MARUDHAR KESARI JAIN COLLEGE FOR WOMEN,VANIYAMBADI
PG & RESEARCH DEPARTMENT OF MATHEMATICS

SUBJECT NAME: NUMERICAL METHODS I

CLASS : 1 B.Sc CS

CODE: 23UECS12A

SYLLABUS:

Unit-IV

Finite differences Operators Δ, $\nabla$and E - relation between them — factorial polynomials. Interpolation with equal intervals: Gregory-Newton forward and backward interpolation formulas.

Finite Difference:

Let $y = f(x)$ be a given function x and let $y_0, y_1, \ldots$ etc upto y_n be the value of y corresponding to $x_0, x_1, \ldots x_n$ be the values of x the independent variable x is called the argument and the corresponding dependent value y is called the entry.

We can write the argument and the entries is

$$x : x_0, x_1, \ldots x_n$$

$$y : y_0, y_1, \cdots y_n$$

Finite order Difference of $f(x)$:

First order Difference:

The difference derived from the sequence of values obtained from a given function $f(x)$ when the variable x changes in Arithmetical progration $x = a, a+h, a+2h, \ldots$ the function takes the values $f(a), f(a)+h, f(a)+2h \ldots$ Here is known as interval of difference. If $f(x)$ is a function of an independent variable x then they change $\Delta f(x)$ in f. correct to a change Δx is known as first order Difference of $f(x)$

$$\Delta f(x) = f(x+h) - f(x)$$

Second order Difference:

$$\Delta^2 f(x) = \Delta[\Delta f(x)]$$
$$= \Delta[f(x+h) - f(x)]$$
$$= [f(x+h+h) - f(x+h) - [f(x+h) - f(x)]$$
$$= [f(x+2h) - f(x+h) - [f(x+h) - f(x)]$$
$$= f(x+2h) - f(x+h) - f(x+h) + f(x)$$

$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

Third order Difference:

$$\Delta^3 f(x) = \Delta[\Delta^2 f(x)]$$
$$= \Delta[f(x+2h) - 2f(x+h) + f(x)]$$
$$= \Delta f(x+2h) - 2[\Delta f(x+h) + [\Delta f(x)]$$
$$= [f(x+2h+h) - f(x+2h) - 2[f(x+h+h) - f(x+h)] + [f(x+h) - f(x)]$$
$$= f(x+3h) - f(x+2h) - 2(f(x+2h)) - f(x+h) + [f(x+h) - f(x)]$$
$$= f(x+3h) - f(x+2h) - 2f(x+2h) \cancel{+2f(x+2h)+} 2f(x+h) + f(x+h) - f(x)$$

$$\Delta^3 f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$

n^{th} order Difference:

$$\Delta^n f(x) = \Delta[\Delta^{n-1} f(x)]$$
$$= \Delta^{n-1}[\Delta f(x)]$$
$$= \Delta^{n-1}[f(x+h) - f(x)]$$
$$\Delta^n f(x) = \Delta^{n-1} f(x+h) - \Delta^{n-1} f(x)$$

Difference $\Delta f(x), \Delta^2 f(x)$ upto $\Delta^n f(x)$ are turned as forward difference and Δ is forward difference operator.

Finite difference forward table:

	Argument x	Entry $y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1.	a	$f(a)$	$\Delta f(a) = f(a+h) - f(a)$	$\Delta^2 f(a) = f(a+2h) - 2f(a+h) + f(a)$	$\Delta^3 f(a) = f(a+3h) - 3f(a+2h) + 3f(a+h) - f(a)$
2.	$a+h$	$f(a+h)$	$\Delta f(a+h) = f(a+2h) - f(a+h)$	$\Delta^2 f(a+h) = f(a+3h) - 2f(a+2h) + f(a+h)$	$\Delta^3 f(a+h) = f(a+4h) - 3f(a+3h) + 3f(a+2h) - f(a+h)$
3.	$a+2h$	$f(a+2h)$	$\Delta f(a+2h) = f(a+3h) - f(a+2h)$	$\Delta^2 f(a+2h) = f(a+4h) - 2f(a+3h) + f(a+2h)$	$\Delta^3 f(a+2h) = f(a+5h) - 3f(a+4h) + 3f(a+3h) - f(a+2h)$

Problems:

1. Form the difference table from the below Table.

x	0	1	2	3	4
$f(x)$	-1	3	19	53	111

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	-1				
		4			
1	3		12		
		16		6	
2	19		18		0
		34		6	
3	53		24		
		58			
4	111				

2. Evalute Forward difference for $(ab)^{cx}$ or $\Delta(ab)^{cx}$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta(ab)^{cx} = (ab)^{c(x+h)} - (ab)^{c(x)}$$

$$= (ab)^{cx+ch} - ab^{cx}$$

$$= (ab)^{cx}\ ab^{ch} - (ab)^{cx}$$

$$= (ab)^{cx}\left[(ab)^{ch} - 1\right]$$

$$\Delta(ab)^{cx} = (ab)^{cx}\left[(ab)^{ch} - 1\right]$$

3. Evaluate $\Delta \tan^{-1}(x)$

$$\Delta \tan^{-1}(x) = \tan^{-1}(x+h) - \tan^{-1}(x)$$

$$\tan^{-1} A - \tan^{-1} B = \tan^{-1}\left[\frac{A-B}{1-AB}\right]$$

$$= \tan^{-1}\left[\frac{\cancel{x}+h-\cancel{x}}{1-(x+h)(x)}\right]$$

$$= \tan^{-1}\left[\frac{h}{1+x^2+xh}\right]$$

4. Evaluate $\Delta(\cot 2^x)$

Let $f(x) = \Delta(\cot 2^x)$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta(\cot 2^x) = \cot 2^{(x+h)} - \cot 2^x \cancel{\cot 2^x}$$

$$\Delta\left[\frac{\cos 2^x}{\sin 2^x}\right] = \frac{\cos 2^{(x+h)}}{\sin 2^{(x+h)}} - \frac{\cos 2^x}{\sin 2^x}$$

$$= \frac{\sin 2^x \cos 2^{(x+h)} - \cos 2^x \sin 2^{(x+h)}}{\sin 2^{(x+h)} \sin 2^x}$$

$$= \frac{\sin(2^x - 2^{x+h})}{\sin 2^{(x+h)} \sin 2^x}$$

$$(\Delta \cot 2^x) = \frac{\sin 2^x[1-2^h]}{\sin 2^{(x+h)} \sin 2^x}$$

5. $\Delta[\cot(a+bx)]$

$$f(x) = \Delta[\cot(a+bx)]$$

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta[\cot(a+bx) = \cot[a+b(x+h)] - \cot[a+b(x)]$$

$$\frac{\cos(a+bx)}{\sin(a+bx)} = \frac{\cos[a+b(x+h)]}{\sin[a+b(x+h)]} - \frac{\cos[a+b(x)]}{\sin[a+b(x)]}$$

$$= \frac{\sin[a+bx]\cos[a+b(x+h) - \cos[a+bx]\sin[a+b(x+h)]}{\sin[a+b(x+h)]\sin[a+bx]}$$

$$= \frac{\sin[a+bx-a+b(x+h)]}{\sin[a+b(x+h)]\sin[a+bx]}$$

$$\Delta\cot(a+bx) = \frac{\sin[a+bx-a+b(x+h)]}{\sin[a+b(x+h)]\sin[a+bx]}$$

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6. If $f(x)$ and $g(x)$ are any function of x prove the following result holds.

i) $\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$

$$\Delta[f(x) \pm g(x)] = [f(x+h) \pm g(x+h)] - [f(x) \pm g(x)]$$

$$= [f(x+h) - f(x)] \pm [g(x+h) - g(x)]$$

$$\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$$

Hence proved.

ii) $\Delta[af(x)] = a[\Delta f(x)]$

$$\Delta[af(x)] = [af(x+h) - af(x)]$$
$$= a[f(x+h) - f(x)]$$
$$= a[\Delta f(x)]$$
$$\Delta[af(x)] = a[\Delta f(x)]$$

iii) $\Delta[f(x)g(x)] = f(x+h)\Delta g(x) + g(x)\Delta f(x)$

$$\Delta[f(x)g(x)] = f(x+h)g(x+h) - f(x)g(x)$$
$$= f(x+h)g(x+h) + f(x+h)g(x) - f(x+h)g(x) - f(x)g(x)$$
$$= f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]$$

$$\boxed{\Delta[f(x)g(x)] = f(x+h)\Delta g(x) + g(x)\Delta f(x)}$$

iv) $\Delta\left[\frac{1}{f(x)}\right] = \frac{-\Delta f(x)}{f(x)f(x+h)}$

$$\Delta\left[\frac{1}{f(x)}\right] = \frac{1}{f(x+h)} - \frac{1}{f(x)}$$
$$= \frac{f(x) - f(x+h)}{f(x+h)\cancel{+}f(x)}$$
$$= \frac{-[f(x+h) - f(x)]}{f(x+h)\,f(x)}$$

$$\boxed{\Delta\left(\frac{1}{f(x)}\right) = \frac{-\Delta f(x)}{f(x)f(x+h)}}$$

Backward Difference:

Backward difference operator ∇ defined as $\nabla f(x) = f(x) - f(x+h)$

It is also known as first order Backward difference.

Second order Backward Difference:

$$\nabla^2 f(x) = \nabla[\nabla f(x)]$$

$$= \nabla[f(x) - f(x-h)]$$

$$= \nabla f(x) - \nabla f(x-h)$$

$$\nabla^2 f(x) = f(x) - f(x-h) - f(x-h) + f(x-2h)$$

$$\nabla^2 f(x) = f(x) - 2f(x-h) + f(x-2h)$$

Third order Backward Difference:

$$\nabla^3 f(x) = \nabla[\nabla^2 f(x)]$$

$$= \nabla[f(x) - 2f(x-h) + f(x-2h)]$$

$$= \nabla f(x) - \nabla^2 f(x-h) + \nabla f(x-2h)$$

$$= f(x) - f(x-h) - 2f(x-h) + 2f(x-2h) + f(x-2h) - f(x-3h)$$

$$\nabla^3 f(x) = f(x+h) - 3f(x-h) + 3f(x-2h) - f(x-3h)$$

n^{th} order Backward Difference:

$$\nabla^n f(x) = \nabla[\nabla^{n+1} f(x)]$$
$$= \nabla^{n+1}[\nabla f(x)]$$
$$= \nabla^{n+1}[f(x-h) + f(x)]$$
$$\nabla^n f(x) = \nabla^{n+1} f(x-h) + \nabla^{n+1} f(x)$$

Shifting operator (E)

$$Ef(x) = f(x+h)$$
$$E^2 f(x) = f(x+2h)$$
$$E^3 f(x) = f(x+3h)$$
$$E^4 f(x) = f(x+4h)$$

Inverse shifting operator:

$$E^{-1} f(x) = f(x-h)$$
$$E^{-2} f(x) = f(x-2h)$$
$$E^{-3} f(x) = f(x-3h)$$
$$E^{-4} f(x) = f(x-4h)$$

Relation between difference operator:

i) Relation between Δ operator and shifting operator (Δ and E).

we know that

$$\Delta f(x) = f(x+h) - f(x)$$
$$= E f(x) - f(x)$$

$$\Delta f(x) = f(x)[E-1]$$

$$\boxed{\Delta = E-1 \text{ (or) } E = \Delta+1}$$

ii) Relation between Backward difference and shifting operator (∇ and E)

we know that.

$$\nabla f(x) = f(x) - f(x-h)$$

$$= f(x) - E^{-1}f(x)$$

$$\nabla f(x) = f(x)[1-E^{-1}]$$

$$\nabla = 1-E^{-1}$$

$$\boxed{E^{-1} = 1-\nabla}$$

1. prove that

$$\Delta = E\nabla$$

$E^{-1} - 1 = -\nabla$

$-(-E^{-1}+1) = -\nabla$

$\boxed{1-E^{-1} = \nabla}$

proof

$$E\nabla = E\nabla f(x)$$

$$= E[f(x) - f(x-h)]$$

$$= Ef(x) - Ef(x-h)$$

$$= f(x+h) - f(x-h+h)$$

$$= f(x+h) - f(x)$$

$$= \Delta$$

$$E\nabla = \Delta$$

2. prove that the following results

i) $\nabla = E^{-1}\Delta$

$$E^{-1}\Delta f(x) = E^{-1}[f(x+h) - f(x)]$$
$$= E^{-1}f(x+h) - E^{-1}f(x)$$
$$= f(x+h-h) - f(x-h)$$
$$= f(x) - f(x-h)$$
$$= \nabla$$
$$\boxed{\nabla = E^{-1}\Delta}$$

ii) $\nabla^r y_k = \Delta^r y_{k-r}$

By using (iii) $\qquad E(y_k) = y_{k+1}$

$$E^{-1} = 1 - \nabla$$

consider $\qquad (a-b)^n = nC_0 a^n - nC_1 a^{n-1} b$

$$\nabla^r y_k = (1 - E^{-1})^r y_k$$
$$= [1 - rC_1 E^{-1} + rC_2 E^{-2} - \dots + (-1)^r E^{-r}] y_k$$
$$= y_k - rC_1 E^{-1} y_k + rC_2 E^{-2} y_k - \dots + (-1)^r E^{-r} y_k$$
$$= y_k - rC_1 y_{k-1} + rC_2 y_{k-2} + \dots + (-1)^r y_{k-r}$$
$$= E^r y_{k-r} - rC_1 E^{r-1} y_{k-r} + rC_2 E^{r-2} y_{k-r} + \dots (-1)^r y_{k-r}$$

y_{k+r-r}

$E\, y_{k-r+r}$

$E^r y_{k-r}$

$$= (E-1)^r y_{k-r}$$
$$= \Delta^r y_{k-r}$$

$$\nabla^r y_k = \Delta^r y_{k-r}.$$

iii) $1-\nabla = E^{-1}$

$$(1-\nabla) f(x) = f(x)-[f(x)-f(x-h)]$$
$$= f(x)-f(x)+f(x-h)$$
$$= f(x-h)$$

$$1-\nabla f(x) = E^{-1} f(x)$$

$$\boxed{1-\nabla = E^{-1}}$$

iv) $E = e^{hD}$

using $Ef(x) = f(x+h)$

$$= f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \ldots$$

$$= f(x) + \frac{h}{1!} Df(x) + \frac{h^2}{2!} D^2 f(x) + \ldots$$

$$= \left[1 + \frac{h}{1!} D + \frac{h^2}{2!} D^2 + \ldots\right] f(x)$$

$$Ef(x) = e^{hD} f(x)$$

$$E = e^{hD}$$

v) $\Delta = 1-e^{-hD}$

By using (iv)

$$E = e^{hD}$$
$$1+\Delta = e^{hD}$$
$$e^{-hD} = \frac{1}{1+\Delta}$$

$$e^{-hD} = (1+\Delta)^{-1}$$

$= (1-\Delta+\Delta^2+\cdots)$

$= 1-\Delta$

$\Delta = 1-e^{-hD}$

prove that

$(1+\Delta)(1-\nabla) = 1$

$(1-\nabla)f(x) = f(x) - \nabla f(x)$

$= f(x) - [f(x) - f(x-h)]$

$= f(x) - f(x) + f(x-h)$

$= f(x-h)$

$(1-\nabla)f(x) = E^{-1}f(x)$

$1-\nabla = E^{-1} \longrightarrow (1)$

$(1+\Delta)f(x) = f(x) + \Delta f(x)$

$= f(x) + [f(x+h) - f(x)]$

$= f(x+h)$

$(1+\Delta)f(x) = Ef(x)$

$1+\Delta = E \longrightarrow (2)$

$(1), (2) \Rightarrow (1+\Delta)(1-\nabla) = EE^{-1}$

$= E\frac{1}{E}$

$= 1$

$(1+\Delta)(1-\nabla) = 1$

prove that

$E\Delta = \Delta E$

$$E\Delta f(x) = E[\Delta f(x)]$$
$$= E[f(x+h) - f(x)]$$
$$= Ef(x+h) - Ef(x)$$
$$= f(x+2h) - f(x+h)$$
$$= \Delta f(x+h)$$
$$= \Delta E$$
$$\boxed{E\Delta = \Delta E}$$

Factorial notation (or) Factorial function (or) Factorial polynomial

Taking 'n' to be positive integral using as interval of differencing we write

$$x^{n} = x(x-1)(x-2)\cdots(x-n+1)$$

In case the interval of differencing is 'h'

$$x^{(n)} = x(x-h)(x-2h)\cdots(x-\overline{n-1}h)$$

$$x^{(0)} = 1$$
$$x^{(1)} = x$$
$$x^{(2)} = (x-h)x$$
$$x^{(3)} = x(x-h)(x-2h)$$

and so on

We shall calculate the differencing of $x^{(n)}$

$$\Delta x^{n} = (x+h)^{(n)} - x^{(n)}$$

$$= [(x+h)(x+h-h)(x+h-2h)\cdots(x+h-\overline{n-1}\,h)] - [x(x-h)\cdots(x-\overline{n-1}\,h)]$$

$$= x(x-h)\cdots(x-\overline{n-2}\,h)\,[(x+h)-(x-\overline{n-1}\,h)]$$

$$= x^{(n-1)}\,n^{h}$$

$$\Delta x^{(n)} = nhx^{(n-1)}$$

$$\Delta^2 x^{(n)} = \Delta[\Delta x^{(n)}]$$

$$= \Delta[nhx^{(n-1)}]$$

$$= nh[\Delta x^{(n-1)}]$$

$$= nh(n-1)hx^{(n-2)}$$

$$\Delta^2 x^{(n)} = n(n-1)h^2 x^{(n-2)}$$

III'y $\Delta^x x^{(n)} = n(n-1)(n-2)\cdots h^n$

$$= n!\,h^n$$

$\Delta[x(x+1)(x+2)(x+3)]$ interval of differening is unity.

We know that

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta[x(x+1)(x+2)(x+3)] = [(x+1)(x+2)(x+3)(x+4)] - [x(x+1)(x+2)(x+3)]$$

$$= (x+1)(x+2)(x+3)[x+4-x]$$

$$\Delta[x(x+1)(x+2)(x+3)] = 4(x+1)(x+2)(x+3)$$

Evaluate:

$$\Delta^2\left[\frac{5x+12}{x^2+5x+6}\right] \text{ with } h=1$$

$$\frac{5x+12}{x^2+5x+6} = \frac{5x+12}{(x+2)(x+3)}$$

$$5x+12 = \frac{A}{(x+2)} + \frac{B}{(x+3)} \longrightarrow ①$$

$$\Rightarrow 5x+12 = A(x+3) + B(x+2)$$

put $x=-2$

$$5(-2)+12 = A(-2+3) + B(0)$$

$$-10+12 = A(1)$$

$$\boxed{A = 2}$$

put $x=-3$

$$5(-3)+12 = A(0) + B(-3+2)$$

$$-15+12 = B(-1)$$

$$-3 = -B$$

$$\boxed{B = 3}$$

Sub A, B values in ①

$$\frac{5x+12}{(x+2)(x+3)} = \frac{2}{(x+2)} + \frac{3}{(x+3)}$$

$$\Delta^2\left[\frac{5x+12}{(x+2)(x+3)}\right] = \Delta^2\left[\frac{2}{(x+2)} + \frac{3}{(x+3)}\right]$$

$$= \Delta\left[\Delta\left(\frac{2}{x+2}\right) + \Delta\left(\frac{3}{x+3}\right)\right]$$

$$= \Delta\left[2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right)\right]$$

$$= \Delta\left[2\left(\frac{x+2-x-3}{(x+3)(x+2)}\right) + 3\left(\frac{x+3-x-4}{(x+4)(x+3)}\right)\right]$$

$$= 2\Delta\left[\left(\frac{\overset{-1}{\cancel{x+2-x-3}}}{(x+3)(x+2)}\right)\right] + 3\Delta\left[\frac{-1}{(x+4)(x+3)}\right]$$

$\boxed{uv' = uv' + vu'}$

$$= -2\left[\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)}\right] - 3\left[\frac{1}{(x+5)(x+4)} - \frac{1}{(x+4)(x+3}\right]$$

$$= -2\left[\frac{x+2-x-4}{(x+2)(x+3)(x+4)}\right] - 3\left[\frac{x+3-x-5}{(x+3)(x+4)(x+5)}\right]$$

$$= \frac{4}{(x+2)(x+3)(x+4} + \frac{6}{(x+3)(x+4)(x+5)}$$

Evaluate $\left(\frac{\Delta^2}{E}\right)x^3$

$$\left(\frac{\Delta^2}{E}\right)x^3 = \left(\frac{(E-1)^2}{E}\right)x^3$$

$$= \left(\frac{E^2-2E+1}{E}\right)x^3$$

$$= \left(E-2+\frac{1}{E}\right)x^3$$

$$= \left(E-2+E^{-1}\right)x^3$$

$$= Ex^3 - 2x^3 + E^{-1}x^3$$

$$= (x+1)^3 - 2x^3 + (x-1)^3$$

$$= 6x$$

$$\left(\frac{\Delta^2}{E}\right)x^3 = 6x$$

Evaluate

$\Delta^n(e^{ax+b})$ with $h=1$

$$\Delta^n(e^{ax+b}) = \Delta^{n-1}\left[\Delta e^{ax+b}\right]$$

$$= \Delta^{n-1}\left[e^{a(x+1)+b} - e^{ax+b}\right]$$

$$= \Delta^{n-1}\left[e^{ax+b}e^{a} - e^{ax+b}\right]$$

$$= \Delta^{n-1}\left[e^{ax+b}(e^{a}-1)\right]$$

$$= \Delta^{n-1}\left[(e^{ax+b})(e^{a+b})\right.$$

$= (e^a - 1)\Delta^{n-1}(e^{ax+b})$

$= (e^a - 1)\Delta^{n-2}[(\Delta e^{ax+b})]$

$= (e^a - 1)\Delta^{n-1}[e^{a(x+1)+b} - e^{ax+b}]$

$= (e^a - 1)\Delta^{n-2}[e^{ax+b}e^a - e^{ax+b}]$

$= (e^a - 1)\Delta^{n-2}[e^{ax+b}(e^a - 1)]$

$= (e^a - 1)^2\Delta^{n-2}[e^{ax+b}]$

Proceeding in this way, we get

$$\Delta^n(e^{ax+b}) = (e^a - 1)^n\Delta^{n-n}(e^{ax+b})$$

$$= (e^a - 1)^n\Delta^0(e^{ax+b})$$

$$= (e^a - 1)^n(e^{ax+b})$$

Evaluate

$$\frac{\Delta^2}{E}\sin(x+h)$$

$$\frac{\Delta^2}{E}\sin(x+h) = \left[\frac{(E-1)^2}{E}\right]\sin(x+h)$$

$$= \left(\frac{E^2 - 2E + 1}{E}\right)\sin(x+h)$$

$$= \left(E - 2 + \frac{1}{E}\right)\sin(x+h)$$

$$= (E - 2 + E^{-1})\sin(x+h)$$

$$= E\sin(x+h) - 2\sin(x+h) + E^{-1}\sin(x+h)$$

$$= \sin(x+2h) - 2\sin(x+h) + \sin(x-h+h)$$

$$= \sin(x+2h) - 2\sin(x+h) + \sin(x)$$

ii) $\delta = E^{1/2} - E^{1/2}$

prove that, $\Delta\nabla = \Delta\nabla = \Delta - \nabla = \delta^2$

consider:

$$\Delta\nabla = \Delta[\nabla f(x)]$$

$$= \Delta[f(x) - f(a-h)]$$

$$= \Delta f(x) - \Delta f(a-h))$$

$$= f(x+h) - f(x) - f(x-h+h) + f(x-h)$$

$$= f(x+h) - f(x) - f(x-h+h) + f(x-h)$$

$$= f(x+h) - f(x) - f(x) + f(x-h)$$

$$\Delta\nabla = f(x+h) - 2f(x) \boxed{\cancel{-f(x)}} + f(x+h) \rightarrow (1)$$

$$\Delta\nabla = \nabla[\Delta f(x)]$$

$$= \nabla[f(x+h) - f(x)]$$

$$= f(x+h) - f(x-h+h) - f(x) + f(x-h)$$

$$\Delta\nabla = f(x+h) - 2f(x) + f(x-h) \rightarrow (2)$$

$$\Delta - \nabla = \Delta f(x) - \nabla f(x)$$

$$= f(x+h) - f(x) - [f(x) - f(x-h)]$$

$$\Delta - \nabla = f(x+h) - 2f(x) + f(x-h) \rightarrow (3)$$

$$\delta^2 = \delta^2 f(x)$$
$$= [E^{1/2} - E^{1/2}]^2 f(x)$$
$$= [(E^{1/2})^2 + (E^{-1/2})^2 - 2(E^{1/2})(E^{-1/2})] f(x)$$
$$= [E + E^{-1} - 2] f(x)$$
$$= E f(x) - 2 f(x) + E^{-1} f(x)$$
$$\delta^2 = f(x+h) - 2f(x) + f(x+h) \to (4)$$

from (1), (2), (3), (4) we get.

$$\cancel{\nabla\Delta = \nabla\Delta}$$
$$\Delta\nabla = \nabla\Delta = \Delta - \nabla = \delta^2$$

ii) $\left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x} = e^x$, h is interval of differencing.

$$\left(\frac{\Delta^2}{E}\right)e^x = \left[\frac{(E-1)^2}{E}\right]e^x$$
$$= \left(\frac{E^2+1-2E}{E}\right)e^x$$
$$= (E - 2 + E^{-1})e^x$$
$$= Ee^x - 2e^x + E^{-1}e^x$$
$$= e^{(x+h)} - 2e^{(x)} + e^{x-h}$$
$$\left(\frac{\Delta^2}{E}\right)e^x = e^x(e^h - 2 + e^{-h})$$

$Ee^x = e^{x+h}$

$\Delta^2 e^x = (E-1)^2 e^x$

$= (E^2+1-2E)e^x$

$= e^{(x+2h)} - 2e^{x+h} + e^x$

$= e^x[e^{2h} - 2e^h + 1]$

$= e^x e^h [e^h - 2 + e^{-h}]$

$\Delta^2 e^x = e^{x+h}[e^h - 2 + e^{-h}]$

$$\left(\frac{\Delta^2}{E}\right) e^x \cdot \frac{Ee^x}{\Delta^2 e^x} = e^x[e^h - 2 + e^{-h}] \cdot \frac{e^{x+h}}{e^{x+h}[e^h - 2 + e^{-h}]}$$

Interpolation

Newton's forward formula

i) $u = \dfrac{x-a}{h}$

ii) $f(x) = f(a) + \dfrac{u^{(1)}}{1!} \Delta f(a) + \dfrac{u^{(2)}}{2!} \Delta^2 f(a) + \dfrac{u^{(3)}}{3!} \cdot \Delta^3 f(a) + \ldots$

Backward formula

i) $u = \dfrac{x-(a+nh)}{h}$

ii) $f(x) = f(a+nh) + \frac{u^{(1)}}{1!} \nabla f(a+nh) + \frac{u^{(2)}}{2!}$

$\nabla^2 f(a+nh) + \cdots u(u+1)(u+2)$

$\cdots (u+1)(n-1) \frac{\nabla^n}{n!} f(a+nh)$

1. Using Backward formula to estimate the population for the year 1925

x	1891	1901	1911	1921	1931
y	46	66	81	93	101

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1891	46				
		20			
			-5		
1901	66			2	
		15			-3
			-3		
1911	81			-1	
		12			
			-4		
1921	93				
		8			
1931	101				

$$n = \frac{1925 - 1931}{10}$$

$$= \frac{-6}{10}$$

$$u = -0.6$$

$$f(x) = 101 + \frac{(-0.6)}{1!}(8) + \frac{(-0.6)^2}{2!}(-4) + \frac{(-0.6)^3}{3!}(-1)$$

$$+ \frac{(-0.6)^4}{4!}(-3) + \dots$$

$$= 101 + \frac{(-0.6)}{1}(8) + \frac{(-0.6)(-0.6+1)}{2}(-4) +$$

$$\frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(-1) +$$

$$\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}(-3)$$

$$= 101 + (-4.8) + (0.48) + 0.056 + 0.4367$$

$$= 96.8396$$

2. Calculate $U_{75} = 2459$ $U_{85} = 1180$ $U_{80} = 2018$ $U_{90} = 402$ using Forward Method.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
75	2459			
		-441		
80	2018		-397	
		-838		457
85	1180		60	
		-778		
90	402			

$$u = \frac{x-a}{h}$$

$$= \frac{82-75}{5}$$

$$= 7/5$$

$$u = 1.4$$

$$f(x) = f(a) + \frac{u^{(1)}}{1!}\Delta f(a) + \frac{u^{(2)}}{2!}\Delta^2 f(a) + \frac{u^{(3)}}{3!} \cdot\cdot \Delta^3 f(a) + \cdots$$

$$= 2459 + \frac{1.4}{1!}(-441) + \frac{(1.4)^2}{2!}(-397) + \frac{(1.4)^3}{3!}(457)$$

$$= 2459 + \frac{1.4}{1}(-441) + \frac{(1.4)(1.4-1)}{2}(-397) + \frac{(1.4)(1.4-1)(1.4-2)}{6}(457)$$

$1 \times 1 = 1$

$1 \times 2 \times 3$ $1 \times 2 = 2$

$1 \times 2 \times 3 = 6$

$$= 2459 + (\overset{-617.4}{\cancel{-3.174}}) + (-111.16) + (\overset{-2.5}{\cancel{-1.225}})$$

$$= 2447.549.$$

$$= 1704.848$$

NEWTON GREGORY FORMULA

Newton Gregory formula for forward interpolation [for equal intervals).

Statement :

Let $y = f(x)$ denotes a function which takes the values of $f(a), f(a+h)$, $f(a+2h) \dots f(a+nh)$ for $(n+1)$ equidistant values, $a+h, a+2h, \dots a+nh$ of independent variable x then.

$$f(x) = f(a) + u^{(1)} \Delta f(a) + \frac{u^{(2)}}{2!} \Delta^2 f(a) + \dots$$

$$+ \frac{u^{(n)}}{n!} \Delta^n f(a)$$

where,

$$u = \frac{x-a}{h} \text{ and } u^{(n)} = u(u-1)(u-2)\dots(u-n-1)$$

Proof :

Let $P_n(x)$ be a polynomial in x of degree n $P_n(x) = A_0 + A_1(x-a) + A_2(x-a)$.

$$(x-a-h) + A_3(x-a)$$

$$(x-a-h)(x-a-2h) + \dots + A_n(x-a)(x-ah).$$

$$[x-a-(n-1)h] \Rightarrow (1)$$

Where we choose the coefficient

$A_0, A_1 \dots A_n$

$P_n(a) = f(a),\ P_n(a+h) = f(a+h) \dots$

$P_n(a+nh) = f(a+nh)$

$P_n(a) = A_0 + A_1(a-a) + \dots + A_n(a-a)(a-a-h)\dots$

$[a-a-(n-1)n]$

put $x = a+h$ in ①

$P_n(a+h) = A_0 + A_1(a+h-a) + A_2(0) \dots + A_n(0)$

$= A_0 + hA_1$

$f(a+h) = f(a) + hA$

$hA_1 = f(a+h) - f(a)$

$A_1 = \dfrac{\Delta f(a)}{h\,1!}$

put $x = a+2h$ in ①

$P_n(a+2h) = A_0 + A_1(a+2h-a) + A_2(a+2h-a)$

$(a+2h-ah) + A_3(0) + \dots$

$+ A_n(0)$

$f(a+2h) = A_0 + A_1 2h + A_2 2h^2$

$= f(a) + 2h\dfrac{\Delta fa}{1!h} + A_2 2h^2$

$$= f(a) + 2[f(a+h) - f(a)] + 2h^2 A_2$$
$$= f(a) + 2f(a+h) - 2f(a) + 2h^2 A_2$$

$$2h^2 A_2 = f(a+2h) - 2f(a+h) + f(x)$$

Similarly we can find

$$A_3 = \frac{\Delta^3 f(a)}{3!\,h^3}$$

$$A_n = \frac{\Delta^n f(a)}{n!\,h^n}$$

Sub these values in $A_0, A \dots A_n$ in (1)

$$P_n(x) = f(a) + \frac{\Delta f(a)}{1!\,h}(x-a) + \frac{\Delta^2 f(a)}{2!\,h^2}(x-a)(x-a-h) + \dots + \frac{\Delta^n f(a)}{n!\,h^n}(x-a)(x-a-h)\dots[x-a-(n-1)h] \quad \to (2)$$

where

$$u = \frac{x-a}{h}$$

$$uh = x - a$$

$$x = uh + a$$

Sub $x = a + uh$ in eq (2)

$$P_n(a+uh) = f(a) + \frac{\Delta f(a)}{1!h}(uh-a-a) + \frac{\Delta^2 f(a)}{2!h^2}$$

$$(uh+a-a) + \cdots + \frac{\Delta^n f(a)}{n!h^n}(uh)(uh-h)\cdots$$

$$(uh-(n-1)h$$

$$f(a+uh) = f(a) + \frac{\Delta f(a)}{1!h}(uh) + \frac{\Delta^2 f(a)}{2!h^2}(uh)$$

$$(uh-h) + \cdots + \frac{\Delta^n f(a)}{n!h^n} h^n [u(u-1)\cdots u-(n-1)]$$

$$f_n(a+uh) = f(a) + \Delta f(a)\, u + \frac{\Delta^2 f(a)}{2!h^2} h^2 u(u-1) +$$

$$\cdots + \frac{\Delta^n f(a)}{n!h^n} h^n [u(u-1)\cdots u-(n-1)]$$

$$f_n(a+uh) = f(a) + \Delta f(a) u^{(1)} + \frac{\Delta^2 f(a)}{2!} u^{(2)} + \cdots$$

$$+ \Delta^n \frac{f(a)}{n!} u^{(n)}.$$

This is Newton gregory formula for forward interpolation.

Representation of a polynomial in factorial notation.

1. Represent the polynomial x^3+2x^2+3x-4 in factorial notation

$$
\begin{array}{c|cccc}
0 & 1 & 2 & 3 & -4 \\
 & 0 & 0 & 0 & 0 \\
\hline
1 & 1 & 2 & 3 & \boxed{-4} \\
 & 0 & 1 & 3 & \\
\hline
2 & 1 & 3 & \boxed{6} & \\
 & 0 & 2 & & \\
\hline
 & 1 & \boxed{5} & &
\end{array}
$$

$$x^3+2x^2+3x-4 = x^3+5x^2-6x-4=0$$

2. Represent the polynomial $2x^3-3x^2+3x-4$ in factorial notation.

$$
\begin{array}{ccccc}
0 & 2 & -3 & 3 & -4 \\
 & 0 & 0 & 0 & 0 \\
1 & 2 & -3 & 3 & -4 \\
 & 0 & 2 & -1 & \\
 & 2 & -1 & 2 & \\
 & 0 & 4 & & \\
 & 2 & 3 & &
\end{array}
$$

$$2x^3+3x^2+2x-4=0$$

3. Represent the polynomial $x^5 + 0x^4 + 0x^3 + 0x^2 + 0x - 3 = 0$ in factorial Notation.

0	1	0	0	0	0	-3
	0	0	0	0	0	0
1	1	0	0	0	0	-3
	0	1	1	1	1	
2	1	1	1	1	1	
	0	2	6	14		
3	1	3	7	15		
	0	3	18			
4	1	6	25			
	0	4				
	1	10				

$$x^5 + 10x^4 + 25x^3 + 15x^2 + x - 3 = 0$$

4. Represent the polynomial $f(x) = x^6 - 6x^4 - 2x^3 - 1$ and find is different in factorial Notation and find is different in factorial notation.

0	1	0	-6	-2	0	0	-1
	0	0	0	0	0	0	0
1	1	0	-6	-2	0	0	$\boxed{-1}$
	0	1	0	-5	-7	-7	
2	1	0	-5	-7	-7	$\boxed{-7}$	
	0	2	6	2	-10		
3	1	3	1	-5	$\boxed{-17}$		
	0	3	18	57			
4	1	6	19	$\boxed{52}$			
	0	4	40				
5	1	10	$\boxed{59}$				
	0	5					
	1	$\boxed{15}$					

$$x^6 + 15x^5 + 59x^4 + 52x^3 - 17x^2 - 7x - 1 = 0$$

$$\Delta f(x) = 6x^5 + 75x^4 + 236x^3 + 156x^2 - 34x - 7 = 0$$

$$\Delta^2 f(x) = 30x^4 + 300x^3 + 708x^2 + 312x - 34 = 0$$

$$\Delta^3 f(x) = 120x^3 + 900x^2 + 1416x + 312 = 0$$

$\Delta^4 f(x) = 360x^2 + 1800x + 1416 = 0$

$\Delta^5 f(x) = 720x + 1800 = 0$

$\Delta^6 f(x) = 720 = 0$

$\Delta^7 f(x) = 0 \qquad n \geq 7$

5. Represent the polynomial $f(x) = 2x^3 - 3x^2 + 3x - 10$ factorial Notation and find is difference in factorial Notation

0	2	-3	3	-10
	0	0	0	0
1	2	-3	3	$\boxed{-0}$
	0	2	-1	
2	2	-1	$\boxed{2}$	
	0	4		
	2	$\boxed{3}$		

$\Delta f(x) = 2x^3 + 3x^2 + 2x - 10 = 10$

$\Delta^2 f(x) = 6x^2 + 6x + 2 = 0$

$\Delta^3 f(x) = 12x + 6 = 0$

$\Delta^4 f(x) = 12x$

$\Delta^5 \quad = 0$

$n \geq 5$

Theorem:

Newton gregory formula for Backward interpolation ~~Statement~~.

Statement: Let $y = f(x)$ denotes a function which takes the value of $f(a), f(a+h), f(a+2h) \ldots$ for $(n+1)$ equidistant values $a, a+h, a+2h, \ldots (a+nh)$ of independent variable x then.

~~ftr~~

$$f_n(x) = f(a+nh) + \frac{u}{1!} \nabla f(a+nh) + \frac{u(u+1)}{2!} \nabla^2 f(a+nh) + \cdots + \frac{u(u+1)(u+2) \cdots (u+(n-1))}{n!}$$

where

$$\boxed{u = \frac{x-a-nh}{h}}$$

Proof:

$p_n(x)$ be a polynomial of degree n in x such that

$$y = f(x_i) = p_n(x_i)$$

Let us assume $p_n(x)$ is

$$P_n(x) = A_0 + A_1(x-a-nh) + A_2(x-a-nh)(x-a-nh+h)$$
$$+ A_3(x-a-nh)(x-a-nh+h)(x-a-nh+2h)$$
$$+ \cdots + A_n[(x-a-nh)(x-a-nh+h)\cdots$$
$$(x-a-nh+(n-1)h \longrightarrow (1)$$

put $x = a+nh-h$ in (1)

$$P_n(a+nh-h) = A_0 + A_1(a+nh-h-a-nh) + A_2(0) + \cdots$$

$$= A_0 + A_1(-h)$$

$$f(a+nh-h) = f(a+nh) - hA$$

$$hA_1 = f(a+nh) - hA$$

$$A_1 = \frac{f(a+nh) - f(a+nh-h)}{1!h} = \frac{\nabla f(a+h)}{1!h}$$

put $x = a+nh-2h$ in (1)

$$p_n(a+nh-2h) = A_0 + A_1(a+nh-2h-a-nh) +$$
$$A_2(a+nh-2h-a-nh)$$
$$(a+nh-2h-nh+h-a)\, A_3(0)$$

$$= A_0 + A_1(-2h) + A_2(-2h)(-h)$$

$$= A_0 - 2hA_1 + 2h^2 A_2$$

$$f(a+nh-2h) = f(a+nh) - 2h\left[\frac{\nabla f(a+nh)}{!h}\right] + 2h^2 A_2$$

$$= f(a+nh) - 2[f(a+nh) - f(a+nh-h)] + 2h^2 A_2$$

$$= f(a+nh) - 2f(a+nh) - 2f(a+nh-h) + 2h^2 A_2$$

$$2h^2 A_2 = f(a+nh-2h) + f(a+nh) - 2f(a+nh-h)$$

$$= f(a+nh) - 2f(a+nh-h) + f(a+nh-2h)$$

$$= \nabla^2 f(a+nh)$$

$$A_2 = \frac{\nabla^2 f(a+nh)}{2!h^2}$$

Similarly

$$A_3 = \frac{\nabla^3 f(a+nh)}{3!h^3}$$

$$\vdots$$

$$A_n = \frac{\nabla^n f(a+nh)}{n!h^n}$$

Sub $A_0, A_1 \ldots A_n$ value in ① we get,

$$p_n(x) = f(a+nh) + \frac{\nabla f(a+h)}{1!h}(x-a-nh) + \frac{\nabla^2 f(a+nh)}{2!h^2}(x-a-nh+h) + \frac{\nabla^3 f(a+nh)}{3!h^3}(x-a-h)(x-a-nh+1)$$

$$(x-a-nh+2h) + \ldots + \frac{\nabla^n f(a+nh)}{n!h^n}(x-a-nh)(x-a-nh+h)\ldots(x-a-nh(n-1)h) \longrightarrow ②$$

When

$$u = \frac{x-a-nh}{h} \Rightarrow uh = x-a-nh$$

$$x = a+uh+nh$$

Sub $x = a+uh+nh$ in equ ②

$$p_n(x) = f(a+nh) + \frac{\nabla f(a+nh)}{1!h} + \frac{\nabla^2 f(a+nh)}{2!h}$$

$$(a+uh+nh-a-nh)(a+uh+nh-a-nh+h) + \ldots +$$

$$\frac{\nabla^n f(a+nh)}{n!h^n}\left[(a+uh-a-nh)(a+uh+nh-a-nh+h)(a+uh-a-nh+(n-1)h)\right]$$

$$= f(a+nh) + \frac{\nabla f(a+nh)}{1!h}(uh) + \frac{\nabla^2 f(a+nh)}{2!h}(uh)(uh+h)$$

$$+ \cdots + \frac{\nabla^n f(a+nh)}{n!h^n}\Big[(uh)(uh+h)(uh+2h) \\ (uh+(n-1)h\Big]$$

$$\boxed{\therefore f(a) = f(a+nh) + \frac{u}{1!}\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \cdots + \frac{u(u+1)\cdots u+(n-1)}{n!}\nabla^n f(a+nh)}$$

This is known as Newton's Gregory formula for Backward Interpolation.