MARDHAR KESARI JAIN COLLEGE FOR WOMEN, VANIYAMBADI

PG AND RESEARCH DEPARTMENT OF MATHEMATICS

CLASS	: I M. Sc. MATHEMATICS
SUBJECT CODE	: 23PMA13
SUBJECT NAME	: ORDINARY DIFFERENTIAL EQUATIONS

SYLLABUS

UNIT V: Existence and uniqueness of solutions to first order equations

Equation with variable separated – Exact equation – method of successive approximations–the Lipschitz condition–convergence of the successive approximations and the existence theorem.

UNIT-V

Existence and Uniqueness of solutions of First order equations:

Introduction:

Consider the general 1^{st} order equation

$$y' = f(x,y) \longrightarrow (1)$$

where f is some continuous function.

If The linear equation is

$$y' = g(x)y + h(x) \longrightarrow (2)$$

where g, h are continuous on some interval I.
then Any solution ϕ of eqn. (2) can be written in the form,

$$\phi(x) = e^{Q(x)} \int_{x_0}^{x} e^{-Q(t)} h(t)\, dt + ce^{Q(x)} \longrightarrow (3)$$

where, $Q(x) = \int_{x_0}^{x} g(t)\,dt$,

x_0 is in I and c is a constant.

Equations with Variable separated:

A first order eqn.

$$y' = f(x,y)$$

is said to have the Variables seperated if f can be written in the form

$$f(x,y) = \frac{g(x)}{h(y)}.$$

Where g, h are fun.'s of a single argument.

We can write the eqn. as

$$h(y)\frac{dy}{dx} = g(x) \Rightarrow \frac{dy}{dx} = f(x,y) = \frac{g(x)}{h(y)}$$

x, y' $\longrightarrow$ (1)

$$\Rightarrow h(y)\,dy = g(x)\,dx.$$

See statement after the proof

Stmt: If ϕ is a real-valued solution of (1) on some interval I containing a point x_0 then.

$$h(\phi(x))\cdot\phi'(x) = g(x)$$

for all x in I.

$$\therefore \int_{x_0}^{x} h(\phi(t))\,\phi'(t)\,dt = \int_{x_0}^{x} g(t)\,dt \longrightarrow (2)$$

for all x in I

$t = x_0$, $u = \phi(x_0)$

$t = x$, $u = \phi(x)$

Let $u = \phi(t)$ in the integral on the left in (2)

$\therefore$ The above equation becomes,

$$\int_{\phi(x_0)}^{\phi(x)} h(u)\,du = \int_{x_0}^{x} g(t)\,dt.$$

Conversely,

suppose x and y are related by the formula,

$$\int_{y_0}^{y} h(u)\,du = \int_{x_0}^{x} g(t)\,dt \longrightarrow (3)$$

and that this defines implicity a differentiable fun. ϕ for x in I.

Then, this fun. satisfies

$$\int_{y_0}^{\phi(x)} h(u)\,du = \int_{x_0}^{x} g(t)\,dt.$$

for all x in I.

And differentiating we get

$$h(\phi(x))\,\phi'(x) = g(x).$$

whis shows that ϕ is a soln. of eqn. (1) on I.

$$\Rightarrow h(y)\,dy = g(x)dx. \rightarrow h(y)\frac{dy}{dx} = g(x).$$

(thus seperating the variables). and integrate we get,

$$\int h(y)dy = \int g(x)dx + c.$$

where c is constant, and the integrals are anti-derivatives. → are the inverse operation of differentiation

Thus,

$$H(y) = \int h(y)dy,$$

$$G(x) = \int g(x)dx,$$

represent any two fun.'s H, G s.t

$$H' = h, \quad G' = g.$$

Then any differentiable fun. ϕ which is defined implicity by the relation,

$$H(y) = G(x) + C. \longrightarrow (4).$$

which is the soln. of eqn. (1).

Theorem:

Statement:

Let g, h be continuous real valued functions for $a \le x \le b$, $c \le y \le d$ respectively and consider the equation $h(y)y' = g(x) \longrightarrow (1)$.

If G and H are any fun.'s s.t $G' = g$, $H' = h$ and c is any constant s.t the relation $H(y) = G(x) + c$ defines a real-valued differentiable fun. ϕ for x in some interval I contained in $a \le x \le b$, then ϕ will be a soln. of $h(y)y' = g(x)$ on I. Conversely, if ϕ is a soln. of eqn. (1) on I, it satisfies the relation $H(y) = G(x) + c$ on I, for some constant c.

Example: 1.

Suppose $h(y) = 1$ then the diffr. eqn. is $y' = g(x)$. $\rightarrow$ Integrating we get,

$$y = \int g(x)\,dx + C = G(x) + C$$

And every solution ϕ has the form,

$$\phi(x) = G(x) + c.$$

where G is defined for the integral any function on $a \le x \le b$ s.t $G' = g$. and c is a constant.

Example : 2.

Suppose $g(x) = 1$ then we have

$$h(y)\,y' = 1$$

$$\Rightarrow \quad y' = \frac{1}{h(y)} \rightarrow (1)$$

$$(\text{or}) \quad \frac{dy}{dx} = \frac{1}{h(y)}$$

$$\Rightarrow h(y)\,dy = dx.$$

Thus, if $H' = h$, any differentiable fun. defined implicitly by the relation,

$$H(y) = x + c \quad \rightarrow (2)$$

where c is constant. then ϕ is a solution of eqn. (1).

consider the eqn.

Example For instance, ~~if~~ $y' = y^2$ then $\frac{dy}{dx} = y^2$

$$\Rightarrow \frac{1}{y^2}\,dy = dx. \quad \hookrightarrow (3)$$

~~$\Rightarrow dy =$~~

Here $h(y) = \frac{1}{y^2}$, which is not continuous at $y = 0$.

$$\frac{dy}{y^2} = dx$$

$$y^{-2}\,dy = dx$$

$\int$ing we get

$$-\frac{1}{y} = x + C.$$

$$\Rightarrow \quad y = -\frac{1}{x+C}.$$

Thus if c is any constant, the fun. ϕ is gn. b.

$$\phi(x) = -\frac{1}{x+C}.$$ is a soln. of eqn. (3)

provided $x \neq -c$.

Remark :

Note that, seperation of variables method of finding solutions may not yield all solutio of an equation.

For Ex: The zero fun. $\psi = 0$ $[\psi(x) = 0 \; \forall x]$ is a soln. of the diff. eqn. $y' = y^2$.

But this cannot be got from the soln.

$$y = -\frac{1}{x+C}.$$

Exact Equation:

Let the first order eqn. $y' = f(x,y)$ is written in the form,

$$y' = \frac{-M(x,y)}{N(x,y)}$$

(or) $M(x,y) + N(x,y)y' = 0 \longrightarrow (1)$.

where M, N are real-valued fun.'s defined for real x, y on some rectangle R. The eqn. (1) is said to be exact in R if ∃ a fun. F having continuous first partial derivatives there such that

$$\frac{\partial F}{\partial x} = M, \ \frac{\partial F}{\partial y} = N \longrightarrow (2)$$

in R.

Theorem: 1

Suppose the equation $M(x,y) + N(x,y)y' = 0 \longrightarrow (1)$ is exact in a rectangle R, and F is a real-valued fun. s.t $\frac{\partial F}{\partial x} = M, \ \frac{\partial F}{\partial y} = N \longrightarrow (2)$ in R. Every differentiable fun. ϕ defined implicitly by a relation $F(x,y) = c$, (c = constant), is a soln. of (1) and every soln. of (1) whose graph lies in R arises..

Proof:

Ans: Suppose $M(x,y) + N(x,y)y' = 0 \longrightarrow (1)$ is exact in R and F is a fun. satisfying

$$\frac{\partial F}{\partial x} = M, \frac{\partial F}{\partial y} = N \text{ in } R$$

Then (1) becomes,

$$\frac{\partial F}{\partial x}(x,y) + \frac{\partial F}{\partial y}(x,y)y' = 0.$$

If ϕ is any soln. on some interval I then

$$\frac{\partial F}{\partial x}(x, \phi(x)) + \frac{\partial F}{\partial y}(x, \phi(x))\phi'(x) = 0 \longrightarrow (3)$$

for all x in I.

If $\Phi(x) = F(x, \phi(x))$ then from eqn. (3)

$$\Rightarrow \Phi'(x) = 0$$

Hence, $F(x, \phi(x)) = c$, where c is some consta

Thus the soln. ϕ must be a fun. which is given implicitly by the relation,

$$F(x,y) = c. \longrightarrow (4)$$

Conversely,

if ϕ is a differentiable fun. on some interval I defined implicitly by the relation $F(x,y) = c$ then $F(x, \phi(x)) = c$. $\forall x \in I$. and differentiating we get eqn. (3).

Thus ϕ is a soln. of eqn. (1).

Theorem : 2

Let M, N be two real-valued functions which have continuous first partial derivatives on some rectangle. $R : |x - x_0| \leq a, |y - y_0| \leq b$

then the eqn. $M(x,y) + N(x,y)y' = 0$ is exact in R if and only if, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ in R.

Proof:

Given: The equation $M(x,y) + N(x,y)y' = 0$ is exact. $\longrightarrow (1)$

To prove: $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. $\longrightarrow (2)$

Suppose the eqn. $M(x,y)dx + N(x,y)dy = 0$ is exact

i.e.) $M(x,y) + N(x,y)y' = 0$ and F is a fun. s.t $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$

$\frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$

Par. Diff w. r. to y. Par. diff. w. r. to x.

$\Rightarrow \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial M}{\partial y}$; $\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial N}{\partial x}$.

since, $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}$.

we get, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Conversely,

given : $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

To find the fun. F satisfying

$\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$.

Consider,

$$F(x,y)-F(x_0,y_0) = F(x,y)-F(x_0,y)+F(x_0,y)-F(x_0,y_0)$$

$$= \int_{x_0}^{x} \frac{\partial F}{\partial x}(s,y)\,ds + \int_{y_0}^{y} \frac{\partial F}{\partial y}(x_0,t)\,dt$$

$$= \int_{x_0}^{x} M(s,y)\,ds + \int_{y_0}^{y} N(x_0,t)\,dt \longrightarrow (3)$$

Illy

$$F(x,y)-F(x_0,y_0) = F(x,y)-F(x,y_0)+F(x,y_0)-F(x_0,y_0)$$

$$= \int_{y_0}^{y} \frac{\partial F}{\partial y}(x,t)\,dt + \int_{x_0}^{x} \frac{\partial F}{\partial x}(s,y_0)\,ds$$

$$F(x,y)-F(x_0,y_0) = \int_{y_0}^{y} N(x,t)\,dt + \int_{x_0}^{x} M(s,y_0)\,ds \rightarrow (4)$$

Let us define F by the formula,

$$F(x,y) = \int_{x_0}^{x} M(s,y)\,ds + \int_{y_0}^{y} N(x_0,t)\,dt \rightarrow (5)$$

$\Rightarrow F(x_0,y_0)=0$ and

par. diff. w.r. to x in eqn (5).

$\frac{\partial F}{\partial x}(x,y) = M(x,y)$ for all (x,y) in R.

|||^{ly} From eqn. (4) F is also given by

$$F(x,y) = \int_{y_0}^{y} N(x,t)\,dt + \int_{x_0}^{x} M(s,y_0)\,ds \longrightarrow (6)$$

partially diff. w.r.to 'y'.

$$\Rightarrow \frac{\partial F}{\partial y}(x,y) = N(x,y). \text{ for all } (x,y) \text{ in } R.$$

$\therefore$ F satisfies $\frac{\partial F}{\partial x} = M$ & $\frac{\partial F}{\partial y} = N$.

Now we show that (5) and (4) are valid, Let us consider the difference.

$$F(x,y) - \left[\int_{y_0}^{y} N(x,t)\,dt + \int_{x_0}^{x} M(s,y_0)\,ds\right]$$

$$= \int_{x_0}^{x} [M(s,y) - M(s,y_0)]\,ds - \int_{y_0}^{y} [N(x,t) - N(x_0,t)]\,dt$$

$$= \int_{x_0}^{x}\left[\int_{y_0}^{y} \frac{\partial M}{\partial y}(s,t)\,dt\right] ds - \int_{y_0}^{y}\left[\int_{x_0}^{x} \frac{\partial N}{\partial x}(s,t)\,ds\right] dt$$

$$= \int_{x_0}^{x}\int_{y_0}^{y}\left[\frac{\partial M}{\partial y}(s,t) - \frac{\partial N}{\partial x}(s,t)\right] ds\,dt$$

which is zero by ~~ver~~ virtue of eqn. (2).

|| Hence the proof.

Example:

Find the soln. of $y' = \dfrac{3x^2 - 2xy}{x^2 - 2y}$,

Soln. :

$$\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 - 2y}$$

$$(x^2 - 2y)\,dy = (3x^2 - 2xy)\,dx.$$

$$\Rightarrow (3x^2 - 2xy)\,dx - (x^2 - 2y)\,dy = 0$$

This is of the form $M\,dx + N\,dy = 0$.

Here, $M = 3x^2 - 2xy$, $N = -(x^2 - 2y)$

$$\frac{\partial M}{\partial y} = -2x \qquad \frac{\partial N}{\partial x} = -2x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x},$$

$\therefore$ this is an exact equation.

To find F:

W.K.T $\dfrac{\partial F}{\partial x} = M$, $\dfrac{\partial F}{\partial y} = N$.

Consider, $\dfrac{\partial F}{\partial x} = 3x^2 - 2xy$.

$\int$ing, we get,

$$F(x,y) = \int 3x^2\,dx - \int 2xy\,dx$$

$$= \frac{\cancel{3}x^3}{\cancel{3}} - \cancel{2}y\frac{x^2}{\cancel{2}} + f(y)$$

$$F(x,y) = x^3 - x^2y + f(y) \longrightarrow (1)$$

consider, $\frac{\partial F}{\partial y} = N$

$\int$ing, we get.

$$\int \frac{\partial}{\partial y} F(x,y) = \int (-x^2 + 2y)\, dy$$

$$F(x,y) =$$

where f is independent of ~~y~~ x.

Now $\frac{\partial F}{\partial y} = N$

$$F = x^3 - x^2 y + f(y)$$

$$\frac{\partial F}{\partial y} = -x^2 + f'(y)$$

$$\cancel{-x^2} + f'(y) = 2y - \cancel{x^2}$$

$$f'(y) = 2y.$$

$\int$ing, we get,

$$f(y) = y^2.$$

Substitute in eqn. ①

$$① \Rightarrow F(x,y) = x^3 - x^2 y + y^2.$$

Any differentiable fun. ϕ which is defined implicitly by the relation

$$x^3 - x^2 y + y^2 = c, \text{ where } c \text{ is a constant.}$$

2. Determine which equations are exact and solve.

(i) $2xy\,dx + (x^2 + 3y^2)\,dy = 0.$

(ii) $(x^2 + xy)\,dx + xy\,dy = 0.$

(iii) $e^x dx + (e^y(y+1))\,dy = 0$

(iv) $\cos x \cos^2 y\, dx - \sin x \sin 2y\, dy = 0$ (v) $x^2 y^3 dx - x^3 y^2 dy$

Soln/:

(i) $2xy\,dx + (x^2+3y^2)\,dy = 0$

$$M = 2xy \,,\quad N = x^2+3y^2$$

$$\frac{\partial M}{\partial y} = 2x \,,\quad \frac{\partial N}{\partial x} = 2x .$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ this is exact eqn/.

To find F :

W.K.T,

$$\frac{\partial F(x,y)}{\partial x} = M \,,\quad \frac{\partial F(x,y)}{\partial y} = N .$$

Consider, $\frac{\partial F(x,y)}{\partial x} = 2xy$

$\int$ing we get,

$$F(x,y) = \int 2xy\, dx = \cancel{2}y\frac{x^2}{\cancel{2}} + f(y)$$

$$F(x,y) = yx^2 + f(y) \longrightarrow (1).$$

where f is an independent of 'x'.

Now $\frac{\partial F}{\partial y} = N$

$$\frac{\partial}{\partial y}(yx^2 + f(y)) = x^2+3y^2$$

$$\cancel{x^2} + f'(y) = \cancel{x^2} + 3y^2$$

$$f'(y) = 3y^2$$

$\int$ing we get,

$$f(y) = \frac{3y^3}{3} = y^3$$

sub in eqn. (1),

$F(x,y) = yx^2 + y^3$

Any diff. fun. ϕ which is defined implicitly by the relation,

$$yx^2 + y^3 = c.$$

(ii) $(x^2 + xy)\,dx + xy\,dy = 0.$

$M = x^2 + xy$, $N = xy$.

$\frac{\partial M}{\partial y} = x$, $\frac{\partial N}{\partial x} = y$.

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

This is not an exact equation.

(iii) $e^x dx + (e^y(y+1))\,dy = 0.$

$M = e^x$, $N = e^y(y+1)$

$\frac{\partial M}{\partial y} = 0$ $\frac{\partial N}{\partial x} = 0.$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$

This is an exact equation.

To find F:

W.K.T

$\frac{\partial F}{\partial x}(x,y) = M$, $\frac{\partial F(x,y)}{\partial y} = N.$

Consider, $\frac{\partial F}{\partial x}(x,y) = M = e^x.$

$\frac{\partial F}{\partial x} = \cancel{x}\, e^x$

$\int$ing, $F(x,y) = \cancel{\frac{x^2}{2}} + f(y) \longrightarrow (1)$

$$\frac{\partial F}{\partial y}(x,y) = e^{y}(y+1)$$

$$\frac{\partial}{\partial y}\left(\frac{x^2}{2}+f(y)\right) = e^{y}(y+1)$$

$$f'(y) = e^{y}(y+1)$$

$\Rightarrow$ Sing, $f(y) = \int e^{y}(y+1)\,dy.$

$$u = y+1, \quad dv = e^{y}$$
$$du = dy \qquad v = e^{y}$$

$$f(y) = (y+1)e^{y} - \int e^{y}\,dy$$

$$f(y) = (y+1)e^{y} - e^{y}$$

$\Rightarrow$ $F(x,y) = \cancel{\frac{x^2}{2}}\, e^{x} + (y+1-1)e^{y}$

$$F(x,y) = \cancel{\frac{x^2}{2}}\, e^{x} + ye^{y}$$

Any diff. fun. ϕ which is defined by the relation $\boxed{\cancel{\frac{x^2}{2}}\, e^{x} + ye^{y} = c}$.

(iv) $\cos x \cos^2 y\, dx - \sin x \sin 2y\, dy = 0$

$M = \cos x \cos^2 y \quad , \quad N = -\sin x \sin 2y$

$\frac{\partial M}{\partial y} = \cos x \, 2\cos y(-\sin y) \quad , \quad \frac{\partial N}{\partial x} = -\sin 2y \cos$

$= -2\cos x \cos y \sin y$

$= -\cos x \sin 2y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}$

To find F: W.R.T $\frac{\partial F(x,y)}{\partial x} = M$, $\frac{\partial F}{\partial y}(x,y) = N$.
consider,

$$\frac{\partial F(x,y)}{\partial x} = M$$

$$\frac{\partial F}{\partial x}(x,y) = \cos x \cos^2 y$$

$\int$ing, we get,

$$F(x,y) = \int \cos x \cos^2 y \, dx = \cos^2 y \int \cos x \, dx$$

$$F(x,y) = \cos^2 y \sin x + f(y) \quad \text{--- (1)}$$

Now,
$$\frac{\partial F}{\partial y}(x,y) = N$$

$$\frac{\partial}{\partial y}\left(\cos^2 y \sin x + f(y)\right) = -\sin x \sin 2y$$

$$-2\sin x \cos y \sin y + f'(y) = -\sin x \sin 2y$$

$$-\cancel{\sin x \sin 2y} + f'(y) = -\cancel{\sin x \sin 2y}$$

$$f'(y) = 0.$$

$$f(y) = \text{Constant } (c)$$

(1) $\Rightarrow F(x,y) = \cos^2 y \sin x + c$

$$\Rightarrow \boxed{\cos^2 y \sin x = c}$$

Equations with Variable seperated:-

1. S.T the solution ϕ of $y'=y^2$ which passes through the point (x_0, y_0) is gn/. by $\phi(x) = \frac{y_0}{1-y_0(x-x_0)}$

Soln/. :

Given : $y' = y^2 \longrightarrow (1)$

$$\frac{dy}{dx} = y^2$$

$$\Rightarrow dy = y^2 dx$$

$$y^{-2} dy = dx.$$

$\int$ing

$$-\frac{1}{y} = x + c \longrightarrow (2)$$

Which passes through the point (x_0, y_0)

$$\Rightarrow -\frac{1}{y_0} = x_0 + c$$

$$-\frac{1}{y_0} - x_0 = c$$

$$-\left(\frac{1}{y_0} + x_0\right) = c$$

$$\Rightarrow -\left(\frac{x_0 y_0 + 1}{y_0}\right) = c.$$

From (2),

$$-\frac{1}{y} = x + c$$

$$-\frac{1}{x+c} = y$$

$$\Rightarrow y = -\frac{1}{x+c}$$

$$\Rightarrow y = -\frac{1}{x - \left(\frac{1 + x_0 y_0}{y_0}\right)}$$

$$= \frac{-y_0}{x y_0 - 1 - x_0 y_0}$$

$$= \frac{-y_0}{-1 - x_0 y_0 + x y_0}$$

$$= \frac{-y_0}{-(1 + x_0 y_0 - x y_0)}$$

$$y = \frac{y_0}{1 - (x - x_0) y_0}$$

Q. Find the soln. of the following eqn.'s.

(i) $y' = x^2 y$.

Soln. :

$$\frac{dy}{dx} = x^2 y$$

$$\frac{dy}{y} = x^2 dx.$$

Sing, $\log y = \frac{x^3}{3} + c$.

Taking exponential on b.s.

$$y = e^{x^3/3 + c} = c e^{x^3/3}.$$

(b) $y' = x^2 y^2 - 4x^2$

$$y = \frac{x^3 y^2}{2} - \frac{4x^3}{3} + c.$$

$$\frac{dy}{dx} = x^2 (y^2 - 4)$$

$$\frac{dy}{y^2 - 4} = x^2 dx$$

$$\int \frac{dy}{y^2 - 2^2} = \int x^2 dx.$$

$$\left[\frac{1}{2 \times 2} \log\left(\frac{y-2}{y+2}\right)\right] = \frac{x^3}{3} + c.$$

$$\left(\because \int \frac{dy}{y^2 - a^2} = \frac{1}{2a} \log\left(\frac{y-a}{y+a}\right) + c\right)$$

(ii) $yy' = x$

Soln:

$$y\frac{dy}{dx} = x$$

$$\Rightarrow y\,dy = x\,dx$$

$\int$ing,

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\Rightarrow \frac{y^2}{2} - \frac{x^2}{2} = c$$

$$\Rightarrow y^2 - x^2 = 2c$$

$$\Rightarrow y^2 - x^2 = c$$

$$\Rightarrow y^2 = x^2 + c$$ //.

(iii) $y' = \frac{e^{x-y}}{1+e^x}$

Soln:

$$\frac{dy}{dx} = \frac{e^{x-y}}{1+e^x}$$

$$\frac{dy}{dx} = \frac{e^x \cdot e^{-y}}{1+e^x}$$

$$e^y dy = \frac{e^x}{1+e^x} dx$$

$\int$ing,

$$e^y = \log(1+e^x) + c$$

(iv) $y' = \frac{x+x^2}{y-y^2}$.

Soln:

$$\frac{dy}{dx} = \frac{x+x^2}{y-y^2}$$

$$(y-y^2)\,dy = (x+x^2)\,dx$$

$\int$ing,

$$\frac{y^2}{2} - \frac{y^3}{3} = \frac{x^2}{2} + \frac{x^3}{3} + c$$

$$\frac{3y^2 - 2y^3}{6} = \frac{3x^2 + 2x^3}{6} + c$$

$$\frac{3y^2 - 2y^3}{6} - \frac{3x^2 + 2x^3}{6} = c$$

$$\Rightarrow 3y^2 - 2y^3 - 3x^2 - 2x^3 = 6c$$

$$\Rightarrow 3y^2 - 2y^3 - 3x^2 - 2x^3 = 6c$$

$$\Rightarrow 3y^2 - 2y^3 = 3x^2 + 2x^3 + c$$

UQ Exact Equation problem:

1. Consider the eqn. $M(x,y)dx + N(x,y)dy = 0$, where M, N have continuous first partial derivatives on some rectangle R. P.T a function u on R, having continuous first partial derivatives, is an integrating factor iff

$$u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y} \text{ on } R.$$

Proof:

The eqn. $M(x,y)dx + N(x,y)dy = 0 \longrightarrow (1)$

Let a fun. u on R having continuous first partial derivatives is an integrating factor.

$\therefore u(x,y)M(x,y)\,dx + u(x,y)N(x,y)dy = 0$

ie.) $uMdx + uNdy = 0$

$\Rightarrow M_1dx + N_1dy = 0$

Let us assume that the eqn. is exact.

ie.) eqn. (1) is exact.

Here $M_1 = uM$, $N_1 = uN$.

$$\frac{\partial M_1}{\partial y} = u\cdot\frac{\partial M}{\partial y} + M\cdot\frac{\partial u}{\partial y}.$$

$$\frac{\partial N_1}{\partial x} = u\cdot\frac{\partial N}{\partial x} + N\cdot\frac{\partial u}{\partial x}.$$

since, the eqn. is exact.

ie, $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$.

$$\therefore u\frac{\partial M}{\partial y} + M\cdot\frac{\partial u}{\partial y} = u\frac{\partial N}{\partial x} + N\frac{\partial u}{\partial x}$$

$$u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}.$$

Conversely,

Given: $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

(i.e.) $u\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\frac{\partial u}{\partial x} - M\frac{\partial u}{\partial y}$.

T.P : F satisfies $\frac{\partial F}{\partial x} = M_1,\ \frac{\partial F}{\partial y} = N_1$.

Write converse proof of theorem (2).

Remark :

(1)(a) If the eqn. $M(x,y)dx + N(x,y)dy = 0 \rightarrow (1)$ has an integrating factor u, which is a fun. of x alone then $p = \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a continuous fun. of x alone.

(b) If p is continuous and independent of y, then the integrating factor is given by

$$u(x) = e^{P(x)}.$$

Where P is any fun. satisfying $P' = p$.

(c) An integrating factor u, which is a fun. of y alone, then $q = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a continuous fun. of y alone.

(d) If q is continuous and independent of x, then an integrating factor is giv. by,

$$u(y) = e^{Q(y)}$$

where Q is any fun. s.t $Q' = q$.

Examples:

1. Solve the eqn. $y\,dx - x\,dy = 0$ — (1)

Soln.:

$$M = y \quad , \quad N = -x$$

$$\frac{\partial M}{\partial y} = 1 \quad , \quad \frac{\partial N}{\partial x} = -1.$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ This is not an exact.

To find the Integrating factor:

Let an integrating factor u, which is a fun. of y alone then $q = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$$q = \frac{1}{y}(-1-1) = \frac{-2}{y}$$

$$\text{I.F } u(y) = e^{\int q\,dy} = e^{\int -\frac{2}{y}dy} = e^{-2\int \frac{1}{y}dy} = e^{-2\log y}$$

$$u(y) = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}.$$

$\times^{ly}$ by $u = \frac{1}{y^2}$ in eqn. (1),

$$\frac{1}{y^2} y\,dx - \frac{1}{y^2} x\,dy = 0$$

$$\Rightarrow \frac{dx}{y} - \frac{x}{y^2}dy = 0$$

Here, $M = \frac{1}{y}$, $N = \frac{-x}{y^2}$

$\frac{\partial M}{\partial y} = -\frac{1}{y^2}$ $\frac{\partial N}{\partial x} = \frac{-1}{y^2}$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ it is an exact.

To find F:-

W.K.T $\frac{\partial F}{\partial x} = M, \frac{\partial F}{\partial y} = N$.

Consider, $\frac{\partial F}{\partial x} = M$

$$\frac{\partial F(x,y)}{\partial x} = \frac{1}{y}$$

$\int$ing w.r to 'x'

$$F(x,y) = \frac{x}{y} + f(y) \longrightarrow (2).$$

Now, $\frac{\partial F}{\partial y}(x,y) = N$

$$\frac{\partial}{\partial y}\left(\frac{x}{y} + f(y)\right) = -\frac{x}{y^2}$$

$$-\cancel{\frac{x}{y^2}} + f'(y) = \cancel{\frac{-x}{y^2}}$$

$$f'(y) = 0.$$

$$f(y) = c \text{ (say)}$$

From (2) $\Rightarrow F(x,y) = \frac{x}{y} + c$

Any diff. soln. ϕ which is defined by the relation, $\frac{x}{y} = c \Rightarrow y = cx$ //.

Find an integrating factor and solve.

(a) $(2y^3+2)\,dx + 3xy^2\,dy = 0$ Ans: $x^2(y^3+1) = c$.

(b) $(5x^3y^2 + 2y)\,dx + (3x^4y + 2x)\,dy = 0$ Ans: $x^5y^3 + x^2y^2 = c$.

(c) $(e^y + xe^y)\,dx + xe^y\,dy = 0$ Ans: $xe^{x+y} = c$.

The Method of successive approximations:

Consider the 1^{st} order differential eqn.

$y' = f(x,y)$ with $y(x_0) = y_0 \longrightarrow (1)$

where f is any continuous real valued function on some rectangle R in the real (x,y) plane.

On some Interval I containing x_0 there is a soln. ϕ of (1) satisfying

$$\phi(x_0) = y_0 \longrightarrow (2)$$

By this we mean there is a real-valued differentiable fun. ϕ satisfying (2) such a points $(x, \phi(x))$ are in R for x in I and

$$\phi'(x) = f(x, \phi(x)) \text{ for all } x \text{ in } I.$$

Thus a fun. ϕ is called a soln. to the initial value problem

$$y' = f(x,y),\ y(x_0) = y_0 \longrightarrow (3)$$

on I.

An integral eqn.

$$y = y_0 + \int_{x_0}^{x} f(t, y)\,dt \rightarrow (4)$$

on I. By a soln. of this eqn. on I is meant a real-valued continuous fun. ϕ on I such that $(x, \phi(x))$ is in R for all x in I and

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt. \ \forall \ x \text{ in } I.$$

Theorem;

A fun. ϕ is a soln. of the initial value Problem $y' = f(x, y)$, $y(x_0) = y_0$ on an interval I iff it is soln. of the integral eqn.

$$y = y_0 + \int_{x_0}^{x} f(t, y)\,dt \text{ on } I.$$

Proof:

Given: ϕ is a soln. of IVP, $y' = f(x, y), y(x_0) = y_0$

T.P: ϕ is a soln. of the integral eqn.

Suppose ϕ is a soln. of IV.P on I

then $\phi'(t) = f(t, \phi(t))$. on I — (1)

Since, ϕ is a continuous on I and f is continuous on R, the fun. F defined by

$f(t) = f(t, \phi(t))$ is continuous on I.

ling eqn. (1) we get from x_0 to x we get,

$$\int_{x_0}^{x} \phi'(t)\,dt = \int_{x_0}^{x} f(t, \phi(t))\,dt$$

$$\left[\phi(t)\right]_{x_0}^{x} = \int_{x_0}^{x} f(t, \phi(t))\,dt$$

$$\phi(x) - \phi(x_0) = \int_{x_0}^{x} f(t, \phi(t))\,dt$$

$$\Rightarrow \quad \phi(x) = \phi(x_0) + \int_{x_0}^{x} f(t, \phi(t))\,dt.$$

since, $\phi(x_0) = y_0$.

$$\Rightarrow y = y_0 + \int_{x_0}^{x} f(t, y)\,dt \rightarrow (2).$$

Hence ϕ is a soln. of the integral eqn.

Conversely,

Suppose ϕ satisfies a soln. of the integral equation.

$$\phi(x) = \phi(x_0) + \int_{x_0}^{x} f(t, \phi(t))\,dt.$$

i.e., $\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt$.

Diff. we get,

$\phi'(x) = f(x, \phi(x))$ and clearly $\phi(x_0) = y_0$.

[By the fundamental thm of integral calculus lower limit is a constant, it will vanish].

Thus ϕ is a soln. of I.V.P

$$y' = f(x, y) \quad , \quad y(x_0) = y_0$$

Successive approximation general formula

We now describe the method of successive approximations to obtain a soln. of the integral eqn. (2).

As a first approximation we consider the fun. is defined by $\phi_0(x) = y_0$.

This fun. satisfies IVP Initial condition $\phi_0(x_0) = y_0$, but does not in general satisfy eqn. (2)

However if we compute,

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, \phi_0(t))\,dt$$

$$\phi_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt.$$

We expect that ϕ_1 is a closer approximation to a soln. than ϕ_0.

Infact, if we continue the process and define successively $\phi_0(x) = y_0$

$$\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\,dt, \quad \rightarrow (3)$$

$k = 0, 1, 2, \ldots$

We might expect, on taking the limit as $k \to \infty$, then we obtain, $\phi_k(x) \to \phi(x)$.

where ϕ would satisfy

$$\phi(x) = y_0 + \int_{x_0}^{x} f(t, \phi(t))\,dt.$$

We call the fun.'s $\phi_0, \phi_1, \ldots$ defined above as successive approximations to a solution of the integral eqn.(2) (or) the initial value problem(1)

Remark:

Since f is continuous on R, it is bounded there, that is, $\exists$ a constant $M > 0$ such that $|f(x, y)| \leqq M$. for all (x, y) in R.

Theorem:

The successive approximation ϕ_k, defined by $\phi_0(x) = y_0$, $\phi_{k+1}(x) = y_0 + \int_{x_0}^{x} f(t, \phi_k(t))\,dt$,

$k = 0, 1, 2 \ldots$ exist as continuous fun. on the int/ ~~by~~ $I : |x - x_0| \leq \alpha = \text{minimum}\{a, b/M\}$, and $(x, \phi_k(x))$ is in R for x in I. Indeed the ϕ_k satisfy $|\phi_k(x) - y_0| \leq M|x - x_0|$ → (3)

for all x in I.

Proof:

clearly ϕ_0 exists on I as a continuous fun/. and satisfy the eqn/.

$$|\phi_k(x) - y_0| \leq M|x - x_0| \text{ for } k = 0.$$

when $k = 1$,

$$|\phi_1(x) - y_0| = \left|\int_{x_0}^{x} f(t, y_0)\,dt\right| \leq \left|\int_{x_0}^{x} |f(t, y_0)|\,dt\right|$$

$$\leq M|x - x_0|. \qquad (\because |f(x,y)| \leq M$$

Hence ϕ_1 satisfies the inequality (1).

since, f is continuous on R, the fun/. F_0 defined by,

$$F_0(t) = f(t, y_0).$$

is continuous on I.

Thus ϕ_1 which is gn/. by,

$$\phi_1(x) = y_0 + \int_{x_0}^{x} F_0(t)\,dt.$$

is continuous on I.

Assume that the thm has been proved for the fun/. $\phi_0, \phi_1, \ldots \phi_k$. We p.T it is valid for ϕ_{k+1}.

W.K.T $(t, \phi_k(t))$ is in R for t in I.

Thus the fun. F_k given by

$$F_k(t) = f(t, \phi_k(t)) \text{ exists for } t \text{ in } I.$$

It is continuous on I. Since f is continuous on R. and ϕ_k is continuous on I.

$$\therefore \phi_{k+1}(x) = y_0 + \int_{x_0}^{x} F_k(t)\,dt.$$

exists as a continuous fun. on I.

Also,
$$|\phi_{k+1}(x) - y_0| \leqq \left| \int_{x_0}^{x} |F_k(t)|\,dt \right|$$
$$\leqq M|x - x_0|$$

$\therefore \phi_{k+1}$ satisfies the inequality.

The thm is proved by induction.

Note:

Since for x in I, $|x - x_0| \leq b/M$, the inequality $|\phi_k(x) - y_0| \leq M|x - x_0| \leq M \cdot \frac{b}{M} = b$. for x in I. This implies the point $(x, \phi_k(x))$ are in R for x in I.

Geomentrically, the graph of each ϕ_k lies in the region bounded by the two

$\frac{M}{k}$ times the p^{th} term of the power series for $e^{k|x-x_0|}$.

Since, the power series for $e^{k|x-x_0|}$

i.e.) $\sum \frac{(k|x-x_0|)^p}{p!}$ is cgt and the series (A) is convergent for x in I.

$\Rightarrow$ the series (1) is cgt on I.

$\therefore$ The k^{th} partial sum of eqn. (1) which is $(\phi_k(x))$ tends to $\cancel{\phi_k(x) \Rightarrow}$ $\phi(x)$ as $k \to \infty$, for each x in I.

Hence, $\{\phi_k(x)\}$ converges to $\phi(x)$.

(b) Properties of the limit ϕ:

This limit fun. ϕ is a soln. of our problem on I.

T.P: ϕ is continuous on I.

If $x_1, x_2 \in I$,

$$|\phi_{k+1}(x_1) - \phi_{k+1}(x_2)| = \left| \int_{x_2}^{x_1} f(t, \phi_k(t)) dt \right|$$