Udaya Baskaran, S. (2014): Allied Mathematics, Vol. – I I,S.Chand& Company Pvt. Ltd.