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PG & RESEARCH DEPARTMENT OF MATHEMATICS

SUBJECT NAME: NUMERICAL METHODS I

CLASS : 1 B.Sc CS

CODE: 23UECS12A

SYLLABUS:

UNIT II

The solution of numerical algebraic and transcendental Equations: Bisection method – Iteration Method – Regula Falsi Method – Newton – Raphson method

Unit 1 : A Solution of algebraic and transcendental

Algebraic equations

$f(x)$ is a polynomial then the equation $f(x)$ is called as algebraic equation.

Eg:

$$f(x) = x^2 - 5x + 6 = 0$$

$$2x^2 - 5x + 1 = 0$$

Transcendental equations.

Equations which involves transcendental functions like $\sin x$, $\cos x$, $\log x$, e^x etc.. are called Transcendental equations.

Eg:

$$2e^x + 1 = 0, \quad 2^x + \cos x - 1 = 0$$

$$\log_{10} x - 2x = 12$$

The following method for obtaining approximate solution for algebraic and transcendental equations.

i) Bisection method.

ii) Iteration method

iii) Regula-falsi method (False position method)

iv) Newton Raphson method

i) Bisection Method

Let $f(x)$ be the continuous function defined on $[a, b]$ such that $f(a)$ and $f(b)$ are opposite sign then the root is of $f(x)=0$ lying between a and b

i) Let $x_0 = \frac{a+b}{2}$ be the first approximation of the required root (x_0 is the mid point of a and b)

ii) If $f(x_0) = 0$ then x_0 is the root of $f(x_0)$ and if $f(x_0) = -ve$. The root lies between a and x_0

If $f(x_0) = +ve$ then the root lies between x_0 and b

iii) Suppose $f(x_0)$ is a position then the root lies between x_0 and b and the root has $x_1 = \frac{x_0 + b}{2}$

iv) Suppose $f(x_1)$ is negative then the root lies between x_0 and x_1 and the root be

$$x_2 = \frac{x_0 + x_1}{2}$$

v) Now $f(x_2)$ is negative then the root lies between x_0 and x_2 and the root be

$$x_3 = \frac{x_0 + x_2}{2}$$ and so on

1) Find the real root of the equation $x^3 - x - 11$ by using bisection method.

Let $f(x) = x^3 - x - 11$

put $x=0$, $f(0) = 0 - 0 - 11 = -11$ (-ve)

$f(1) = 1 - 1 - 11 = 1 - 12 = -11$ (-ve)

$f(2) = 8 - 2 - 11 = 8 - 13 = -5$ (-ve)

$f(3) = 27 - 3 - 11 = 27 - 14 = 13$ (+ve)

The root lies between 2 & 3

Let $a = 2$, $b = 3$

First approximation

$$x_0 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$$

$f(2.5) = (2.5)^3 - 2.5 - 11 = 2.125$ (+ve)

The root lies between 2.5 & 2

2nd approximation

$$x_1 = \frac{2+2.5}{2} = \frac{4.5}{2} = 2.25$$

$f(2.25) = (2.25)^3 - 2.25 - 11$

$= -1.85937$ (-ve)

The root lies btw 2.25 & 2.5

3^{rd} approximation

$$x_2 = \frac{2.25+2.5}{2} = 2.375$$

$$f(2.375) = (2.375)^3 - 2.375 - 11$$

$$f(2.375) = 0.02148 \ (+ve)$$

The root lies btw 2.375 & 2.25

4^{th} approximation

$$x_3 = \frac{2.375+2.25}{2} = \frac{4.625}{2} = 2.3125$$

$$f(2.315) = (2.315)^3 - 2.315 - 11$$

$$= 12.366455 - 2.315 - 11$$

$$= 12.3664 - 13.315$$

$$\doteq -0.9486 \ (-ve)$$

The root lies btw 2.3125 & 2.375

5^{th} approximation.

$$x_4 = \frac{2.3125+2.375}{2} = 2.34375$$

$$f(2.34375) = (2.34375)^3 - 2.34375 - 11$$

$$= 12.87460 - 13.34375$$

$$= -0.46915 (-ve)$$

The root lies btw 2.34375 & 2.375

6^{th} approximation

$$x_5 = \frac{2.34375 + 2.375}{2} = 2.359375$$

$$f(2.359375) = (2.359375)^3 - 2.359375 - 11$$

$$= -0.2255 \text{ (-ve)}$$

The root lies btw 2.359375 & 2.375

7^{th} approximation

$$x_6 = \frac{2.359375 + 2.375}{2} = 2.3671$$

$$f(2.3671) = (2.3671)^3 - 2.3671 - 11$$

$$= -0.103854 \text{ (-ve)}$$

The root lies btw 2.3671 & 2.375

8^{th} approximation

$$x_7 = \frac{2.3671 + 2.375}{2} = 2.37105$$

$$f(2.37105) = (2.37105)^3 - 2.37105 - 11$$

$$= -0.04129 \text{ (-ve)}$$

The root lies btw 2.37105 & 2.375

9^{th} approximation

$$x_8 = \frac{2.37105 + 2.375}{2} = 2.3730$$

$$f(2.3730) = (2.3730)^3 - 2.3730 - 11$$

$= -0.00990$ (-ve)

The root lies btw 2.3730 & 2.375

10th approximation

$$x_9 = \frac{2.3730 + 2.375}{2} = 2.374$$

$f(2.374) = 0.0055$ (+ve)

The root lies btw 2.374 & 2.3730

11th approximation

$$x_{10} = \frac{2.374 + 2.3730}{2} = 2.3735$$

$$f(2.3735) = (2.3735)^3 - 2.3735 - 11$$

$$= -0.0023 \text{ (-ve)}$$

The root lies btw 2.3735 & 2.374

12th approximation

$$x_{11} = \frac{2.3735 + 2.374}{2} = 2.37375$$

$$f(2.37375) = (2.37375)^3 - 2.37375 - 11$$

$$= 0.00159 \text{ (+ve)}$$

The root lies btw 2.37375 & 2.3735

13th approximation

$$x_{12} = \frac{2.37375 + 2.3735}{2} = 2.373625$$

$$f(2.3736) = (2.3736)^3 - 2.3736 - 11$$

$$= -0.00039 \ (-ve)$$

The root lies btw 2.373625 & 2.37355

14th approximation.

$$x_{13} = \frac{2.373625 + 2.37355}{2} = 2.373587$$

$$f(2.3735) = -0.00238 \ (-ve)$$

The root lies btw 2.3735 & 2.37355

15th approximation

$$x_{14} = \frac{2.3735 + 2.37355}{2} = 2.3735$$

Find the positive root $x \log_{10} x = 1.2$ using the bisection method for four iteration.

$x \log_{10} x = 1.2$

0.30103

$f(x) = x \log x - 1.2$

Let $x = 0$, $f(0) = 0 - 1.2 = -1.2$ (-ve)

$x = 1$, $f(1) = (1) \log(1) - 1.2 = -1.2$ (-ve)

$x = 2$, $f(2) = 2 \log(2) - 1.2$

$= 2(0.3010) - 1.2$

$= 0.6020 - 1.2$

$= -0.5979$ (-ve)

$x = 3$, $f(3) = 3 \log 3 - 1.2$

$= 3(0.4771) - 1.2 = 1.431 - 1.2$

$= 0.21313$ (+ve)

The root lies btw 2, 3

1st approximation

$$x_0 = \frac{a+b}{2} = \frac{2+3}{2} = 2.5$$

$f(2.5) = (2.5) \log(2.5) - 1.2$

$= 2.5(0.3979) - 1.2$

$= 0.9948 - 1.2$

$= -0.2051$ (-ve)

The root btw 3 & 2.5

2^{nd} approximation

$$x_1 = \frac{3+2.5}{2} = 2.75$$

$$f(2.75) = 2.75 \log(2.75) - 1.2$$

$$= 2.75(0.4393) - 1.2$$

$$= 1.2081 - 1.2 = 0.0081 \ (+ve)$$

The root lies btw 2.75 & 2.5

3^{rd} approximation

$$x_2 = \frac{2.75+2.5}{2} = 2.625$$

$$f(x_2) = (2.625)\log(2.625) - 1.2$$

$$= -0.09978 \ (-ve)$$

The root btw 2.625 & 2.75

4^{th} approximation

$$x_3 = \frac{2.625+2.75}{2} = 2.6875$$

$$f(x_3) = (2.6875)\log(2.6875) - 1.2$$

$$= -0.046125 \ (-ve)$$

The root lies btw 2.6875 & 2.75

Iteration method (or) Successive approximation method

* Let, $f(x) = 0$ be the given function (or) Equation

* Suppose, the given equation can be expressed in the form of $x = \phi(x) \longrightarrow (1)$

* Where, $\phi(x)$ is the continuous function. Let x_0 be an approximate value of the desired roots

* We define $x_1 = \phi(x_0)$

$$x_2 = \phi(x_1)$$

$$\vdots$$

$$x_n = \phi(x_{n-1})$$

* The equation $\{x_0, x_1 \dots x_n\}$ is called the Sequence of successive approximation.

* Suppose the sequence $\{x_n\}$ converges to α

$$\therefore \lim_{n \to \infty} \{x_n\} = \alpha \longrightarrow (2)$$

* Since, $\phi(x)$ is the continuous function we have $\lim_{n \to \alpha} \phi(x_n) = \phi(\alpha)$

$$\lim_{n \to \alpha} \phi(x_{n+1}) = \phi(\alpha)$$

$$\lim_{n \to \alpha} \alpha = \phi(\alpha) \text{ by } (2)$$

* Thus α. is the root of a given equation $\therefore x = \phi(x)$ and hence it a root of $f(x) = 0$

1) Using the method of Iteration to find the real root lying between 1 and 2 of the equation $x^3 - 3x + 1 = 0$

$f(x) = x^3 - 3x + 1 = 0$

$f(1) = 1 - 3 + 1 = -1 \; (-ve)$

$f(2) = 8 - 6 + 1 = 3 \; (+ve)$

The root lies btw (1) & (2)

$x^3 - 3x + 1 = 0$

$x^3 = 3x - 1$

$x = (3x - 1)^{1/3}$

$x = \phi(x)$

$[\phi(x) = (3x-1)^{1/3}]$

$\phi'(x) = \frac{1}{3}(3x - 1)^{1/3 - 1} \; (3)$

$= (3x - 1)^{-2/3} = \frac{1}{(3x-1)^{2/3}}$

$\phi'(1) = \frac{1}{(3-1)^{2/3}} = \frac{1}{(2)^{2/3}} = \frac{1}{(2)^{2 \, 1/3}} = \frac{1}{(4)^{1/3}}$

$= \frac{1}{1.5874} = 0.6299$

$\phi'(2) = \frac{1}{(6-1)^{2/3}} = \frac{1}{(5)^{2/3}} = \frac{1}{2.92401}$

$\phi'(2) = 0.34199$

$\therefore x_0 = 2$ [since $1 > 2$]

$x_1 = \phi(x_0) = \phi(2) = [6-1]^{1/3} = (5)^{1/3} = 1.7099$

$x_2 = \phi(x_1) = \phi(1.7099) = [3(1.7099)-1]^{1/3}$

$= 1.6043$

$x_3 = \phi(x_2) = [3(1.6043)-1]^{1/3}$

$= 1.5622$

$x_4 = \phi(x_3) = [3(1.5622)-1]^{1/3}$

$= 1.5448$

$x_5 = \phi(x_4) = [3(1.5448)-1]^{1/3} = 1.5374$

$x_6 = \phi(x_5) = [3(1.5374)-1]^{1/3} = 1.5343$

$x_7 = \phi(x_6) = [3(1.5343)-1]^{1/3} = 1.5330$

$x_8 = \phi(x_7) = [3(1.5330)-1]^{1/3} = 1.5324$

$x_9 = \phi(x_8) = [3(1.5324)-1]^{1/3} = 1.5322$

$x_{10} = \phi(x_9) = [3(1.5322)-1]^{1/3} = 1.5321$

$x_{11} = \phi(x_{10}) = [3(1.5321)-1]^{1/3} = 1.53209$

$= 1.5321$

$\therefore$ The better approximation is 1.5321

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2) Solve $e^x - 3x = 0$ by using Iteration method

$f(x) = e^x - 3x = 0$

$f(0) = e^{(0)} - 0 = 1$ (+ve)

$f(1) = e^1 - 3 = -0.28$ (-ve)

The root lies btw 0 & 1

$$e^x - 3x = 0$$
$$3x = e^x$$
$$x = \frac{e^x}{3}$$
$$x = \phi(x)$$
$$\phi(x) = \frac{e^x}{3}$$
$$\phi'(x) = \frac{e^x}{3}$$

$$\phi(0) = \frac{1}{3} = 0.333$$

$$\phi(1) = \frac{2.71}{3} = 0.9060$$

$$\therefore x_0 = 0 \ [\text{since } 0 < 1]$$

$$x_1 = \phi(x_0) = \phi(0) = \frac{e^0}{3} = 0.333$$
$$x_2 = \phi(x_1) = \phi(0.333) = \frac{e^{0.333}}{3} = 0.4650$$
$$x_3 = \phi(x_2) = \phi(0.4650) = \frac{e^{0.4650}}{3} = 0.5306$$
$$x_4 = \phi(x_3) = \phi(0.5306) = \frac{e^{0.5306}}{3} = 0.5666$$
$$x_5 = \phi(x_4) = \phi(0.5666) = \frac{e^{0.5666}}{3} = 0.5874$$
$$x_6 = \phi(x_5) = \phi(0.5874) = \frac{e^{0.5874}}{3} = 0.5997$$
$$x_7 = \phi(x_6) = \phi(0.5997) = \frac{e^{0.5997}}{3} = 0.6071$$
$$x_8 = \phi(x_7) = \phi(0.6071) = \frac{e^{0.6071}}{3} = 0.6117$$
$$x_9 = \phi(x_8) = \phi(0.6117) = \frac{e^{0.6117}}{3} = 0.6145$$

$x_{10} = \phi(x_9) = \phi(0.6145) = \frac{e^{0.6145}}{3} = 0.6162$

$x_{11} = \phi(x_{10}) = \phi(0.6162) = \frac{e^{0.6162}}{3} = 0.6172$

$x_{12} = \phi(x_{11}) = \phi(0.6172) = \frac{e^{0.6172}}{3} = 0.6179$

$x_{13} = \phi(x_{12}) = \phi(0.6179) = \frac{e^{0.6179}}{3} = 0.6183$

$x_{14} = \phi(x_{13}) = \phi(0.6183) = \frac{e^{0.6183}}{3} = 0.6185$

$x_{15} = \phi(x_{14}) = \phi(0.6185) = \frac{e^{0.6185}}{3} = 0.6187$

$x_{16} = \phi(x_{15}) = \phi(0.6187) = \frac{e^{0.6187}}{3} = 0.6188$

$x_{17} = \phi(x_{16}) = \phi(0.6188) = \frac{e^{0.6188}}{3} = 0.6188$

$\therefore$ The better approximation is 0.6188

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3) Solve $\cos x = 3x - 1$ correct to three decimal places b, the real roots of the equation lies between 0 & $\pi/2$

$f(x) = \cos x - 3x + 1 = 0$

$f(0) = \cos - 3(0) + 1 = 1+1 = 2\ (+ve)$

$f(\pi/2) = \cos \pi/2 - 3(\pi/2) + 1 = 0 - 3(\pi/2) + 1$

$= (-ve)$

The point lies btw 0 & $\pi/2$

$\cos x + 1 = 3x$

$\phi(x) = x = \frac{\cos x + 1}{3}$

$\phi'(x) = \frac{1}{3}[-\sin x]$

$\phi'(0) = 1/3[-\sin 0] = 0$ (+ve)

$\phi'(\pi/2) = 1/3[-\sin \pi/2] = -1/3 = -0.33$ (-ve)

$x_0 = 0$

$x_1 = \phi(x_0) = \frac{\cos(0)+1}{3}$

$x_2 = \phi(x_1) = \frac{\cos(0.6666)+1}{3} = 0.5953$

$x_3 = \phi(x_2) = \frac{\cos(0.5953)+1}{3} = 0.6093$

$x_4 = \phi(x_3) = \frac{\cos(0.6093)+1}{3} = 0.6066$

$x_5 = \phi(x_4) = \frac{\cos(0.6066)+1}{3} = 0.6071$

b,

$x_6 = \phi(x_5) = \frac{\cos(0.6071)+1}{3}$

$= 0.6071$

$\therefore$ The better approximation is 0.6071

4) Use the method of Iteration to solve

$3x - \log_{10} x = 6$

$f(x) = 3x - \log x - 6$

$f(0) = -6 = (-ve)$

$f(1) = 3 - 0 - 6 = -3\ (-ve)$

$f(2) = 6 - \log 2 - 6 = -0.3010\ (-ve)$

$f(3) = 9 - \log 3 - 6 = 9 - 0.4770 - 6$

$= 2.522\ (+ve)$

The root lies btw 2 & 3

$3x = \log_{10} x + 6$

$x = \frac{1}{3}[\log x + 6] = \frac{1}{3}[\log_e x \log_{10} e + 6]$

$\phi(x) = \frac{1}{3}[\log_e x \log_{10} e + 6]$

$\phi'(x) = \frac{1}{3}\left[\frac{1}{x} \log_{10} e\right] = \frac{\log_{10} e}{3x}$

$\phi'(x) = \frac{0.4343}{3x}$ $\qquad \log_{10} e = 0.434343$

$\phi'(2) = \frac{0.4343}{6} = 0.0723$

$\phi'(3) = \frac{0.4343}{9} = 0.04825$

$x_0 = 3$

$x_1 = \phi(x_0) = \frac{1}{3}[\log 3 + 6] = 2.1590$

$x_2 = \phi(x_1) = \frac{1}{3}[\log(2.1590) + 6] = 2.1114$

$x_3 = \phi(x_2) = \frac{1}{3}[\log(2.1114)+6] = 2.1081$

$x_4 = \phi(x_4) = \frac{1}{3}[\log(2.1081)+6] = 2.1079$

$x_5 = \phi(x_5) = \frac{1}{3}[\log(2.1079)+6] = 2.1079$

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5) Solve $x^3 = 2x+5$ by Iteration method

2.0945

$f(x) = x^3 - 2x - 5$

$f(0) = -5$ (-ve)

$f(1) = 1-2-5 = -6$ (-ve)

$f(2) = 8-4-5 = -1$ (-ve)

$f(3) = 27-6-5 = 16$ (+ve)

$$\begin{array}{r} 27 \\ 11 \\ \hline 16 \end{array}$$

The root lies btw 2 & 3

$\cancel{2x = x^3 - 5}$ $\qquad$ $x^3 = 2x+5$

$\cancel{x = \frac{x^3-5}{2}}$ $\qquad$ $x = (2x+5)^{1/3}$

$\phi(x) = (2x+5)^{1/3}$

$\phi'(x) = \frac{1}{3}(2x+5)^{-2/3}$

$= \frac{2}{3(2x+5)^{2/3}}$

$\phi'(x) = \frac{1}{3(2+5)^{2/3}} = 0.0740 = \frac{1}{3(4.3267)}$

$\phi'(2) = \frac{1}{12.9802} = 0.1540 = 0.0770404$

$\phi'(3) = \frac{1}{3(11)^{2/3}} = \frac{1}{14.8382} = 0.1347$

$= 0.0673936$

$x_0 = 3$

$x_1 = \phi(x_0) = [6+5]^{1/3} = (11)^{1/3} = 2.2239$

$x_2 = \phi(x_1) = [2(2.2239)+5]^{1/3} = 2.1140$

$x_3 = \phi(x_2) = [2(2.1140)+5]^{1/3} = 2.0975$

$x_4 = \phi(x_3) = [2(2.0975)+5]^{1/3} = 2.0949$

$x_5 = \phi(x_4) = [2(2.0949)+5]^{1/3} = 2.0946$

$x_6 = \phi(x_5) = [2(2.0946)+5]^{1/3} = 2.09455$
$= 2.0946$

$\therefore$ The better approximation is

2.0946.

Regula falsi method (False position method)

* Consider a equation $f(x) = 0$ where $f(x)$ is a continues function choose two point a and b, Such that $f(a)$ and $f(b)$ are positive Sign. Hence they exist a root lies between a and b in this method the approximate curve of the function $f(x)$ by a chord

* Equation of the chord joining the point $(a, f(a))$ & $(b, f(b))$ is given

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a) \longrightarrow (1)$$

* The point of intersection of the chord with the x axis this taken as

first approximation.

* x_1, x_2, two roots which is obtained by putting $y=0$ in equation ①,

we get

put $x = x_1$, $y = 0$

$$-f(a) = \frac{f(b)-f(a)}{b-a}(x_1-a)$$

$$\frac{-f(a)(b-a)}{f(b)-f(a)} = (x_1-a)$$

$$x_1 = a - \frac{f(a)(b-a)}{f(b)-f(a)}$$

$$= \frac{af(b) - af(a) - bf(a) + af(a)}{f(b)-f(a)}$$

$$x_1 = \frac{af(b) - bf(a)}{f(b)-f(a)} \longrightarrow ②$$

* Now, If $f(a)$ and $f(x_1)$ are all +ve Signs, the root lies between a and x_1 So, replace by b by ② and get the next approximation x_2

* But if $f(a)$ and $f(x_1)$ are all same sign, then $f(1)$ and $f(b)$ will be opposite sign, hence the root lies btw x_1 and b, so replace $\frac{a}{x_1}$

1. Find the real root lying between 1 & 2 of the equation $x^3 - 3x + 1 = 0$ upto 3 decimal places by using Regula-falsi method.

$f(x) = x^3 - 3x + 1$

$f(1) = 1 - 3 + 1 = -1$ (-ve)

$f(2) = 8 - 6 + 1 = 3$ (+ve)

The root lies btw 1 & 2

$a = 1, \ b = 2, \ f(b) = 3, \ f(a) = -1$

1st approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{1(3) - 2(-1)}{3+1} = \frac{3+2}{4} = \frac{5}{4} = 1.25$$

$x_1 = 1.25$

$f(x_1) = (1.25)^3 - 3(1.25) + 1$

$= -0.7968$ (-ve)

The roots be 1.25 and 2

2nd approximation

$a = 1.25, \ b = 2, \ f(1.25) = -0.7968$

$f(2) = 3$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{(1.25)(3) - 2(-0.7968)}{3 + 0.7968} = 1.4073$$

$f(1.4073) = (1.4073)^3 - 3(14073) + 1$

$= -0.4347$ (-ve)

The roots 2 & 1.4073

3rd approximation

$a = 2$, $b = 1.4073$, $f(a) = 3$, $f(b) = -0.4347$

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{2(-0.4347) - 1.4073(3)}{-0.4347 - 3}$$

$x_3 = 1.4823$

$f(1.4823) = (1.4823)^3 - 3(1.4823) + 1$

$= -0.1898$ (-ve)

The roots 2 & 1.4823

4th approximation

$a = 2$, $b = 1.4823$, $f(a) = 3$, $f(b) = -0.1898$

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-4.8265}{-3.1898}$$

$x_4 = 1.5131$

$f(1.5131) = (1.5131)^3 - 3(1.5131) + 1$

$= -0.0751$ (-ve)

The roots are 2 & 1.5131

5^{th} approximation

$a = 2$, $b = 1.5131$, $f(a) = 3$, $f(b) = -0.0751$

$$x_5 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-4.6895}{-3.0751}$$

$$x_5 = 1.5249$$

$$f(1.5249) = (1.5249)^3 - 3(1.5249) + 1$$

$$= -0.0281 \text{ (-ve)}$$

The roots are 2 & 1.5249

6^{th} approximation.

$a = 2$, $b = 1.5249$, $f(a) = 3$, $f(b) = -0.0281$

$$x_6 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-4.6309}{-3.0281}$$

$$x_6 = 1.5293$$

$$f(1.5293) = (1.5293)^3 - 3(1.5293) + 1$$

$$= -0.0112 \text{ (-ve)}$$

The roots are 2 & 1.5293

7^{th} approximation

$a = 2$, $b = 1.529$, $f(a) = 3$, $f(b) = -0.0112$

$$x_7 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-4.6094}{-3.0112}$$

$$x_7 = 1.5307 = 1.531$$

$f(1.5307) = (1.5307)^3 - 3(1.5307) + 1$

$= -0.005604$ (-ve)

The roots are 2 & 1.5307

8th approximation

$a = 2,\ b = 1.5307,\ f(a) = 3,\ f(b) = -0.0056$

$$x_8 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-4.6041}{-3.0056}$$

$x_8 = 1.531$

∴ The better approximation is 1.531

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2. $x^3 - 3x - 5$ using Regula - falsi method.

$f(x) = x^3 - 3x - 5$

27
14
13

$f(0) = -5 = $ (-ve)

$f(1) = 1 - 3 - 5 = $ (-ve)

$f(2) = 8 - 6 - 5 = -3$ (-ve)

$f(3) = 27 - 9 - 5 = 13$ (+ve)

The roots are lies btw 2 & 3

$a = 2,\ b = 3,\ f(a) = -3,\ f(b) = 13$

1st approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{35}{16} = 2.1875$$

$f(2.1875) = (2.1875)^3 - 3(2.1875) - 5$

$= -1.0949$ (-ve)

The roots are 2.1875 & 3

2^{nd} approximation

$a = 2.1875$, $b = 3$, $f(a) = -1.0949$

$f(b) = 13$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{31.7222}{14.0949}$$

$$= 2.2506$$

$$f(2.2506) = (2.2506)^3 - 3(2.2506) - 5$$

$$= -0.3520 \ (-ve)$$

The roots are 2.2506 & 3

3^{rd} approximation

$a = 2.2506$, $b = 3$, $f(a) = -0.3520$, $f(b) = 13$

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{30.3138}{13.352}$$

$$= 2.2703$$

$$f(2.2703) = (2.2703)^3 - 3(2.2703) - 5$$

$$= -0.1091 \ (-ve)$$

The roots are 2.2703 & 3

4^{th} approximation

$a = 2.2703$, $b = 3$, $f(a) = -0.1091$, $f(b) = 13$

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{29.8412}{13.1091} = 2.2763$$

$f(2.2763) = (2.2763)^3 - 3(2.2763) - 5$

$= -0.0341$ (-ve)

The roots are 2.2763 & 3

<u>5th approximation</u>

$a = 2.2763,\ b = 3,\ f(a) = -0.0341,\ f(b) = 13$

$$x_5 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{29.6942}{13.0341}$$

$$= 2.2781$$

$f(2.2781) = (2.2781)^3 - 3(2.2781) - 5$

$= -0.0115$ (-ve)

The roots are 2.2781 & 3

<u>6th approximation</u>

$a = 2.2781,\ b = 3,\ f(a) = -0.0115,\ f(b) = 13$

$$x_6 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{29.6498}{13.0115}$$

$$= 2.2787$$

$\therefore$ The better approximation is 2.278

3 Find the positive root of $xe^x = 2$ by using Regula - falsi method

$f(x) = xe^x - 2$

$f(0) = 0 - 2 = -2$ (-ve)

$f(1) = e^1 - 2 = 2.278281 - 2$

$= 0.718281$ (+ve)

The root 0 & 1

1st approximation

$a = 0$, $b = 1$, $f(a) = -2$, $f(b) = 0.718281$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = 0.7357$$

$f(x) = xe^x - 2$

$f(0.7357) = (0.7357)e^{0.7357} - 2$

$= -0.4646$ (-ve)

The root btw 1 & 0.7357

2nd approximation

$a = 1$, $b = 0.7357$, $f(a) = 0.718281$,

$f(b) = -0.4646$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-0.9930}{-1.1828} = 0.8395$$

$f(0.8395) = (0.8395)e^{0.8395} - 2$ 0.85261

$= -0.056381$ (-ve)

The roots btw 1 & 0.8395

3rd approximation

$a = 1$, $b = 0.8395$, $f(a) = 0.718281$

$f(b) = -0.056381$

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-0.6593}{-0.7746}$$

$= 0.8511$

$f(0.8511) = (0.8511)e^{0.8511} - 2$

$= -0.0065349$ (-ve)

The roots btw 1 & 0.8511

4th approximation

$a = 1$, $b = 0.8511$, $f(a) = 0.718281$,

$f(b) = -0.0065349$ 0.8511

$$x_4 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-0.6786}{-0.7248} = 0.8524$$

$f(0.8634) = (0.8524)e^{0.8524} - 2$

$= -0.000891$

The roots btw 0.8524 & 1

5^{th} approximation

$a = 1,\ b = 0.8524,\ f(a) = 0.718281,$

$f(b) = -0.000891$

$$x_5 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-0.6131}{-0.7191} = 0.8525$$

$f(0.8525) = \cancel{(0.8525)^3 - 3(0.8523) - 5}$

$f(0.8525) = (0.8525)e^{0.8525} - 2$

$= -0.000458$

$\therefore$ The better approximation is 0.852

4 Find the positive root of $x - \cos x = 0$ by using regula-falsi method.

$f(x) = x - \cos x$

$f(0) = 0 - 1 = -1$ (-ve)

$f(1) = 1 - \cos 1 = 1 - 0.5403$ (+ve)

$f(1) = 0.4597$ (+ve)

$\therefore$ The root lies btw 0 & 1

1^{st} approximation

$a = 0,\ b = 1,\ f(a) = -1,\ f(b) = 0.4597$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{1}{1.4597} = 0.6850$$

$f(0.6850) = 0.6850 - \cos(0.6850)$

$= -0.08941$ (-ve)

The roots lies btw 0.6850 & 1

2nd approximation

$a = 0.6850$, $b = 1$, $f(a) = -0.08941$,

$f(b) = 0.4597$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.4043}{0.5491} = 0.7362$$

$f(0.7362) = 0.7362 - \cos(0.7362)$

$= -0.004825$ (+ve)

The root lies btw 0.7362 & 1

3rd approximation

$a = 0.7362$, $b = 1$, $f(a) = -0.004825$

$f(b) = 0.4597$

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.3432}{0.4645} = 0.7388$$

$= 0.739$

$f(0.7388) = 0.7388 - \cos(0.7388)$

$= -0.000477$ (-ve)

The root lies btw 0.7388 & 1

4th approximation

$a = 0.7388$, $b = 1$, $f(a) = -0.000477$,

$f(b) = 0.4597$

$$x_4 = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{0.3401}{0.4601} = 0.7391$$

$\therefore$ The better approximation is 0.7391

NEWTON - RAPHSON METHOD or

NEWTON'S METHOD

* Assuming that x_0 is a approximate value of a, real root of the equation $f(x)=0$ and x_0+h is the exact root h being small.

* We have $f(x_0+h)=0 \rightarrow (1)$

* Expanding (1) by Taylor's series we have,

$$f(x_0+h) = f(x_0) + \frac{h}{1!}f'(x_0) + \frac{h'}{2!}f''(x_0) + \cdots \rightarrow (2)$$

* Neglecting 2^{nd} and higher order terms of h, h be small

$$(2) \Rightarrow f(x_0+h) = f(x_0) + \frac{h}{1!}f'(x_0)$$

$$f(x_0) + \frac{h}{1!}f'(x_0) = 0$$

$$hf'(x_0) = -f(x_0)$$

$$h = \frac{-f(x_0)}{f'(x_0)} \rightarrow (3)$$

Now (3) gives a value of h when adding to x_0 would be a better approoximation to the root by x_1, we have,

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{—(4)}$$

where $h_2 = -\frac{f(x_1)}{f(x_1)}$

$$x_2 = x_1 + h_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

|||ly

$$x_{n+1} = x_n + h_{n+1}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

---

1. Using Newton-Raphson method find the correct decimal places, the root lies btw 0.6 & 0.7 of the equation $x^3 - 6x + 4$

$$f(x) = x^3 - 6x + 4$$

$$f(0.6) = (0.6)^3 - 6(0.6) + 4$$

$$= 0.616$$

$$f(0.7) = (0.7)^3 - 6(0.7) + 4 = 0.143$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f'(x) = 3x^2 - 6$$

Let $x_0 = 0.7$

1st approximation

$$x_1 = 0.7 - \frac{0.143}{3(0.7)^2 - 6}$$

$$= 0.7 + \frac{0.143}{4.53}$$

$$= 0.7 + 0.0315$$

$$x_1 = 0.7315.$$

2nd approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.7315 - \frac{(0.7315)^3 - 6(0.7315) + 4}{3(0.7315)^2 - 6}$$

$$= 0.7315 + \frac{0.002419}{4.3947}$$

$$x_2 = 0.7320.$$

3rd approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.7320 - \frac{(0.7320)^3 - 6(0.7320) + 4}{3(0.7320)^2 - 6}$$

$$= 0.7320 - \frac{0.0002}{4.3925}$$

$$x_3 = 0.7320$$

$\therefore$ The better approximation is 0.7320

2. Find the 1^{st} approximation of the root lies btw 0 & 1 of the equation $x^3+3x-1=0$ by using Newton Raphson method.

$$f(x) = x^3+3x-1$$

$$f(0) = -1 \quad (-ve)$$

$$f(1) = 1+3-1 = 3$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$f'(x) = 3x^2+3$$

Let $x_0 = 0$

1^{st} approximation

$$x_1 = 0 - \frac{(-1)}{3}$$

$$= 0+\frac{1}{3}$$

$$= 0.3$$

(-ve Not Valid)

Let $x_0 = 1$

1^{st} approximation

$$x_1 = 1 - \frac{3}{6}$$

$$= 1-\frac{1}{2}$$

$$= \frac{1}{2}$$

$$\boxed{x_1 = 0.5}$$

3 By using Newton raphson method to find the root of $x \tan x = 1/2$ which lies btw 0.6 and 0.7

$$f(x) = x\tan x - \frac{1}{2}$$

$$f(x) = \frac{2x\tan x - 1}{2}$$

$$f'(x) = \frac{1}{2}\left[2(1)\tan x + 2x\sec^2 x - 0\right]$$

$$= \frac{1}{2}\left[2\tan x + 2x\sec^2 x\right]$$

$$f'(x) = \tan x + x\sec^2 x.$$

4 Write Newton Raphson formula to obtain the cube root of N

Let $x = \sqrt[3]{N}$

$x = (N)^{1/3}$

$x^3 = N$

$x^3 - N = 0$

$f(x) = x^3 - N$

$f'(x) = 3x^2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 - \frac{x_0^3 - N}{3x_0^2}$$

$$x_2 = x_1 - \frac{x_1^3 - N}{3x_1^2}$$

$\vdots$

$$x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2}$$

$\therefore$ The Newton Raphson formula $\sqrt[3]{N}$ =

$$x_n - \frac{x_n^3 - N}{3x_n^2}$$

5- Find the Iteration formula for $\sqrt{N}$ where N is a positive number and hence find $\sqrt{5}$

Let $x = \sqrt{N}$

$x = (N)^{1/2}$

$x^2 = N$

$f(x) = x^2 - N = 0$

$f'(x) = 2x$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= x_0 - \frac{x_0^2 - N}{2x_0}$$

$$x_2 = x_1 - \frac{x_1^2 - N}{2x_1}$$

$\vdots$

$$x_{n+1} = x_n - \frac{x_n^2 - N}{2x_n}$$

Let $x = \sqrt{5}$

$x^2 = 5$

$f(x) = x^2 - 5$

$f'(x) = 2x$

$x = 0 \Rightarrow f(0) = -5$ (-ve)

$x = 1 \Rightarrow f(1) = -4$ (-ve)

$x = 2 \Rightarrow f(2) = -1$ (-ve)

$x = 3 \Rightarrow f(3) = 4$ (+ve)

$\therefore$ The root lies btw 2 & 3

First approximation

$x_0 = 2$

$$x_1 = x_0 - \frac{x_0^2 - N}{2x_0} = 2 - \frac{2^2 - 5}{4} = 2 + \frac{1}{4}$$

$x_1 = 2.25$

Second approximation

$$x_2 = x_1 - \frac{x_1^2 - N}{2x_1}$$

$$= 2.25 - \frac{(2.25)^2 - 5}{2(2.25)}$$

$$x_2 = 2.25 - \frac{0.0625}{4.5}$$

$$= 2.25 - 0.01388$$

$$x_2 = 2.23612$$

<u>Third approximation</u>

$$x_3 = x_2 - \frac{x_2^3 - N}{2x_2}$$

$$= (2.23612) - \frac{(2.23612)^2 - 5}{(2.23612)^2}$$

$$= 2.23612 - \frac{0.0002326}{4.4722}$$

$$= 2.23612 - 0.00005053$$

$$= 2.236069 = 2.2361$$

∴ The better approximation is 2.2361

---

(3) Find the real root of the equation $x - \cos x = 0$ by using bisection method

$$f(x) = x - \cos x$$

$f(0) = -1$ (-ve)

$f(1) = 1 - \cos 1 = 0.4596$ (+ve)

The root lies btw 0 & 1

<u>1st approximation</u>

$a = 0,\ b = 1$

$$x_0 = \frac{a+b}{2} = 0.5$$

$f(0.5) = 0.5 - \cos 0.5$

$= -0.3775 \text{(-ve)}$

The root lies btw 1 & 0.5

2nd approximation

$x_1 = \frac{1+0.5}{2} = 0.75$

$f(0.75) = 0.75 - \cos 0.75$

$= 0.01831 \text{ (+ve)}$

The root lies btw 0.75 & 0.5

3rd approximation

$x_2 = \frac{0.75+0.5}{2} = 0.625$

$f(0.625) = 0.625 - \cos(0.625)$

$= -0.1859 \text{ (-ve)}$

The root lies btw 0.625 & 0.75

4th approximation

$x_3 = \frac{0.625+0.75}{2} = 0.6875$

$f(0.6875) = 0.6875 - \cos(0.6875)$

$= -0.08533 \text{ (-ve)}$

The root lies btw 0.6875 & 0.75

5th approximation

$x_4 = \frac{0.6875+0.75}{2} = 0.71875$

$f(0.71875) = 0.71875 - \cos(0.71875)$

$= -0.0338$ (-ve)

The roots lies btw 0.71875 & 0.75

6th approximation

$x_5 = \frac{0.71875 + 0.75}{2} = 0.7343$

$f(0.7343) = 0.7343 - \cos(0.7343)$

$= -0.0079$ (-ve)

The root lies btw 0.7343 & 0.75

7th approximation

$x_6 = \frac{0.7343 + 0.75}{2} = 0.7421$

$f(0.7421) = 0.7421 - \cos(0.7421)$

$= -0.00504$ (-ve)

The root lies btw 0.7421 & 0.75

8th approximation

$x_7 = \frac{0.7421 + 0.75}{2} = 0.7460$

$f(0.7460) = 0.7460 - \cos(0.7460)$

$= 0.0115$ (+ve)

The root lies btw 0.7460 & 0.7421

9th approximation

$x_8 = \frac{0.7460 + 0.7421}{2} = 0.7440$

$f(0.7440) = 0.7440 - \cos(0.7440)$

$= -0.00823$ (-ve)

The root lies btw 0.7440 & 0.7460

10th approximation

$$x_9 = \frac{0.7440 + 0.7460}{2} = 0.745$$

$f(0.745) = 0.745 - \cos(0.745)$

$= -0.00991$ (-ve)

The root lies btw 0.745 & 0.7460

11th approximation

$$x_{10} = \frac{0.745 + 0.7460}{2} = 0.7455$$

$f(0.7455) = 0.7455 - \cos(0.7455)$

$= 0.01075$ (+ve)

The root lies btw 0.7455 & 0.745

12th approximation

$$x_{11} = \frac{0.7455 + 0.745}{2} = 0.7452$$

$f(0.7452) = 0.7452 - \cos(0.7452)$

$= 0.01024$ (+ve)

The root lies btw 0.7452 & 0.745

13th approximation

$$x_{12} = \frac{0.7452 + 0.745}{2} = 0.7452$$

The better approximation is 0.7452

4) Find the real root of the equation $x^3 - 9x$ ... by using bisection method

$f(x) = x^3 - 9x + 1$

$f(0) = 1$ (+ve)

$f(1) = -7$ (+ve)

The root lies btw 0 & 1

1st approximation

$x_0 = \frac{0+1}{2} = 0.5$

$f(0.5) = (0.5)^3 - 9(0.5) + 1$

$= -3.375$ (-ve)

The root lies btw 0.5 & 0

2nd approximation

$x_1 = \frac{0+0.5}{2} = 0.25$

$f(0.25) = (0.25)^3 - 9(0.25) + 1$

$= -1.2543$ (-ve)

The root lies btw 0.25 & 0

3rd approximation

$x_2 = \frac{0+0.25}{2} = 0.125$

$f(0.125) = (0.125)^3 - 9(0.125) + 1$

$= -0.12304$ (-ve)

The root lies btw 0.125 & 0

4th approximation

$x_3 = \frac{0.125}{2} = 0.0625$

$f(0.0625) = (0.0625)^3 - 9(0.0625) + 1$

$= 0.4377$ (+ve)

The root lies btw 0.0625 & 0.125

5th approximation

$x_4 = \frac{0.0625 + 0.125}{2} = 0.0937$

$f(0.0937) = (0.0937)^3 - 9(0.0937) + 1$

$= 0.15752$ (+ve)

The root lies btw 0.0937 & 0.125

6th approximation

$x_5 = \frac{0.0937 + 0.125}{2} = 0.10935$

$f(0.10935) = (0.10935)^3 - 9(0.10935) + 1$

$= 0.01715$ (+ve)

The root lies btw 0.10935 & 0.125

7th approximation

$x_6 = \frac{0.10935 + 0.125}{2} = 0.11717$

$f(0.11717) = (0.11717)^3 - 9(0.11717) + 1$

$= -0.05292$ (-ve)

The root lies btw 0.11717 & 0.10935

8th approximation

$$x_7 = \frac{0.11717 + 0.10935}{2} = 0.11326$$

$$f(0.11326) = (0.11326)^3 - 9(0.11326) + 1$$

$$= -0.0178 \text{ (-ve)}$$

The root lies btw 0.11326 & 0.10925

9th approximation

$$x_8 = \frac{0.11326 + 0.10925}{2} = 0.11123$$

$$f(0.11123) = (0.11123)^3 - 9(0.11123) + 1$$

$$= -0.000306 \text{ (-ve)}$$

The root lies btw 0.11123 & 0.10925

10th approximation

$$x_9 = \frac{0.11123 + 0.10925}{2} = 0.11024$$

$$f(0.11024) = (0.11024)^3 - 9(0.11024) + 1$$

$$= 0.00917 \text{ (+ve)}$$

The root lies btw 0.11024 & 0.11123

11^{th} approximation

$$x_{10} = \frac{0.11024 + 0.11123}{2} = 0.11073$$

$$f(0.11073) = (0.11073)^3 - 9(0.11073) + 1$$

$$= 0.00478 \ (+ve)$$

The root lies btw 0.11073 & 0.11123

12^{th} approximation

$$x_{11} = \frac{0.11073 + 0.11123}{2} = 0.11098$$

$$f(0.11098) = (0.11098)^3 - 9(0.11098) + 1$$

$$= 0.00254 \ (+ve)$$

The root lies btw 0.11098 & 0.11123

13^{th} approximation

$$x_{12} = \frac{0.11098 + 0.11123}{2} = 0.111105$$

$$f(0.111105) = (0.111105)^3 - 9(0.111105) + 1$$

$$= 0.00142 \ (+ve)$$

The root lies btw 0.111105 & 0.11123

14^{th} approximation

$$x_{13} = \frac{0.111105 + 0.11123}{2} = 0.111167$$

$\therefore$ The better approximation is 0.1111

Newton Raphson Method

b) Find the real roots $x^3 - 3x + 1 = 0$ lying between 1 & 2 currect two three decimal places by newton Raphson method

$f(x) = x^3 - 3x + 1 = 0$

$f(1) = x^3 - 3x + 1 = 0$

$f(2) = 8 - 6 + 1 = 3$

$f'(x) = 3x^2 - 3$

$x_0 = 2$

1st approximation

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{3}{3(2)^2 - 3}$$

$$= 2 - \frac{3}{9} = 2 - \frac{1}{3}$$

$x_1 = 1.66666$

2nd approximation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.66666 - \frac{(1.66666)^3 - 3(1.66666) + 1}{3(1.66666)^2 - 3}$$

$$= 1.66666 - 0.11804$$

$x_2 = 1.54862$

3rd approximation

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 1.54862 - \frac{(1.54862)^3 - 3(1.54862) + 1}{3(1.54862)^2 - 3}$$

$$= 1.54862 - 0.01622$$

$$x_3 = 1.5324$$

4th approximation

$$x_4 = 1.5324 - \frac{(1.5324)^3 - 3(1.5324) + 1}{3(1.5324)^2 - 3}$$

$$= 1.5324 - 0.00031$$

$$x_4 = 1.5321$$

$\therefore$ The better approximation is 1.532