MARUDHAR KESARI JAIN COLLEGE FOR WOMEN, VANIYAMBADI

PG & RESEARCH DEPARTMENT OF MATHEMATICS

CLASS : I B.Sc PHYSICS

SUBJECT CODE : MATHEMATICS I

SUBJECT NAME : 23UEMA10C

SYLLABUS

UNIT- I

Summation of series

Binomial series -Exponential series - Logarithmic series -Simple Problems.

Binomial series

When n is a rational number,

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \dots$$

$\forall x$, such that $-1 < x < 1$.

Results :

1. $(1-x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \dots$

2. $(1-x)^{-n} = 1 - \frac{(-n)}{1}x + \frac{(-n)((-n)-1)}{1\cdot 2}x^2 - \dots$
   when $-1 < x < 1$ and n is a positive integer.

3. $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

4. $\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

5. $\frac{1}{(1-x)^3} = (1-x)^{-3} = \frac{1}{1\cdot 2}\left[1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + 4\cdot 5x^3 + \dots\right]$

6. $\frac{1}{(1-x)^4} = (1-x)^{-4} = \frac{1}{1\cdot 2\cdot 3}\left[1\cdot 2\cdot 3 + 2\cdot 3\cdot 4\cdot x + 3\cdot 4\cdot 5\cdot x^2 + \dots\right]$

7. $\frac{1}{(1-x)^n} = (1-x)^{-n} = \frac{1}{1\cdot 2\cdot 3\cdots(n-1)}\left[1\cdot 2\cdots(n-1) + 2\cdot 3\cdot 4\cdots n\cdot x + \dots\right]$

8. $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

9. $\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

10. $\frac{1}{(1+x)^3} = (1+x)^{-3} = \frac{1}{1\cdot 2}\left[1\cdot 2 - 2\cdot 3x + 3\cdot 4x^2 - \dots\right]$

11. $\frac{1}{(1+x)^4} = (1+x)^{-4} = \frac{1}{1\cdot2\cdot3}\left[1\cdot2\cdot3 - 2\cdot3\cdot4\,x + 3\cdot4\cdot5\,x^2 \ldots\right.$

12. $\frac{1}{(1+x)^n} = (1+x)^{-n} = \frac{1}{1\cdot2\cdot3\cdots n}\left[1\cdot2\cdot3\cdots(n-1) - 2\cdot3\cdots n\,x + \cdots\right]$

when n is positive number.

Problems

1. Find the co-eff of x^n in the expansion of $\frac{1}{1-x^2}$

U.Q

Soln:

$\frac{1}{1-x^2} = (1-x^2)^{-1}$

$(1-x)^{-1} = 1+x+x^2+x^3+\cdots+x^n+\cdots$

Put $x = x^2$

$(1-x^2)^{-1} = 1+x^2+(x^2)^2+(x^2)^3+\cdots(x^2)^n+\cdots$

$= 1+x^2+x^4+x^6+\cdots+x^{2n}+\cdots$

$\therefore$ co-eff of $x^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

2. Find the co-eff of x^{2n} in the expansion of $(1-x^2)^{-1}$.

U.Q

Soln:

$(1-x^2)^{-1}$

$(1-x)^{-1} = 1+x+x^2+x^3+\cdots+x^n+\cdots$

Put $x = x^2$ in formula

$(1-x^2)^{-1} = 1+x^2+(x^2)^2+(x^2)^3+\cdots+(x^2)^n+\cdots$

$= 1 + x^2 + x^4 + x^6 + \cdots + (x^{2n}) + \cdots$

$\therefore$ co-eff of $x^{2n} = 1$

3\. Find the co-eff of x^2 in expansion of $(1+x)^{-3}$

u.a

Soln:

$(1+x)^{-3} = \frac{1}{1\cdot 2}[1\cdot 2 - 2\cdot 3x + 3\cdot 4x^2 + \cdots]$

co-eff of $x^2 = \frac{1}{1\cdot 2}(3\cdot 4)$

co-eff of $x^2 = 6$

4\. Find the co-eff of x^n in $\frac{1}{1-2x} + \frac{1}{1-3x}$

u.a Soln:

$\frac{1}{1-2x} = (1-2x)^{-1}$; $\frac{1}{1-3x} = (1-3x)^{-1}$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

Put $x = 2x$ & $x = 3x$

$(1-2x)^{-1} + (1-3x)^{-1} = [1 + 2x + (2x)^2 + (2x)^3 + \cdots + (2x)^n + \cdots]$
$+ [1 + 3x + (3x)^2 + (3x)^3 + \cdots + (3x)^n + \cdots]$

u.a $\therefore$ co-eff of $x^n = 2^n + 3^n$

5\. Find the sums of the following series

i) $1 + 2(1/2) + 3(1/2)^2 + \cdots + \infty$

ii) $1 + 2(1/3) + 3(1/3)^2 + 4(1/3)^3 + \cdots \infty$

Soln:

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Put $x = \frac{1}{2}$

$$1 + 2\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right)^2 + \cdots = \left(1 - \tfrac{1}{2}\right)^{-2}$$

$$= \left(\frac{2-1}{2}\right)^{-2}$$

$$= \left(\tfrac{1}{2}\right)^{-2}$$

$$= \left(\tfrac{1}{4}\right)$$

$$= 4$$

ii) $$1 + 2\left(\tfrac{1}{3}\right) + 3\left(\tfrac{1}{3}\right)^2 + \cdots = \left(1 - \tfrac{1}{3}\right)^{-2}$$

$$= \left(\frac{3-1}{3}\right)^{-2}$$

$$= \frac{1}{\left(\frac{2}{3}\right)^2}$$

$$= \left(\frac{1}{4/9}\right)$$

$$1 + 2\left(\tfrac{1}{3}\right) + 3\left(\tfrac{1}{3}\right)^2 + \cdots = \tfrac{9}{4}$$

6. Find the co-eff of x^n in expansion of $(2+3x)^{-1}$ in ascending power of x.

Soln:

$$(2+3x)^{-1} = 2^{-1}\left(1 + \frac{3x}{2}\right)^{-1}$$

$$= \frac{1}{2}\left(1 + \frac{3}{2}x\right)^{-1}$$

W.K.T

$$\left[(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots\right]$$

$$\frac{1}{2}\left[1+\frac{3}{2}x\right]^{-1} = \frac{1}{2}\left[1-\frac{3}{2}x+\left(\frac{3}{2}x\right)^2-\left(\frac{3}{2}x\right)^3-\dots + (-1)^n\left(\frac{3}{2}x\right)^n+\dots\right]$$

$$= \frac{1}{2}\left[1-\frac{3}{2}x+\frac{3^2}{2^2}x^2+\dots+(-1)^n\frac{3^n}{2^n}x^n+\dots\right]$$

$\therefore$ Co-eff of $x^n = \frac{1}{2}\times\left[(-1)^n\left(\frac{3^n}{2^n}\right)\right]$

$\therefore$ co-eff of $x^n = (-1)^n \frac{3^n}{2^{n+1}}$

T. Find the co-eff of x^n in expansion of $[1+2x+3x^2+4x^3+\dots]^2$

Soln:

$$[1+2x+3x^2+4x^3+\dots]^2 = \left[(1-x)^{-2}\right]^2$$

$$= \left[\frac{1}{(1-x)^2}\right]^2$$

$$= \frac{1}{(1-x)^4}$$

$$= (1-x)^{-4}$$

W.K.T

$$(1-x)^{-4} = \frac{1}{1\cdot2\cdot3}\left[1\cdot2\cdot3+2\cdot3\cdot4\cdot x+3\cdot4\cdot5\cdot x^2+\dots + (n+1)(n+2)(n+3)x^n+\dots\right]$$

$\therefore$ co-eff of $x^n = \frac{1}{1\cdot2\cdot3}\left[(n+1)(n+2)(n+3)\right]$

$\therefore$ Co-eff of $x^n = \frac{1}{6}\left[(n+1)(n+2)(n+3)\right]$

8. Write the $(n+1)^{th}$ term in expansion of $(3-2x)^{-2}$, where x is small.

Soln:

$$(3-2x)^{-2} = \frac{1}{(3-2x)^2}$$

$$= \frac{1}{3^2\left[1-\frac{2}{3}x\right]^2}$$

$$= \frac{1}{9}\left[1-\frac{2}{3}x\right]^{-2}$$

$$(1-x)^{-2} = 1+2x+3x^2+\cdots+(n+1)x^n+\cdots$$

$$\frac{1}{9}\left[1-\frac{2}{3}x\right]^{-2} = \frac{1}{9}\left[1+2\left(\frac{2}{3}x\right)+3\left(\frac{2}{3}x\right)^2+\cdots+(n+1)\left(\frac{2}{3}x\right)^n+\cdots\right]$$

$$= \frac{1}{9}\left[1+2\left(\frac{2}{3}\right)x+3\left(\frac{2^2}{3^2}x^2\right)+\cdots+(n+1)\left(\frac{2^n}{3^n}x^n\right)\right]$$

$(n+1)^{th}$ term $= \left[\frac{2^n}{3^n}x^n\right]\frac{1}{9}$

$\therefore (n+1)^{th}$ term $= \frac{2^n}{3^{n+2}}(n+1) = \frac{2^n}{3^{n+2}}(n+1)x^n.$

9. Find the co-efficient of x^2 in the expansion of $\left(1+\frac{2}{3}x\right)^{3/2}$

H.W

Soln:

W.K.T

$$(1+x)^n = 1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3+\cdots$$

Put $x = \frac{2}{3}x$; $n = 3/2$

$$\left(1+\frac{2}{3}x\right)^{3/2} = \left[1+\frac{\frac{3}{2}}{1}\left(\frac{2}{3}\right)x+\frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{1\cdot 2}\left(\frac{2}{3}x\right)^2+\cdots\right]$$

$$\text{co-efficient of } x^2 = \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{1.2} \quad \frac{4}{9}$$

$$= \frac{\frac{3}{2}\left(\frac{1}{2}\right)}{\cancel{2}} \times \frac{\cancel{4}^{2}}{9}$$

$$= \frac{\cancel{3}}{\cancel{4}_{2}} \times \frac{\cancel{2}}{\cancel{9}_{3}}$$

$$\therefore \text{co-eff of } x^2 = \frac{1}{6}$$

10. Find the co-eff of x^6 in the expansion of $\frac{1}{(1-x^2)^3}$

Soln:

W.K.T

$$(1-x)^{-3} = \frac{1}{1\cdot 2}\left[1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + 4\cdot 5x^3 + 5\cdot 6x^4 + 6\cdot 7x^5 + 7\cdot 8x^6 + \cdots\right]$$

Put $x = x^2$

$$(1-x^2)^{-3} = \frac{1}{1\cdot 2}\left[1\cdot 2 + 2\cdot 3(x^2) + 3\cdot 4(x^2)^2 + 4\cdot 5(x^2)^3 + 5\cdot 6(x^2)^4 + 6\cdot 7(x^2)^5 + 7\cdot 8(x^2)^6 + \cdots\right]$$

$$= \frac{1}{1\cdot 2}\left[1\cdot 2 + 2\cdot 3x^2 + 3\cdot 4x^4 + 4\cdot 5x^6 + 5\cdot 6x^8 + 6\cdot 7x^{10} + 7\cdot 8x^{12} + \cdots\right]$$

$$\text{co-eff of } x^6 = \frac{1}{\cancel{1\cdot 2}}\left[\cancel{4}^{2}\cdot 5\right]$$

$$\therefore \text{co-eff of } x^6 = 10$$

11. If x is small, what is the value of a if $\sqrt{x^2+4} - \sqrt{x^2+1} = 1 - ax^2$ nearly.

U.Q

Soln:

$$\sqrt{x^2+4} - \sqrt{x^2+1} = (x^2+4)^{1/2} - (x^2+1)^{1/2}$$

$$= 4^{1/2}\left(1+\frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

$$= 2\left(1+\frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

W.K.T

$$(1+x)^n = \left[1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \cdots\right]$$

Put $x = \frac{x^2}{4}$; $n = \frac{1}{2}$

$$= 2\left[1 + \frac{\frac{1}{2}}{1}\left(\frac{x^2}{4}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}\left(\frac{x^2}{4}\right)^2 + \cdots\right]$$

$$- \left[1 + \frac{\frac{1}{2}}{1}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}(x^2)^2 + \cdots\right]$$

$$\simeq 2\left[1 + \frac{1}{2}\left(\frac{x^2}{4}\right)\right] - \left[1 + \frac{1}{2}x^2\right] \text{ nearly}$$

$$= 2\left[1 + \frac{x^2}{8}\right] - 1 - \frac{1}{2}x^2$$

$$= 2 + \frac{\cancel{2}x^2}{\cancel{8}\,4} - 1 - \frac{1}{2}x^2$$

$$= 1 + \frac{x^2}{4} - \frac{x^2}{2}$$

$$= 1 + \frac{x^2 - 2x^2}{4}$$

$$= 1 - \frac{x^2}{4}$$

$$\therefore \sqrt{x^2+4} - \sqrt{x^2+1} = 1 - \frac{x^2}{4}$$

$$a = \frac{1}{4}$$

12. When x is small P.T

$$(1-x)^{-1/2} + (1+x)^{1/2} = 2 + x + \frac{x^2}{4} \text{ (nearly)}$$

Soln:

W.K.T

$$(1+x)^{-n} = 1 - \frac{(-n)x}{1} + \frac{(-n)((-n)-1)}{} x^2 + \dots$$

$$(1-x)^{-n} = 1 + \frac{n}{1}x + \frac{n(n+1)}{1\cdot 2} x^2 + \dots$$

$$(1+x)^{+n} = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2} x^2 + \dots$$

$$(1-x)^{-1/2} + (1+x)^{1/2} = \left[1 + \frac{\frac{1}{2}}{1}x + \frac{\frac{1}{2}(\frac{1}{2}+1)}{1\cdot 2}x^2 + \dots\right]$$

$$+ \left[1 + \frac{\frac{1}{2}}{1}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}x^2 + \dots\right]$$

$$= \left[1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{3}{2})}{1\cdot 2}x^2 + \dots\right] + \left[1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{1\cdot 2}x^2 \dots\right]$$

$$= \left[1 + \frac{1}{2}x + \frac{3}{4\times 2}x^2\right] + \left[1 + \frac{1}{2}x - \frac{\frac{1}{4}}{2}x^2\right]$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + 1 + \frac{1}{2}x - \frac{1}{8}x^2 \text{ (nearly)}$$

$$= 2 + x + \frac{\cancel{2}}{\cancel{8}\,4}x^2 \text{ (nearly)}$$

$$(1-x)^{-1/2} + (1+x)^{1/2} = 2 + x + \frac{x^2}{4} \text{ (nearly)}$$

13. When x is small, P.T

H.W

$\sqrt{x^2+4} - \sqrt{x^2+1} = 1 - \frac{1}{4}x^2 + \frac{7}{64}x^4$ nearly.

Soln:

$$\sqrt{x^2+4} - \sqrt{x^2+1} = (x^2+4)^{1/2} - (x^2+1)^{1/2}$$

$$= 4^{1/2}\left(1+\frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

$$= 2\left(1+\frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

W.K.T

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

put $x = \frac{x^2}{4}$ and $x = x^2$; $n = \frac{1}{2}$

$$= 2\left[1 + \frac{\frac{1}{2}}{1}\left(\frac{x^2}{4}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}\left(\frac{x^2}{4}\right)^2 + \cdots\right]$$

$$- \left[1 + \frac{\frac{1}{2}}{1}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\cdot 2}(x^2)^2 + \cdots\right]$$

$$= 2\left[1 + \frac{x^2}{8} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\frac{x^4}{16} + \cdots\right]$$

$$- \left[1 + \frac{1}{2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^4 + \cdots\right]$$

$$= 2\left[1 + \frac{x^2}{8} - \frac{1}{8}\frac{x^4}{16}\right] - \left[1 + \frac{1}{2}x^2 - \frac{1}{8}x^4\right]$$

$$= 2 + \frac{\cancel{2}x^2}{\cancel{8}4} - \frac{\cancel{2}}{\cancel{8}4}\frac{x^4}{16} - 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4$$

$$= 1 + \frac{x^2 - 2x^2}{4} + \left(\frac{8-1}{64}\right)x^4 \text{ nearly}$$

$$= 1 - \frac{1}{4}x^2 + \frac{7}{64}x^4 \text{ nearly}$$

14. Find the co-efficient of x^n in the expansion of $(1+x+x^2+x^3+\ldots)^{-n}$

H.W

Soln:

$$(1+x+x^2+x^3+\ldots)^{-n} = ((1-x)^{-1})^{-n}$$

$$= (1-x)^n$$

$$= 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot2}x^2 + \ldots \frac{(-1)^n n(n-1)(n-2)\ldots 2\cdot1}{1\cdot2\ldots(n-2)(n-1)n}x^n$$

$$= 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot2}x^2 + \ldots + (-1)x^n$$

$\therefore$ co-eff of $x^n = (-1)^n$.

SUMMATION: Binomial series

The formula for finding the sums of Binomial Series is

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2 + \ldots$$

15. Sum the series $1 + \frac{1}{3} + \frac{1\cdot3}{3\cdot6} + \frac{1\cdot3\cdot5}{3\cdot6\cdot9} + \ldots$

U.Q

Soln:

W.K.T

$$(1-x)^{-p/q} = 1 + \boxed{\frac{p}{1}\left(\frac{x}{q}\right)} + \frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2 + \frac{p(p+q)(p+2q)}{1\cdot2\cdot3}\left(\frac{x}{q}\right)^3 + \ldots$$

Denote the Sum by S.

Let

$$S = 1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \cdots$$

$$\phantom{S = 1 + \frac{1}{3} +}\ \ 1.3\ \ 2.3 \qquad 1.3\ \ 2.3\ \ 3.3$$

$p = 1$; $q = 2$

$p + q = 3$

$1 + q = 3$

$q = 3 - 1 = 2$

$$S = 1 + \frac{1}{1}\left(\frac{1}{3}\right) + \frac{1.3}{1.2}\left(\frac{1}{3}\right)^2 + \frac{1.3.5}{1.2.3}\left(\frac{1}{3}\right)^3 + \cdots$$

Here $\frac{x}{q} = \frac{1}{3}$

$$x = \frac{q}{3}$$

$$x = 2/3$$

$$\therefore (1-x)^{-p/q} = (1 - 2/3)^{-1/2}$$

$$= \left(\frac{3-2}{3}\right)^{-1/2}$$

$$= (1/3)^{-1/2}$$

$$= (3^{-1})^{-1/2}$$

$$= 3^{1/2}$$

$$\boxed{S = \sqrt{3}}$$

16. Sum the series $\frac{1}{10} + \frac{1.4}{10.20} + \frac{1.4.7}{10.20.30} + \cdots$

Soln:

Denote the sum by S, then

$$S = \frac{1}{10} + \frac{1\cdot 4}{10\cdot 20} + \frac{1\cdot 4\cdot 7}{10\cdot 20\cdot 30} + \cdots$$

$1\cdot 10 \quad 1\cdot 10\ 2\cdot 10 \quad 1\cdot 10\ 2\cdot 10\ 3\cdot 10$

$$= \frac{1}{1}\left(\frac{1}{10}\right) + \frac{1\cdot 4}{1\cdot 2}\left(\frac{1}{10}\right)^2 + \frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}\left(\frac{1}{10}\right)^3 + \cdots$$

W.K.T

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2 + \cdots$$

Add and sub~~tract~~ 1 in R.H.S

$$= \left[1 + \frac{1}{1}\left(\frac{1}{10}\right) + \frac{1\cdot 4}{1\cdot 2}\left(\frac{1}{10}\right)^2 + \frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}\left(\frac{1}{10}\right)^3 + \cdots\right] - 1$$

$p = 1$; $p + q = 4$

$q = 4 - 1 = 3$

$\frac{x}{q} = \frac{1}{10}$

$\frac{x}{3} = \frac{1}{10} \Rightarrow x = \frac{3}{10}$

$$S = \left(1 - \frac{3}{10}\right)^{-1/3} - 1$$

$$= \left(\frac{10-3}{10}\right)^{-1/3} - 1$$

$$= \left(\frac{7}{10}\right)^{-1/3} - 1$$

$$= \frac{1}{(7/10)^{1/3}} - 1$$

$$\boxed{S = \left(\frac{10}{7}\right)^{1/3} - 1}$$

U.Q

17 Sum the Series

$$\frac{5}{1\cdot 2}\left(\frac{1}{3}\right) + \frac{5\cdot 7}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^2 + \frac{5\cdot 7\cdot 9}{1\cdot 2\cdot 3\cdot 4}\left(\frac{1}{3}\right)^3 + \cdots$$

Soln:

Denote the sum by S, then

$$S = \frac{5}{1\cdot 2}\left(\frac{1}{3}\right) + \frac{5\cdot 7}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^2 + \frac{5\cdot 7\cdot 9}{1\cdot 2\cdot 3\cdot 4}\left(\frac{1}{3}\right)^3 + \cdots$$

W.K.T

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2 + \cdots$$

Multiplying by 3 on both sides.

$$3S = \frac{3\cdot 5}{1\cdot 2}\left(\frac{1}{3}\right) + \frac{3\cdot 5\cdot 7}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^2 + \frac{3\cdot 5\cdot 7\cdot 9}{1\cdot 2\cdot 3\cdot 4}\left(\frac{1}{3}\right)^3 + \cdots$$

Multiplying by $\frac{1}{3}$ on both sides

$$\left(\frac{1}{\cancel{3}}\right)\cancel{3}S = \frac{3\cdot 5}{1\cdot 2}\left(\frac{1}{3}\right)^2 + \frac{3\cdot 5\cdot 7}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^3 + \frac{3\cdot 5\cdot 7\cdot 9}{1\cdot 2\cdot 3\cdot 4}\left(\frac{1}{3}\right)^4 + \cdots$$

Add and subtract $1 + \frac{3}{1}\left(\frac{1}{3}\right)$ in R.H.S

$$S = \left[1 + \frac{3}{1}\left(\frac{1}{3}\right) + \frac{3\cdot 5}{1\cdot 2}\left(\frac{1}{3}\right)^2 + \frac{3\cdot 5\cdot 7}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^3 + \cdots\right] - \left[1 + \frac{3}{1}\left(\frac{1}{3}\right)\right]$$

$p = 3$; $p + q = 5$; $q = 2$; $\frac{x}{q} = \frac{1}{3}$; $x = \frac{2}{3}$

$$S = (1-x)^{-p/q} - (1+1)$$

$$= \left(1 - \frac{2}{3}\right)^{-3/2} - 2$$

$$= \left(\frac{3-2}{3}\right)^{-3/2} - 2$$

$$= \left(\frac{1}{3}\right)^{-3/2} - 2$$

$$3^{3/2} = 3^{1/2} \cdot 3$$

$$= (3)^{3/2} - 2$$

$$\boxed{S = 3\sqrt{3} - 2}$$

18. Sum to infinity the series

U.Q

$$\frac{7}{9} + \frac{7 \cdot 9}{9 \cdot 12} + \frac{7 \cdot 9 \cdot 12}{9 \cdot 12 \cdot 15} + \ldots$$

Soln.

Denote the sum by S, then

$$S = \frac{7}{9} + \frac{7 \cdot 9}{9 \cdot 12} + \frac{7 \cdot 9 \cdot 12}{9 \cdot 12 \cdot 15} + \ldots$$

$3 \cdot 3$ $\quad 3 \cdot 3 \; 3 \cdot 4$ $\quad 3 \cdot 3 \; 3 \cdot 4 \; 3 \cdot 5$

$$= \frac{7}{3}\left(\frac{1}{3}\right) + \frac{7 \cdot 9}{3 \cdot 4}\left(\frac{1}{3}\right)^2 + \frac{7 \cdot 9 \cdot 12}{3 \cdot 4 \cdot 5}\left(\frac{1}{3}\right)^3 + \ldots$$

multiplying by bothsides $\frac{1}{1 \cdot 2}$

$$\frac{S}{1 \cdot 2} = \frac{7}{1 \cdot 2 \cdot 3}\left(\frac{1}{3}\right) + \frac{7 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{1}{3}\right)^2 + \frac{7 \cdot 9 \cdot 12}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\left(\frac{1}{3}\right)^3 + \ldots$$

~~multiplying bothsides 3.5~~

Multiplying bothsides by 3·5

$$\frac{3 \cdot 5}{1 \cdot 2} S = \frac{3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3}\left(\frac{1}{3}\right) + \frac{3 \cdot 5 \cdot 7 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4}\left(\frac{1}{3}\right)^2 + \ldots$$

Multiplying by $(1/3)^2$ on both sides

$$\frac{155}{2}(1/3)^2 = \frac{3\cdot5\cdot7}{1\cdot2\cdot3}(1/3)^3 + \frac{3\cdot5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}(1/3)^4 + \cdots$$

Add and substract by $1 + \frac{3}{1}(1/3) + \frac{3\cdot5}{1\cdot2}(1/3)^2$ in R.H.S

$$\frac{\cancel{155}^{5}}{2} \times \frac{1}{\cancel{9}_{3}} = \left[1 + \frac{3}{1}(1/3) + \frac{3\cdot5}{1\cdot2}(1/3)^2 + \frac{3\cdot5\cdot7}{1\cdot2\cdot3}(1/3)^3 + \cdots\right]$$

$$- \left[1 + \boxed{\frac{3}{1}(1/3) + \frac{3\cdot5}{1\cdot2}(1/3)^2}\right]$$

$$\frac{5}{6}s = \left[1 + \frac{3}{1}(1/3) + \frac{3\cdot5}{1\cdot2}(1/3)^2 + \frac{3\cdot5\cdot7}{1\cdot2\cdot3}(1/3)^3 + \cdots\right]$$

$$- \left[1 + 1 + \frac{\cancel{15}^{5}}{2} \times \frac{1}{\cancel{9}_{3}}\right]$$

$p = 3$; $p+q = 5$; $\frac{x}{q} = 1/3$

$q = 5-3$

$q = 2$; $x = \frac{2}{3}$

$$\frac{5}{6}s = \left(1 - \frac{2}{3}\right)^{-3/2} - \left[2 + \frac{5}{6}\right]$$

$$= \left(\frac{3-2}{3}\right)^{-3/2} - \left(\frac{12+5}{6}\right)$$

$$= \left(\frac{1}{3}\right)^{-3/2} - \frac{17}{6}$$

$$= (3)^{3/2} - \frac{17}{6}$$

$$\frac{5}{6}s = 3\sqrt{3} - \frac{17}{6}$$

~~Multiplying both sides~~

$$s = \left(3\sqrt{3} - \frac{17}{6}\right)\frac{6}{5}$$

$$= \frac{3\sqrt{3}}{5} \times 6 - \frac{17}{\cancel{6}} \times \frac{\cancel{6}}{5}$$

$$S = \frac{18\sqrt{3}}{5} - \frac{17}{5}$$

H.W

19. Sum the series $\frac{1}{3.6} + \frac{1.3}{3.6.9} + \frac{1.3.5}{3.6.9.12} + \cdots$

Soln:

Denote the sum by S, then

W.K.T

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1.2}\left(\frac{x}{q}\right)^2 + \cdots$$

$$S = \frac{1}{3.6} + \frac{1.3}{3.6.9} + \frac{1.3.5}{3.6.9.12} + \cdots$$

$$= \frac{1}{1.2}\left(\frac{1}{3}\right)^2 + \frac{1.3}{1.2.3}\left(\frac{1}{3}\right)^3 + \frac{1.3.5}{1.2.3.4}\left(\frac{1}{3}\right)^4 + \cdots$$

multiply both sides by -1

$$-S = \left\{\frac{-1.1}{1.2}\left(\frac{1}{3}\right)^2 + \frac{-1.1.3}{1.2.3}\left(\frac{1}{3}\right)^3 + \frac{-1.1.3.5}{1.2.3.4}\left(\frac{1}{3}\right)^4 + \cdots\right.$$

Add & subtract $1 + \frac{-1}{1}\left(\frac{1}{3}\right)$ in R.H.S

$$-S = 1 + \frac{-1}{1}\left(\frac{1}{3}\right) + \frac{-1.1}{1.2}\left(\frac{1}{3}\right)^2 + \cdots\Big] - \left[1 + \frac{-1}{1}\left(\frac{1}{3}\right)\right]$$

$p = -1$; $p + q = 1$

$-1 + q = 1$ $\quad \frac{x}{q} = \frac{1}{3} \Rightarrow x = \frac{2}{3}$

$q = 2$

$$-S = (1-x)^{-p/q} - \left(1 - \frac{1}{3}\right)$$

$$= \left(1 - \frac{2}{3}\right)^{1/2} - \frac{2}{3}$$

$$= \left(\frac{3-2}{3}\right)^{1/2} - \frac{2}{3}$$

$$= \left(\frac{1}{3}\right)^{1/2} - \frac{2}{3}$$

$$-S = \frac{1}{\sqrt{3}} - \frac{2}{3} \Rightarrow \boxed{S = \frac{2}{3} - \frac{1}{\sqrt{3}}}$$

20. S.T $\sqrt{8} = 1 + \frac{3}{4} + \frac{3\cdot5}{2\cdot4^2} + \frac{3\cdot5\cdot7}{2\cdot3\cdot4^3} + \cdots$

U.Q

Soln:

R.H.S

$$\Rightarrow 1 + \frac{3}{4} + \frac{3\cdot5}{2\cdot4^2} + \frac{3\cdot5\cdot7}{2\cdot3\cdot4^3} + \cdots$$

$$\Rightarrow 1 + \frac{3}{1}(1/4) + \frac{3\cdot5}{1\cdot2}\left(\frac{1}{4}\right)^2 + \frac{3\cdot5\cdot7}{1\cdot2\cdot3}\left(\frac{1}{4}\right)^3 + \cdots$$

$p = 3$; $p+q = 5$ $\quad \frac{x}{q} = \frac{1}{4}$

$q = 5-3$ $\quad x = \frac{\cancel{2}}{\cancel{4}2} = 1/2$

$q = 2$

$$\Rightarrow (1-x)^{-p/q}$$

$$\Rightarrow (1-1/2)^{-3/2}$$

$$\Rightarrow \left(\frac{2-1}{2}\right)^{-3/2}$$

$$\Rightarrow \left(\frac{1}{2}\right)^{-3/2}$$

$$\Rightarrow (2)^{3/2}$$

$$\Rightarrow (2^3)^{1/2} = (8)^{1/2}$$

$$\Rightarrow \sqrt{8} = L.H.S$$

L.H.S = R.H.S

21. Sum the series $1 + \frac{2}{1}\left(\frac{1}{72}\right) + \frac{2\cdot5}{1\cdot2}(1/64) + \cdots$

Soln:

Denote the sum by S, then

$$S = 1 + \frac{2}{1}(1/72) + \frac{2\cdot5}{1\cdot2}(1/74) + \frac{2\cdot5\cdot8}{1\cdot2\cdot3}(1/76) + \cdots$$

$$= 1 + \frac{2}{1}(1/49) + \frac{2\cdot5}{1\cdot2}(1/49)^2 + \frac{2\cdot5\cdot8}{1\cdot2\cdot3}\left(\frac{1}{49}\right)^3 + \cdots$$

$p = 2$; $p + q = 5$

$2 + q = 5$

$q = 5 - 2$

$q = 3$

$\frac{x}{q} = \frac{1}{49}$

$x = \frac{3}{49}$

$S = (1-x)^{-p/q}$

$= \left(1 - \frac{3}{49}\right)^{-2/3}$

$S = \left(\frac{49-3}{49}\right)^{-2/3} = \left(\frac{46}{49}\right)^{-2/3} = \left(\frac{49}{46}\right)^{2/3}$

22. Sum the series $\frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \cdots$

Soln:

Denote the sum by S, then

$S = \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \cdots$

$1.3 \quad 2.3 \qquad 1.3 \; 2.3 \; 3.3$

$S = \frac{1}{1}(1/3) + \frac{1.3}{1.2}(1/3)^2 + \frac{1.3.5}{1.2.3}(1/3)^3 + \cdots$

Add & Sub 1

$S = \left[1 + \frac{1}{1}(1/3) + \frac{1.3}{1.2}(1/3)^2 + \frac{1.3.5}{1.2.3}(1/3)^3 + \cdots\right] - 1$

$p = 1$; $p + q = 3$

$1 + q = 3$

$q = 3 - 1$

$q = 2$

$\frac{x}{q} = \frac{1}{3}$

$x = 2/3$

$S = (1-x)^{-p/q} - 1$

$= \left(1 - \frac{2}{3}\right)^{-1/2} - 1$

$= \left(\frac{3-2}{3}\right)^{-1/2} - 1 = (1/3)^{-1/2} - 1$

$\boxed{S = \sqrt{3} - 1}$

U.Q

23. Find the sum to infinity series

$$\frac{15}{16}+\frac{15\cdot 21}{16\cdot 24}+\frac{15\cdot 21\cdot 27}{16\cdot 24\cdot 32}+\dots$$

Soln:

W.K.T

$$(1-x)^{-p/q}=1+\frac{p}{1}\left(\frac{x}{q}\right)+\frac{p(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2+\dots$$

Denote the sum by S

$$S=\frac{15}{16}+\frac{15\cdot 21}{16\cdot 24}+\frac{15\cdot 21\cdot 27}{16\cdot 24\cdot 32}+\dots$$

$2\cdot 8$ $\quad 2\cdot 8\ 3\cdot 8$ $\quad 2\cdot 8\ 3\cdot 8\ 4\cdot 8$

$$=\frac{5}{2}\left(\frac{3}{8}\right)+\frac{5\cdot 7}{2\cdot 3}\left(\frac{3}{8}\right)^2+\frac{5\cdot 7\cdot 9}{2\cdot 3\cdot 4}\left(\frac{3}{8}\right)^3+\dots$$

multiply by $\frac{3}{1}$ on b.s

$$3S=\frac{3\cdot 5}{1\cdot 2}\left(\frac{3}{8}\right)+\frac{3\cdot 5\cdot 7}{1\cdot 2\cdot 3}\left(\frac{3}{8}\right)^2+\frac{3\cdot 5\cdot 7\cdot 9}{1\cdot 2\cdot 3\cdot 4}\left(\frac{3}{8}\right)^3+\dots$$

multiplying by $\frac{3}{8}$ on b.s

$$\frac{9S}{8}=\frac{3\cdot 5}{1\cdot 2}\left(\frac{3}{8}\right)^2+\frac{3\cdot 5\cdot 7}{1\cdot 2\cdot 3}\left(\frac{3}{8}\right)^3+\frac{3\cdot 5\cdot 7\cdot 9}{1\cdot 2\cdot 3\cdot 4}\left(\frac{3}{8}\right)^4+\dots$$

Add & sub $1+\frac{3}{1}\left(\frac{3}{8}\right)$ on R.H.S

$$\frac{9S}{8}=\left[1+\frac{3}{1}\left(\frac{3}{8}\right)+\frac{3\cdot 5}{1\cdot 2}\left(\frac{3}{8}\right)^2+\frac{3\cdot 5\cdot 7}{1\cdot 2\cdot 3}\left(\frac{3}{8}\right)^3+\dots\right]$$
$$-\left[1+\frac{3}{1}\left(\frac{3}{8}\right)\right]$$

$p=3;\ p+q=5$; $\frac{x}{q}=\frac{3}{8}$

$q=5-3$

$q=2$ $\qquad x=\frac{\cancel{3}\times 2}{\cancel{8}\,4}=\frac{3}{4}$

$$\frac{9S}{8}=\left(1-\frac{3}{4}\right)^{-3/2}-\left[1+\frac{9}{8}\right]$$

$$= \left(\frac{4-3}{4}\right)^{-3/2} - \frac{8+9}{8}$$

$$= \left(\frac{1}{4}\right)^{-3/2} - \frac{17}{8}$$

$$= 4^{3/2} - \frac{17}{8}$$

$$= 4\sqrt{4} - \frac{17}{8} = 4\times 2 - \frac{17}{8}$$

$$\frac{9S}{8} = 8 - \frac{17}{8}$$

$$= \left(8 - \frac{17}{8}\right)\times\frac{8}{9}$$

$$= \frac{64}{9} - \frac{17}{\cancel{8}}\times\frac{\cancel{8}}{9}$$

$$= \frac{64}{9} - \frac{17}{9}$$

$$\boxed{S = \frac{47}{9}}$$

24. sum the series $1 - \frac{1}{4} + \frac{1\cdot 3}{4\cdot 8} - \frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12} + \cdots$

Soln:

Denote the sum by S, then

$$S = 1 - \frac{1}{4} + \frac{1\cdot 3}{\underset{1\cdot 4\ \ 2\cdot 4}{4\cdot 8}} - \frac{1\cdot 3\cdot 5}{\underset{1\cdot 4\ \ 2\cdot 4\ \ 3\cdot 4}{4\cdot 8\cdot 12}} + \cdots$$

W.K.T

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2 + \cdots$$

$$S = 1 + \frac{1}{1}\left(\frac{-1}{4}\right) + \frac{1\cdot 3}{1\cdot 2}\left(\frac{-1}{4}\right)^2 + \frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\left(\frac{-1}{4}\right)^3 + \cdots$$

$p = 1$, $p+q = 3$

$1+q = 3$

$q = 3-1 = 2$

$\frac{x}{q} = \frac{-1}{4}$

$x = \frac{-2}{4} = \frac{-1}{2}$

$$S = (1-x)^{-p/q}$$

$$= (1+1/2)^{-1/2}$$

$$= \left(\frac{2+1}{2}\right)^{-1/2} = (3/2)^{-1/2}$$

$$\boxed{S = \left(\frac{2}{3}\right)^{1/2}}$$

25. Sum the series $\frac{1\cdot4}{5\cdot10} - \frac{1\cdot4\cdot7}{5\cdot10\cdot15} + \frac{1\cdot4\cdot7\cdot10}{5\cdot10\cdot15\cdot20} - \cdots$

Soln:

Denote the sum by S, then

$$S = \frac{1\cdot4}{\underset{1\cdot5\ 2\cdot5}{5\cdot10}} - \frac{1\cdot4\cdot7}{\underset{1\cdot5\ 2\cdot5\ 3\cdot5}{5\cdot10\cdot15}} + \frac{1\cdot4\cdot7\cdot10}{\underset{1\cdot5\ 2\cdot5\ 3\cdot5\ 4\cdot5}{5\cdot10\cdot15\cdot20}} - \cdots$$

w.k.T

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1\cdot2}\left(\frac{x}{q}\right)^2 + \cdots$$

$$S = \frac{1\cdot4}{1\cdot2}\left(\frac{-1}{5}\right)^2 + \frac{1\cdot4\cdot7}{1\cdot2\cdot3}\left(-\frac{1}{5}\right)^3 + \frac{1\cdot4\cdot7\cdot10}{1\cdot2\cdot3\cdot4}\left(-1/5\right)^4 + \cdots$$

Add & sub $1 + \frac{1}{1}(-1/5)$ in R.H.S

$$S = \left[1 + \frac{1}{1}(-1/5) + \frac{1\cdot4}{1\cdot2}\left(\frac{-1}{5}\right)^2 + \frac{1\cdot4\cdot7}{1\cdot2\cdot3}(-1/5)^3 + \cdots\right] - \left[1 + \frac{1}{1}(-1/5)\right]$$

$p = 1$, $p+q = 4$, $\frac{x}{q} = -1/5$

$q = 4-1$, $x = -3/5$

$q = 3$

$$S = (1-x)^{-p/q} - \left(1 - \frac{1}{5}\right)$$

$$= \left(1 + \frac{3}{5}\right)^{-1/3} - \frac{4}{5}$$

$$= \left(\frac{5+3}{5}\right)^{-1/3} - \frac{4}{5}$$

$$= \left(\frac{8}{5}\right)^{-1/3} - 4/5$$

$$\boxed{S = \left(\frac{5}{8}\right)^{1/3} - 4/5}$$

26. Sum the series

$$1-\frac{1}{2}(1/2)+\frac{1\cdot 3}{2\cdot 4}(1/2)^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(1/2)^3+\cdots$$

Soln:

Denote the sum by S, then

$$S = 1-\frac{1}{2}(1/2)+\frac{1\cdot 3}{2\cdot 4}(1/2)^2-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}(1/2)^3+\cdots$$

$$S = 1+\frac{1}{1}(-1/4)+\frac{1\cdot 3}{1\cdot 2}\left(\frac{-1}{4}\right)^2+\frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}(-1/4)^3+\cdots$$

$p=1$; $p+q=3$; $\frac{x}{q}=-1/4$

$q=3-1$; $x=\frac{-2}{4}=-1/2$

$q=2$

$$S=(1-x)^{-p/q}$$

$$=\left(1+\frac{1}{2}\right)^{-1/2}$$

$$=\left(\frac{2+1}{2}\right)^{-1/2}$$

$$=(3/2)^{-1/2}$$

$$S=\left(\frac{2}{3}\right)^{1/2}$$

EXPONENTIAL SERIES

Results:

For all values of x

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

2. $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

3. $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

4. $\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

5. $e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

6. $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$

7. $\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots$

8. $\frac{e - e^{-1}}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots$

Problems:

1. Sum the series $\frac{1+3x}{1!} + \frac{(1+3x)^2}{2!} + \frac{(1+3x)^3}{3!} + \cdots$

U.Q

Soln:

Let $y = 1 + 3x$

$$S = \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

Add and sub 1

$$S = \left[1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots\right] - 1$$

$$S = e^{y} - 1$$

$$\boxed{S = e^{1+3x} - 1}$$

2\. U.Q Sum the series $1 - \log_e 2 + \frac{(\log_e 2)^2}{2!} - \frac{(\log_e 2)^3}{3!} + \cdots$

Soln:

Let $x = \log_e 2$

$$S = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$= e^{-x}$$

$$= e^{-\log_e 2} \qquad e^{\log x} = x$$

$$= e^{\log_e 2^{-1}}$$

$$= 2^{-1}$$

$$S = \frac{1}{2}$$

3\. U.Q S.T $\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$

Soln:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$$

$$e - e^{-1} = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots\right)$$

$$= \cancel{1} + \frac{1}{1!} + \cancel{\frac{1}{2!}} + \frac{1}{3!} + \cdots - \cancel{1} + \frac{1}{1!} - \cancel{\frac{1}{2!}} + \frac{1}{3!} - \cdots$$

$$= 2\frac{1}{1!} + 2\frac{1}{3!} + 2\frac{1}{5!} + \cdots$$

$$= 2\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right)$$

$$\frac{e - e^{-1}}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

$$\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

4. S.T $\dfrac{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots} = \dfrac{e^2+1}{e^2-1}$

U.Q

Soln:-

L.H.S

$$\frac{1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots}{\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots} = \frac{\frac{e+e^{-1}}{2}}{\frac{e-e^{-1}}{2}}$$

$$= \frac{e+e^{-1}}{\cancel{2}} \times \frac{\cancel{2}}{e-e^{-1}}$$

$$= \frac{e+e^{-1}}{e-e^{-1}} = \frac{e+\frac{1}{e}}{e-\frac{1}{e}}$$

$$= \frac{\frac{e^2+1}{e}}{\frac{e^2-1}{e}} = \frac{e^2+1}{\cancel{e}} \times \frac{\cancel{e}}{e^2-1}$$

$$= \frac{e^2+1}{e^2-1} = R.H.S$$

L.H.S = R.H.S

5. S.T $\dfrac{\dfrac{1}{2!}+\dfrac{1}{4!}+\dfrac{1}{6!}+\cdots}{\dfrac{1}{1!}+\dfrac{1}{3!}+\dfrac{1}{5!}+\cdots}=\dfrac{e-1}{e+1}$

U.Q

Soln:

W.K.T

$$\frac{e+e^{-1}}{2}=1+\frac{1}{2!}+\frac{1}{4!}+\cdots$$

$$\frac{e-e^{-1}}{2}=\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots$$

L.H.S

$$=\frac{\left[\frac{e+e^{-1}}{2}\right]-1}{\left[\frac{e-e^{-1}}{2}\right]}$$

$(a-b)^2=a^2+b^2-2ab$

$$=\frac{\left[\frac{e+\frac{1}{e}}{2}\right]-1}{\frac{e-\frac{1}{e}}{2}}=\frac{\frac{e^2+1}{2e}-1}{\frac{e^2-1}{2e}}=\frac{e^2+1-2e}{\cancel{2e}}\times\frac{\cancel{2e}}{e^2-1}$$

$$=\frac{(e-1)^2}{e^2-1}$$

$a^2-b^2=(a+b)(a-b)$

$$=\frac{(e-1)^{\cancel{2}}}{(e+1)(e-1)}$$

$$=\frac{e-1}{e+1}$$

6. S.T $\dfrac{1+\dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{2^2}{4!}+\cdots}{1+\dfrac{1}{2!}+\dfrac{1}{4!}+\dfrac{1}{6!}+\cdots}=e/2.$

Soln:

L.H.S

$$= \frac{1+\frac{1}{2!}+\frac{2}{3!}+\frac{2^2}{4!}+\cdots}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}$$

Multiply and divided by 2^2 on the numerator

$$= \frac{\frac{1}{2^2}\left[\frac{2^2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\cdots\right]}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}$$

Write the first term $\frac{2^2}{1!}$ as $(1+\frac{2}{1!})+1$.

$$= \frac{\frac{1}{2^2}\left[1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\cdots\right]+1}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}$$

$$= \frac{\frac{1}{2^2}(e^2+1)}{\frac{1}{2}(e+e^{-1})}$$

$$= \frac{2}{2^2}\left(\frac{e^2+1}{e+1/e}\right) = \frac{1}{2}\left(\frac{e^2+1}{\frac{e^2+1}{e}}\right)$$

$$= \frac{\cancel{e^2+1}}{2} \times \frac{e}{\cancel{e^2+1}}$$

$$= \frac{e}{2} = R.H.S$$

L.H.S = R.H.S

H.W

7. S.T $1+\frac{x\log_e a}{1!}+\frac{(x\log_e a)^2}{2!}+\frac{(x\log_e a)^3}{3!}+\cdots=a^x$

Soln:

$$S=1+\frac{x\log_e a}{1!}+\frac{(x\log_e a)^2}{2!}+\frac{(x\log_e a)^3}{3!}+\cdots$$

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$S=e^{x\log_e a} \qquad e^{\log x}=x$$

$$=e^{\log_e a^x}$$

$$S=a^x$$

8. S.T $\log_e 2+\frac{(\log_e 2)^2}{2!}+\frac{(\log_e 2)^3}{3!}+\cdots=1$

Soln:

$$S=\log_e 2+\frac{(\log_e 2)^2}{2!}+\frac{(\log_e 2)^3}{3!}+\cdots$$

Add & sub 1 on R.H.S

$$S=\left[1+\frac{\log_e 2}{1!}+\frac{(\log_e 2)^2}{2!}+\frac{(\log_e 2)^3}{3!}+\cdots\right]-1$$

Let $x=\log_e 2$

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots$$

$$S=e^x-1$$

$$=e^{\log_e 2}-1$$

$$=2-1$$

$$\boxed{S=1}$$

9. S.T $\log_e 3 - \frac{(\log_e 3)^2}{2!} + \frac{(\log_e 3)^3}{3!} + \cdots = \frac{2}{3}$

Soln:

$S = \log_e 3 - \frac{(\log_e 3)^2}{2!} + \frac{(\log_e 3)^3}{3!} - \cdots$

Multiply by -1 on bothsides

$-S = -\log_e 3 + \frac{(\log_e 3)^2}{2!} - \frac{(\log_e 3)^3}{3!} + \cdots$

Add & sub by 1 on R.H.S

$-S = \left[1 - \log_e 3 + \frac{(\log_e 3)^2}{2!} - \frac{(\log_e 3)^3}{3!} + \cdots\right] - 1$

Let $x = \log_e 3$

$-S = \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right] - 1$

W.K.T

$= e^{-x} - 1$ $\quad e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

$= e^{-\log_e 3} - 1$

$= e^{\log_e 3^{-1}} - 1$

$= 3^{-1} - 1$

$= \frac{1}{3} - 1$

$= \frac{1-3}{3}$

$-S = -\frac{2}{3}$

$\boxed{S = \frac{2}{3}}$

10. S.T $1+\frac{(\log_e n)^2}{2!}+\frac{(\log_e n)^4}{4!}+\dots=\frac{n^2+1}{2n}$

Soln:

$$S=1+\frac{(\log_e n)^2}{2!}+\frac{(\log_e n)^4}{4!}+\dots$$

Let $x=\log_e n$, then

$$S=\frac{x^2}{2!}+\frac{x^4}{4!}+\dots$$

Here

$$\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\dots$$

$$S=\frac{e^x+e^{-x}}{2}$$

$$=\frac{e^{\log_e n}+\frac{1}{e^{\log_e n}}}{2}=\frac{n+\frac{1}{n}}{2}$$

$$=\frac{n^2+1}{2}$$

11. Sum the series $1+\frac{3^2}{2!}+\frac{3^4}{4!}+\frac{3^6}{6!}+\dots$

Soln:

$$S=1+\frac{3^2}{2!}+\frac{3^4}{4!}+\frac{3^6}{6!}+\dots$$

Put $x=3$

$$S=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots$$

W.k.T

$$\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\dots$$

$$S=\frac{e^x+e^{-x}}{2}$$

$$= \frac{e^x + \frac{1}{e^x}}{2} = \frac{e^3 + \frac{1}{e^3}}{2}$$

$$= \frac{e^6 + 1}{2e^3}$$

12. Find the Co-eff of x^n in e^{ax+b}

Soln:

$$e^{ax+b} = e^{ax} \cdot e^b \qquad e^{m+n} = e^m \cdot e^n$$

$$= e^b \left[1 + \frac{ax}{1!} + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \cdots + \frac{(ax)^n}{n!} + \cdots \right]$$

$$= e^b \left[1 + \frac{ax}{1!} + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \cdots + \frac{a^nx^n}{n!} + \cdots \right]$$

coeff of x^n $= e^b \left(\frac{a^n}{n!}\right)$

Co-efficient of x^n in exponential series

1. Find the Co-eff of x^n in $\frac{1+2x+3x^2}{e^x}$

Soln:

$$\frac{1+2x+3x^2}{e^x} = (1+2x+3x^2)\, e^{-x}$$

W.K.T

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$= (1+2x+3x^2)\left[1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \right]$$

$$= \left[1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots \frac{(-1)^n x^n}{n!} + \cdots \right]$$

$$+ 2x\left[1 - \frac{x}{1!} + x^2/2! - x^3/3! + \cdots \frac{(-1)^n x^n}{n!} + \cdots \right]$$

$$+3x^2\left[1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots\frac{(-1)^n x^n}{n!}\right]$$

$$=1-\frac{x}{1!}+\frac{x^2}{2!}+\cdots\frac{(-1)^n x^n}{n!}+\cdots+2x-\frac{2x^2}{1!}+\cdots\frac{(-1)^n 2x^{n+1}}{n!}+\cdots$$

$$+3x^2-\frac{3x^3}{1!}+\frac{3x^4}{2!}-\frac{3x^5}{3!}+\cdots+\frac{(-1)^n 3x^{n+2}}{n!}+\cdots$$

$$=1-\frac{x}{1!}+\frac{x^2}{2!}+\cdots\frac{(-1)^n x^n}{n!}+\cdots+2x-\frac{2x^2}{1!}+\cdots\frac{(-1)^{n-1}2x^n}{(n-1)!}+\frac{(-1)^n 2x^{n+1}}{n!}$$

$$+\cdots+3x^2-\frac{3x^3}{1!}+\frac{3x^4}{2!}+\cdots\frac{(-1)^{n-2}}{(n-2)!}3x^n+\frac{(-1)^{n-1}}{(n-1)!}3x^{n+1}+\cdots$$

$$\therefore \text{Coeff of } x^n=\frac{(-1)^n}{n!}+\frac{2(-1)^{n-1}}{(n-1)!}+\frac{3(-1)^{n-2}}{(n-2)!}$$

$$=(-1)^n\left[\frac{1}{n!}\right]+\frac{2(-1)^n(-1)^{-1}}{(n-1)!}+\frac{3(-1)^n(-1)^{-2}}{(n-2)!}$$

$$=(-1)^n\left[\frac{1}{n!}-\frac{2}{(n-1)!}+\frac{3}{(n-2)!}\right]$$

$$=(-1)^n\left[\frac{1}{n!}-\frac{2n}{n(n-1)!}+\frac{3n(n-1)}{n(n-1)(n-2)!}\right]$$

$$=\frac{(-1)^n}{n!}\left[1-2n+3n(n-1)\right]$$

$$=\frac{(-1)^n}{n!}\left[1-2n+3n^2-3n\right]$$

$$=\frac{(-1)^n}{n!}\left[3n^2-5n+1\right]$$

2 Find the co-eff of x^n in $\frac{2+5x}{e^{2x}}$.

Soln:

$$\frac{2+5x}{e^{2x}} = (2+5x)\, e^{-2x}$$

W.K.T

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

put $x = 2x$

$$e^{-2x} = 1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \dots + \frac{(-1)^n (2x)^n}{n!} + \dots$$

$$\frac{2+5x}{e^{2x}} = (2+5x)\left[1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} + \dots + \frac{(-1)^n (2x)^n}{n!} + \dots\right]$$

$$= \left[2 - \frac{2\cdot 2x}{1!} + \frac{2\cdot 2^2 x^2}{2!} + \dots + \frac{2\cdot (-1)^n 2^n x^n}{n!} + \dots\right]$$

$$+ \left[5x - \frac{5\cdot 2x^2}{1!} + \frac{5\cdot 2^2 x^3}{3!} + \dots + \frac{5\cdot (-1)^n 2^n x^{n+1}}{n!} + \dots\right]$$

$$= \left[2 - \frac{2\cdot 2x}{1!} + \frac{2\cdot 2^2 x^2}{2!} + \dots + \frac{2^{n+1}(-1)^n x^n}{n!} + \dots\right]$$

$$+ \left[5x - \frac{5\cdot 2x^2}{1!} + \dots + \frac{5(-1)^{n-1} 2^{n-1} x^n}{(n-1)!} + \frac{5(-1)^n 2^n x^{n+1}}{n!} + \dots\right]$$

Co-eff of x^n $= \frac{2^{n+1}(-1)^n}{n!} + \frac{5(-1)^{n-1} 2^{n-1}}{(n-1)!}$

$$= \frac{2^n \cdot 2(-1)^n}{n!} + \frac{5(-1)^n (-1)^{-1} 2^n \cdot 2^{-1}}{(n-1)!}$$

$$= 2^n (-1)^n \left[\frac{2}{n!} - \frac{5}{2(n-1)!}\right]$$

$$= \frac{2^n(-1)^n}{\lfloor n}\left[\frac{2}{n!} - \frac{5n}{2n(n-1)!}\right]$$

$$= \frac{2^n(-1)^n}{n!}\left[2 - \frac{5n}{2}\right]$$

$$= \frac{2^n(-1)^n}{n!}\left[\frac{4-5n}{2}\right]$$

$$= \frac{2^{n-1}(-1)^n(4-5n)}{n!}$$

3. Find the co-eff of x^n in the series

$$1 + \frac{3-2x}{1!} + \frac{(3-2x)^2}{2!} + \frac{(3-2x)^3}{3!} + \cdots$$

Soln:

$$1 + \frac{3-2x}{1!} + \frac{(3-2x)^2}{2!} + \frac{(3-2x)^3}{3!} + \cdots$$

W.K.T

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Put $x = 3-2x$

$$e^{3-2x} = e^3 \cdot e^{-2x}$$

$$= e^3\left[1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots + (-1)^n\frac{(2x)^n}{n!} + \cdots\right]$$

$$\text{Co-eff of } x^n = \frac{e^3(-1)^n 2^n}{n!}$$

H.W

4. Find the co-eff of x^n in e^{2x-3}

Soln:

$$e^{2x-3} = e^{2x} \cdot e^{-3}$$

$$= e^{-3}\left[1 + \frac{(2x)}{1!} + \frac{(2x)^2}{2!} + \cdots \frac{(2x)^n}{n!} + \cdots\right]$$

$$\text{Co-eff of } x^n = e^{-3}\frac{2^n}{n!}$$

5. Find the co-eff of x^n in e^{e^x}

Soln.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

$$e^{e^x} = 1 + \frac{e^x}{1!} + \frac{(e^x)^2}{2!} + \frac{(e^x)^3}{3!} + \cdots$$

$$e^{e^x} = 1 + \frac{e^x}{1!} + \frac{e^{2x}}{2!} + \frac{e^{3x}}{3!} + \cdots$$

$$= 1 + \frac{1}{1!}\left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right] + \frac{1}{2!}\left[1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots \frac{(2x)^n}{n!} + \cdots\right] + \cdots$$

$$\text{Co-eff of } x^n = \frac{1}{1!}\frac{1}{n!} + \frac{1}{2!}\frac{2^n}{n!} + \frac{1}{3!}\frac{3^n}{n!} + \cdots$$

$$= \frac{1}{n!}\left[\frac{1^n}{1!} + \frac{2^n}{2!} + \frac{3^n}{3!} + \cdots\right]$$

6. Find the co-eff of x^n in the expansion of $\frac{1+x+x^2}{e^x}$ in ascending powers of x.

Soln:

$$\frac{(1+x+x^2)}{e^x} = (1+x+x^2)e^{-x}$$

$$(1+x+x^2)\,e^{-x} = (1+x+x^2)\left(1-\frac{x}{1!}+\frac{x^2}{2!}+\cdots\frac{(-1)^n x^n}{n!}+\cdots\right)$$

$$= \left(1-\frac{x}{1!}+\frac{x^2}{2!}+\cdots\frac{(-1)^n x^n}{n!}+\cdots\right)+x\left(1-\frac{x}{1!}+\frac{x^2}{2!}+\cdots\frac{(-1)^n x^n}{n!}+\cdots\right)$$

$$+x^2\left[1-\frac{x}{1!}+\frac{x^2}{2!}+\cdots\frac{(-1)^n x^n}{n!}+\cdots\right]$$

$$= 1-\frac{x}{1!}+\frac{x^2}{2!}+\cdots\frac{(-1)^n x^n}{n!}+\cdots+\left(x-\frac{x^2}{1!}+\frac{x^3}{2!}+\cdots\frac{(-1)^n x^{n+1}}{n!}+\cdots\right)$$

$$+\left(x^2-\frac{x^3}{1!}+\frac{x^4}{2!}+\cdots+\frac{(-1)^n x^{n+2}}{n!}+\cdots\right)$$

$$= 1-\frac{x}{1!}+\frac{x^2}{2!}+\cdots\frac{(-1)^n x^n}{n!}+\cdots+x-\frac{x^2}{1!}+\frac{x^3}{2!}+\cdots+\frac{(-1)^{n-1}}{(n-1)}x^n+\frac{(-1)^n x^{n+1}}{n!}+\cdots$$

$$+x^2-\frac{x^3}{1!}+\cdots+\frac{(-1)^{n-2}x^n}{(n-2)!}+\frac{(-1)^{n-1}x^{n+1}}{(n-1)!}+\frac{(-1)^n x^{n+2}}{n!}+\cdots$$

Co-eff of $x^n = \dfrac{(-1)^n}{n!}+\dfrac{(-1)^{n-1}}{(n-1)!}+\dfrac{(-1)^{n-2}}{(n-2)!}$

$$= \frac{(-1)^n}{n!}+\frac{(-1)^n(-1)^{-1}}{(n-1)!}+\frac{(-1)^n(-1)^{-2}}{(n-2)!}$$

$$= (-1)^n\left[\frac{1}{n!}-\frac{n}{n(n-1)!}+\frac{n(n-1)}{n(n-1)(n-2)!}\right]$$

$$= \frac{(-1)^n}{n!}\left[1-n+n(n-1)\right]$$

$$= \frac{(-1)^n}{n!}\left[1-n+n^2-n\right]$$

$$= \frac{(-1)^n}{n!}\left[n^2-2n+1\right] = \frac{(-1)^n}{n!}(n-1)^2$$

7. Find the co-eff of x^3 in $\dfrac{1+x-x^2}{e^{2x}}$

Soln:

$$\frac{1+x-x^2}{e^{2x}} = (1+x-x^2)\, e^{-2x}$$

$$e^{-2x} = 1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots$$

$$(1+x-x^2)\, e^{-2x} = (1+x-x^2)\left(1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots + \frac{(-1)^n (2x)^n}{n!} + \cdots\right]$$

$$= \left[1 - \frac{2x}{1!} + \frac{2^2x^2}{2!} + \cdots \frac{(-1)^n 2^n x^n}{n!} + \cdots\right] + x\left[1 - \frac{2x}{1!} + \frac{(2x)^2}{2!} + \cdots \frac{(-1)^n (2x)^n}{n!} + \cdots\right]$$

$$-x^2\left[1 - \frac{2x}{1!} + \frac{2^2x^2}{2!} + \cdots \frac{(-1)^n 2^n x^n}{n!} + \cdots\right]$$

$$= \left[1 - \frac{2x}{1!} + \frac{2^2x^2}{2!} - \frac{2^3x^3}{3!} + \cdots\right] + \left[x - \frac{2x^2}{1!} + \frac{2^2x^3}{2!} + \cdots\right]$$

$$- x^2 + \frac{2x^3}{1!} - \frac{2^2x^4}{2!} + \cdots$$

$$\text{co-eff of } x^3 = -\frac{2^3}{3!} + \frac{2^2}{2!} + \frac{2}{1}$$

$$= -\frac{\cancel{8}^{4}}{\cancel{6}_{3}} + \frac{\cancel{4}^{2}}{\cancel{2}} + \frac{2}{1}$$

$$= -\frac{4}{3} + 2 + 2$$

$$= -\frac{4}{3} + 4$$

$$= \frac{-4+12}{3}$$

$$= 8/3$$

SUMMATION: Exponential series

1. Sum the series $\frac{2^2}{1!} + \frac{3^2}{2!} + \frac{4^2}{3!} + \frac{5^2}{4!} + \cdots$

Soln:

Let t_n be the n^{th} term

$\therefore t_n = \frac{(n+1)^2}{n!}$

Degree of numerator is 2.

Let $(n+1)^2 = A + Bn + Cn(n-1)$

$n^2+2n+1 = A + Bn + Cn(n-1)$ — 1

Equate the co-eff of n^2

$\boxed{1 = C}$

Equate the co-eff of n

$2 = B - C$

$2 = B - 1$

$\boxed{B = 3}$

Equate the constant term

$\boxed{1 = A}$

Sub A, B, C value in 1

$(n+1)^2 = 1 + 3n + n(n-1)$

$t_n = \frac{(n+1)^2}{n!}$

$= \frac{1+3n+n(n-1)}{n!}$

$= \frac{1}{n!} + \frac{3n}{n!} + \frac{n(n-1)}{n!}$

$$= \frac{1}{n!} + \frac{3n}{n(n-1)!} + \frac{n(n-1)}{n(n-1)(n-2)!}$$

$$t_n = \frac{1}{n!} + \frac{3}{(n-1)!} + \frac{1}{(n-2)!}$$

Put $n = 1, 2, 3 \ldots$

$$t_1 = \frac{1}{1!} + \frac{3}{0!}$$

$$t_2 = \frac{1}{2!} + \frac{3}{1!} + \frac{1}{0!}$$

$$t_3 = \frac{1}{3!} + \frac{3}{2!} + \frac{1}{1!}$$

$$t_4 = \frac{1}{4!} + \frac{3}{3!} + \frac{1}{2!}$$

$- - - -$

Adding vertically

$$S = \left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right] + 3\left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right] + \left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots\right]$$

$$= (e^1 - 1) + 3e^1 + e^1$$

$$= e - 1 + 3e + e$$

$$\boxed{S = 5e - 1}$$

2) Sum the series $\sum_{n=0}^{\infty} \frac{5n+1}{(2n+1)!}$

soln:

Let $t_n = \frac{5n+1}{(2n+1)!}$, $n = 0, 1, 2, 3 \ldots$

Degree of numerator is 1

$$5n+1 = A + B(2n+1) \quad \text{--- (1)}$$

Equate the co-eff of n

$$5 = 2B$$

$$\boxed{B = 5/2}$$

Equate the constant term

$$1 = A + B$$

$$1 = A + 5/2$$

$$A = 1 - 5/2$$

$$A = 2 - 5/2 = -3/2$$

$$\boxed{A = \frac{-3}{2}}$$

Sub A, B value in ①

$$5n+1 = \frac{-3}{2} + \frac{5}{2}(2n+1)$$

$$\frac{5n+1}{(2n+1)!} = \frac{-\frac{3}{2} + \frac{5}{2}(2n+1)}{(2n+1)!}$$

$$= \frac{\frac{-3}{2}}{(2n+1)!} + \frac{5/2\,(2n+1)}{(2n+1)!}$$

$$= \frac{\frac{-3}{2}}{(2n+1)!} + \frac{5/2\,(2n+1)}{(2n+1)\,2n!}$$

$$t_n = \frac{-3/2}{(2n+1)!} + \frac{5/2}{2n!}$$

Put n = 0, 1, 2, 3 ...

$$t_0 = \frac{-3/2}{1!} + \frac{5/2}{0!}$$

$$t_1 = \frac{-3/2}{3!} + \frac{5/2}{2!}$$

$$t_2 = \frac{-3/2}{5!} + \frac{5/2}{4!}$$

$$t_3 = \frac{-3/2}{7!} + \frac{5/2}{6!}$$

- - - - -

Adding vertically

$$S = \left[\frac{-\frac{3}{2}}{1!} + \frac{-3/2}{3!} + \frac{-3/2}{5!} + \cdots\right] + \left[\frac{5/2}{0!} + \frac{5/2}{2!} + \frac{5/2}{4!} + \cdots\right]$$

$$= -3/2\left[\frac{1}{1!} + 1/3! + 1/5! + \cdots\right] + 5/2\left[\frac{1}{0!} + 1/2! + \frac{1}{4!} + \cdots\right]$$

$$= -\frac{3}{2}\left[\frac{e-e^{-1}}{2}\right] + \frac{5}{2}\left[\frac{e+e^{-1}}{2}\right]$$

$$= -\frac{3}{2}\left[\frac{e-1/e}{2}\right] + \frac{5}{2}\left[\frac{e+1/e}{2}\right] = -\frac{3}{2}\left[\frac{e^2-1}{2e}\right] + \frac{5}{2}\left[\frac{e^2+1}{2e}\right]$$

$$= \frac{1}{4e}\left[-3e^2+3+5e^2+5\right]$$

$$= \frac{1}{4e}\left[2e^2+8\right] = \frac{2(e^2+4)}{4e} = \frac{e^2+4}{2e}$$

$$S = \frac{e^2}{2e} + \frac{4}{2e}$$

$$\boxed{S = \frac{e}{2} + \frac{2}{e}}$$

3. P.T $\frac{2}{1!} + \frac{3}{2!} + \frac{4}{3!} + \cdots = 2e-1$.

HW Soln:

$$t_n = \frac{n+1}{n!} \quad -① $$

Degree of numerator is 1

$$n+1 = A+Bn$$

put $n=0$

$$\boxed{1 = A}$$

Equate the co-eff of n

$$\boxed{1 = B}$$

$$t_n = \frac{n+1}{n!} = \frac{A+Bn}{n!}$$

$= \frac{1+n}{n!}$

$= \frac{1}{n!} + \frac{\cancel{n}(1)}{\cancel{n}(n-1)!}$

$t_n = \frac{1}{n!} + \frac{1}{(n-1)!}$

Put $n = 1, 2, 3 \ldots$

$t_1 = \frac{1}{1!} + \frac{1}{0!}$

$t_2 = \frac{1}{2!} + \frac{1}{1!}$

$t_3 = \frac{1}{3!} + \frac{1}{2!}$

Adding vertically,

$S = \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\right) + \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots\right)$

$= (e^1 - 1) + e^1 = e^1 + e^1 - 1 = 2e^1 - 1$

$\boxed{S = 2e - 1}$

4. Sum the series $\frac{1^2}{3!} + \frac{2^2}{5!} + \frac{3^2}{7!} + \ldots$

Soln:

Let t_n be the n^{th} term

$t_n = \frac{n^2}{(2n+1)!}$

Let $n^2 = A + B(2n+1) + C\,2n(2n+1)$

$4n^2 + 2n$

Equate the coeff of n^2

$1 = 4C$

$\boxed{C = \frac{1}{4}}$

Equate the coeff of n

$0 = 2B + 2C$

$0 = 2B + \cancel{2}\left(\frac{1}{\cancel{4}_2}\right)$

$2B + \frac{1}{2} = 0$

$2B = -\frac{1}{2}$

$\boxed{B = -\frac{1}{4}}$

Equate the co-eff of constant

$0 = A + B$

$0 = A - \frac{1}{4}$

$\boxed{A = \frac{1}{4}}$

$t_n = \frac{n^2}{(2n+1)!}$

$= \frac{\frac{1}{4} - \frac{1}{4}(2n+1) + \frac{1}{4} 2n(2n+1)}{(2n+1)!}$

$= \frac{\frac{1}{4}}{(2n+1)!} - \frac{\frac{1}{4}\cancel{(2n+1)}}{\cancel{(2n+1)}2n!} + \frac{\frac{1}{4}\cancel{2n}\cancel{(2n+1)}}{\cancel{(2n+1)}\cancel{2n}(2n-1)!}$

$t_n = \frac{\frac{1}{4}}{(2n+1)!} - \frac{\frac{1}{4}}{2n!} + \frac{\frac{1}{4}}{(2n-1)!}$

Put $n = 1, 2, 3 \ldots$

$t_1 = \frac{\frac{1}{4}}{3!} - \frac{\frac{1}{4}}{2!} + \frac{\frac{1}{4}}{1!}$

$t_2 = \frac{\frac{1}{4}}{5!} - \frac{\frac{1}{4}}{4!} + \frac{\frac{1}{4}}{3!}$

$t_3 = \frac{\frac{1}{4}}{7!} - \frac{\frac{1}{4}}{6!} + \frac{\frac{1}{4}}{5!}$

$\ldots$

Adding vertically

$$S = \frac{1}{4}\left[\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots\right] - \frac{1}{4}\left[\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\right]$$

$$+\frac{1}{4}\left[\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots\right]$$

$$= \frac{1}{4}\left[\frac{e-e^{-1}}{2} - \frac{1}{1!}\right] - \frac{1}{4}\left[\frac{e+e^{-1}}{2}-1\right] + \frac{1}{4}\left[\frac{e-e^{-1}}{2}\right]$$

$$= \frac{1}{4}\left\{\frac{e-\frac{1}{e}}{2} - 1 - \left[\left(\frac{e+1/e}{2}\right)-1\right] + \frac{e-1/e}{2}\right\}$$

$$= \frac{1}{4}\left\{\frac{e^2-1}{2e} - \cancel{1} - \left[\left(\frac{e^2+1}{2e}\right) - \cancel{1}\right] + \frac{e^2-1}{2e}\right\}$$

$$= \frac{1}{8}\left\{\frac{e^2-1}{e} - \cancel{2} - \frac{e^2-1}{\cancel{2}e} + \cancel{2} + \frac{e^2-1}{e}\right\}$$

$$= \frac{1}{8e}\left(\cancel{e^2}-1-\cancel{e^2}-1+e^2-1\right)$$

$$= \frac{1}{8e}\left(e^2-3\right)$$

$$= \frac{e^{\cancel{2}}}{8\cancel{e}} - \frac{3}{8e}$$

$$\boxed{S = \frac{e-3e^{-1}}{8}}$$

5. Sum the series $\frac{1}{1!}+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\cdots\infty$

Soln:

Let $t_n = \frac{2n-1}{n!}$

Numerator degree is 1

$$2n-1 = A+Bn$$

Equate the co-eff of n

$$\boxed{2 = B}$$

Equate the constant term

$$\boxed{-1 = A}$$

HW

$$t_n = \frac{2n-1}{n!} = \frac{A+Bn}{n!}$$

$$= \frac{-1+2n}{n!}$$

$$= \frac{-1}{n!} + \frac{2n}{n(n-1)!}$$

$$t_n = \frac{-1}{n!} + \frac{2}{(n-1)!}$$

put $n = 1, 2, 3 \ldots$

$$t_1 = \frac{-1}{1!} + \frac{2}{0!}$$

$$t_2 = \frac{-1}{2!} + \frac{2}{1!}$$

$$t_3 = \frac{-1}{3!} + \frac{2}{2!}$$

$\ldots\ldots$

Adding vertically

$$S = \left[-\frac{1}{1!} - \frac{1}{2!} - \frac{1}{3!} - \cdots\right] + \left[\frac{2}{0!} + \frac{2}{1!} + \frac{2}{2!} + \cdots\right]$$

$$= -1\left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right] + 2\left[1 + \frac{1}{1!} + \frac{1}{3!} + \cdots\right]$$

$$= -1(e-1) + 2e$$

$$= -e + 1 + 2e$$

$$\boxed{S = e+1}$$

6. S.T $\frac{1}{1!}+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\cdots = \frac{3e}{2}$

Soln:

Let t_n be the n^{th} term

$$t_n = \frac{1+2+3+\cdots+n}{n!}, \quad n=1,2,3\ldots$$

$$t_n = \frac{n(n+1)}{2}\cdot\frac{1}{n!}$$

$$= \frac{\cancel{n}(n+1)}{2\cancel{n}(n-1)!} = \frac{n+1}{2(n-1)!}$$

Numerator degree is one

$$n+1 = A + B\,\underbrace{2(n-1)}_{2Bn-2B}$$

Equate the co-eff of n 4

$$1 = 2B$$

$$\boxed{B=\frac{1}{2}}$$

Equate the co-eff of constant

$$1 = A - 2B$$

$$1 = A - \cancel{2}(\tfrac{1}{\cancel{2}})$$

$$1 = A - 1$$

$$\boxed{A=2}$$

$$t_n = \frac{n+1}{2(n-1)!} = \frac{2+\frac{1}{\cancel{2}}\cancel{2}(n-1)}{2(n-1)!}$$

$$= \frac{2+n-1}{2(n-1)!} = \frac{\cancel{2}}{\cancel{2}(n-1)!} + \frac{n-1}{2(n-1)!}$$

$$= \frac{1}{(n-1)!} + \frac{\cancel{n-1}}{2(\cancel{n-1})(n-2)!}$$

$$t_n = \frac{1}{(n-1)!} + \frac{1}{2(n-2)!}$$

put $n = 1, 2, 3, \ldots$

$$t_1 = \frac{1}{0!}$$

$$t_2 = \frac{1}{1!} + \frac{1}{0!}\left(\tfrac{1}{2}\right)$$

$$t_3 = \frac{1}{2!} + \frac{1}{1!}\left(\tfrac{1}{2}\right)$$

$$t_4 = \frac{1}{3!} + \frac{1}{2!}\cdot\tfrac{1}{2}$$

. . . .

Adding vertically

$$S = \left[\frac{1}{0!} + \frac{1}{1!} + \tfrac{1}{2!} + \tfrac{1}{3!} + \cdots\right] + \tfrac{1}{2}\left[\tfrac{1}{0!} + \tfrac{1}{1!} + \tfrac{1}{2!} + \cdots\right]$$

$$= e + \tfrac{1}{2}e$$

$$= e\left(1 + \tfrac{1}{2}\right)$$

$$= e\left(\frac{2+1}{2}\right) = \frac{3e}{2}$$

$$\boxed{S = \frac{3e}{2}}$$

7. S.T $1 + \frac{1+3}{2!} + \frac{1+3+3^2}{3!} + \frac{1+3+3^2+3^3}{4!} + \cdots = \frac{e(e^2-1)}{2}$

Soln:

Let t_n be the n^{th} term

$$t_n = \frac{1+3+3^2+\cdots+3^{n-1}}{n!}$$

$$\boxed{1+x+x^2+\cdots+x^{n-1} = \frac{x^n-1}{x-1}}$$

$$\frac{3^n-1}{3-1} = \frac{3^n-1}{2}$$

$$\therefore t_n = \frac{3^n - 1}{(3-1)n!} = \frac{3^n - 1}{2n!}$$

$$t_n = \frac{3^n}{2n!} - \frac{1}{2n!}$$

$$t_n = \frac{1}{2}\frac{3^n}{n!} - \frac{1}{2}\cdot\frac{1}{n!}$$

put $n = 1, 2, 3 \ldots$

$$t_1 = \frac{1}{2}\cdot\frac{3}{1!} - \frac{1}{2}\cdot\frac{1}{1!}$$

$$t_2 = \frac{1}{2}\frac{3^2}{2!} - \frac{1}{2}\frac{1}{2!}$$

$$t_3 = \frac{1}{2}\frac{3^3}{3!} - \frac{1}{2}\frac{1}{3!}$$

$\ldots$

Adding vertically

$$S = \frac{1}{2}\left[\frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \cdots\right] - \frac{1}{2}\left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right]$$

$$= \frac{1}{2}[e^3 - 1] - \frac{1}{2}[e - 1]$$

$$= \frac{1}{2}[e^3 \cancel{-1} - e + 1]$$

$$= \frac{1}{2}(e^3 - e)$$

$$S = \frac{e(e^2 - 1)}{2}$$

Hw 8. Sum the series $\frac{1}{1!} + \frac{1+5}{2!} + \frac{1+5+5^2}{3!} + \cdots$

Soln.

Let t_n be the n^{th} term

$$t_n = \frac{1 + 5 + 5^2 + \cdots + 5^{n-1}}{n!} = \frac{5^n - 1}{(5-1)n!}$$

$$t_n = \frac{5^n - 1}{4n!}$$

$$t_n = \frac{5^n}{4n!} - \frac{1}{4n!}$$

$$t_n = \frac{1}{4}\frac{5^n}{n!} - \frac{1}{4}\frac{1}{n!}$$

put $n = 1, 2, 3 \ldots$

$$t_1 = \frac{1}{4}\frac{5}{1!} - \frac{1}{4}\frac{1}{1!}$$

$$t_2 = \frac{1}{4}\frac{5^2}{2!} - \frac{1}{4}\frac{1}{2!}$$

$$t_3 = \frac{1}{4}\frac{5^3}{3!} - \frac{1}{4}\frac{1}{3!}$$

$$- \; - \; - \; -$$

Adding vertically

$$S = \frac{1}{4}\left[\frac{5}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \cdots\right] - \frac{1}{4}\left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right]$$

$$= \frac{1}{4}\left[e^5 - 1\right] - \frac{1}{4}\left[e - 1\right]$$

$$= \frac{1}{4}\left[e^5 \cancel{-1} - e \cancel{+1}\right]$$

$$\boxed{S = \frac{e(e^4 - 1)}{4}}$$

9. Sum the series $\frac{1 \cdot 2}{1!} + \frac{2 \cdot 3}{2!} + \frac{3 \cdot 4}{3!} + \cdots$

Soln:

Let t_n be the n^{th} term

$$t_n = \frac{n(n+1)}{n!} = \frac{n^2 + n}{n!}$$

Degree of numerator is 2

$n^2+n = A + Bn + C\,\underbrace{n(n-1)}_{n^2-n}$

Equate the co-eff of n^2

$\boxed{1 = C}$

Equate the co-eff of n

$1 = B - C$

$1 = B - 1$

$\boxed{B = 2}$

Equate the co-eff of constant term

$\boxed{0 = A}$

$n^2+n = 0 + 2n + n(n-1)$

$$\frac{n^2+n}{n!} = \frac{2n + n(n-1)}{n!}$$

$$= \frac{2\cancel{n}}{\cancel{n}(n-1)!} + \frac{\cancel{n}\cancel{(n-1)}}{\cancel{n}\cancel{(n-1)}(n-2)!}$$

$$t_n = \frac{2}{(n-1)!} + \frac{1}{(n-2)!}$$

$$t_n = 2\frac{1}{(n-1)!} + \frac{1}{(n-2)!}$$

Put $n = 1, 2, 3 \ldots$

$t_1 = 2\frac{1}{0!}$

$t_2 = 2\frac{1}{1!} + \frac{1}{0!}$

$t_3 = 2\cdot\frac{1}{2!} + \frac{1}{1!}$

$t_4 = 2\cdot\frac{1}{3!} + \frac{1}{2!}$

$\ldots$

Adding vertically

$$S = 2\left[\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots\right]+\left[\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\cdots\right]$$

$$S = 2e+e$$

$$\boxed{S = 3e}$$

10. Sum the Series $\frac{5}{1!}+\frac{7}{3!}+\frac{9}{5!}+\cdots$

Soln:

Let t_n be the n^{th} term

$$t_n = \frac{2n+3}{(2n-1)!}$$

$$2n+3 = A+B(2n-1)$$

Equate the co-eff of n.

$$2 = 2B$$

$$\boxed{B=1}$$

Equate the co-eff of constants

$$3 = A-B$$

$$3 = A-1$$

$$\boxed{A=4}$$

$$2n+3 = 4+(2n-1)$$

$$\frac{2n+3}{(2n-1)!} = \frac{4+2n-1}{(2n-1)!}$$

$$= \frac{4}{(2n-1)!}+\frac{\cancel{2n-1}}{\cancel{(2n-1)}(2n-2)!}$$

$$t_n = \frac{4}{(2n-1)!}+\frac{1}{(2n-2)!}$$

Put $n = 1, 2, 3 \ldots$

$t_1 = 4 \cdot \frac{1}{1!} + \frac{1}{0!}$

$t_2 = 4 \cdot \frac{1}{3!} + \frac{1}{2!}$

$t_3 = 4 \cdot \frac{1}{5!} + \frac{1}{4!}$

$\ldots$

Adding vertically

$S = 4\left[\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \ldots\right] + \left[1 + \frac{1}{2!} + \frac{1}{4!} + \ldots\right]$

$= 4\left[\frac{e - e^{-1}}{2}\right] + \left[\frac{e + e^{-1}}{2}\right]$

$= 4\left[\frac{e - 1/e}{2}\right] + \left[\frac{e + 1/e}{2}\right]$

$= 4\left[\frac{e^2 - 1}{2e}\right] + \left[\frac{e^2 + 1}{2e}\right]$

$= \frac{4e^2 - 4 + e^2 + 1}{2e} = \frac{5e^2 - 3}{2e}$

$\boxed{S = \frac{5e}{2} - \frac{3}{2e}}$

Put $n = 1, 2, 3 \ldots$

$t_1 = 4 \frac{1}{1!} + \frac{1}{0!}$

$t_2 = 4 \cdot \frac{1}{3!} + \frac{1}{2!}$

$t_3 = 4 \cdot \frac{1}{5!} + \frac{1}{4!}$

$\cdots$

Adding vertically

$$S = 4\left[\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right] + \left[1 + \frac{1}{2!} + \frac{1}{4!} + \cdots\right]$$

$$= 4\left[\frac{e - e^{-1}}{2}\right] + \left[\frac{e + e^{-1}}{2}\right]$$

$$= 4\left[\frac{e - 1/e}{2}\right] + \left[\frac{e + 1/e}{2}\right]$$

$$= 4\left[\frac{e^2 - 1}{2e}\right] + \left[\frac{e^2 + 1}{2e}\right]$$

$$= \frac{4e^2 - 4 + e^2 + 1}{2e} = \frac{5e^2 - 3}{2e}$$

$$\boxed{S = \frac{5e}{2} - \frac{3}{2e}}$$

Logarithmic Series

Def:

The infinite series

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots$$ is called

the logarithmic series. If the value of x is such that $-1<x<1$.

The sum is

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

Formulae

1. $\log(1-x)=-\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots\right]$
2. $-\log(1-x)=x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots$
3. $\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$
4. $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)=x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots$

Problems

1. If $|x|>1$ then find $\boxed{\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\cdots}$

Soln:

W.K.T

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)=x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots$$

$$\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\cdots=\frac{1/x}{1}+\frac{1/x^3}{3}+\frac{1/x^5}{5}+\cdots$$

$$=\frac{1}{2}\log\left(\frac{1+1/x}{1-1/x}\right)$$

$$=\frac{1}{2}\log\left(\frac{\frac{x+1}{x}}{\frac{x-1}{x}}\right)$$

$= \frac{1}{2}\log\left(\frac{1+x}{\cancel{x}} \times \frac{\cancel{x}}{1-x}\right)$

$= \frac{1}{2}\log\left(\frac{1+x}{x-\cancel{1}}\right)$

2 S.T $\log 10 = 3\log 2 + \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4^2} + \frac{1}{3}\cdot\frac{1}{4^3} + \cdots \infty$

Soln:

Let $x = \frac{1}{4}$

$R.H.S = 3\log 2 + x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$= 3\log 2 + \left[x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right]$

$= 3\log 2 + \log(1+x)$

$= 3\log 2 + \log\left(1+\frac{1}{4}\right)$

$= 3\log 2 + \log\left(\frac{4+1}{4}\right)$

$= 3\log 2 + \log\left(\frac{5}{4}\right)$ $\quad \log a + \log b = \log(ab)$

$= \log 2^3 + \log\left(\frac{5}{4}\right)$

$= \log\left(2^3 \times \frac{5}{4}\right)$

$= \log\left(\overset{2}{\cancel{8}} \times \frac{5}{\cancel{4}}\right)$

$= \log 10 = L.H.S$

$L.H.S = R.H.S$

3. If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$, S.T

$x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots$

Soln.

Given $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$y = \log(1+x)$

Taking exponential on bothsides

$e^y = e^{\log(1+x)}$

$e^y = 1+x$

$1+x = e^y$

$\cancel{1} + x = \cancel{1} + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

$x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$

~~4. Sum the series~~ $\cancel{\log_{10} e - \log_{10} 2e + \log_{10} 3e \cdots}$

4. Sum the series

$\log_{10} e - \log_{10^2} e + \log_{10^3} e - \cdots$

Soln.

W.K.T

$$\log_{10^n} e = \left[\frac{\log_{10} e}{\log_{10} 10^n}\right]$$

$(\log_{10} 10 = 1)$

$\log_{10} e - \log_{10^2} e + \log_{10^3} e - \cdots$

$= \log_{10} e - \left(\frac{\log_{10} e}{\log_{10} 10^2}\right) + \left(\frac{\log_{10} e}{\log_{10} 10^3}\right) + \cdots$

$= \log_{10} e - \left(\frac{\log_{10} e}{2\log_{10} 10}\right) + \left(\frac{\log_{10} e}{3\log_{10} 10}\right)$

$= \log_{10} e - \frac{\log_{10} e}{2} + \frac{\log_{10} e}{3} - \cdots$

$= \log_{10} e \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right]$

$= \log_{10} e \cdot \log_e 2$ $\qquad [\log_{10} x = \log_{10} e \cdot \log_e x]$

$= \log_{10} 2$

5. S.T $\frac{a-x}{a} + \frac{1}{2}\left(\frac{a-x}{a}\right)^2 + \frac{1}{3}\left(\frac{a-x}{a}\right)^3 + \cdots = \log a - \log x$

Soln:-

Let $y = \frac{a-x}{a}$

L.H.S $= y + \frac{y^2}{2} + \frac{y^3}{3} + \cdots$

$= -\log(1-y)$

$= -\log\left(1 - \frac{a-x}{a}\right) = -\log\left(\frac{\cancel{a} - \cancel{a} + x}{a}\right)$

$= -\log\left(\frac{x}{a}\right)$

$= -\{\log x - \log a\} = -\log x + \log a$

$= \log a - \log x = R.H.S$

L.H.S = R.H.S

6. S.T $\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \cdots = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots$

Soln:

Let $x = \frac{1}{n+1}$

L.H.S $= x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$= -\log(1-x)$

$= -\log\left(1 - \frac{1}{n+1}\right)$

$= -\log\left(\frac{n+1-1}{n+1}\right) = -\log\left(\frac{n}{n+1}\right)$

$$= \log\left(\frac{n}{n+1}\right)^{-1}$$

$$= \log\left(\frac{n+1}{n}\right)$$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots$

$$= \log\left(\frac{n}{n} + \frac{1}{n}\right) = \log\left(1 + \frac{1}{n}\right)$$

$$= \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} + \dots$$

L.H.S = R.H.S

7. S.T $\log_3 e - \log_9 e + \log_{27} e - \dots = \frac{\log_e 2}{\log_e 3}$

Soln:

W.K.T $\log_a b = \frac{1}{\log_b a}$

$$\log_3 e = \frac{1}{\log_e 3}$$

$$\log_9 e = \frac{1}{\log_e 9}$$

$$= \frac{1}{\log_e 3^2} = \frac{1}{2\log_e 3}$$

$$\log_{27} e = \frac{1}{\log_e 27} = \frac{1}{\log_e 3^3}$$

$$= \frac{1}{3\log_e 3}$$

$$\text{L.H.S} = \frac{1}{\log_e 3} - \frac{1}{2\log_e 3} + \frac{1}{3\log_e 3} + \dots$$

$$= \frac{1}{\log_e 3}\left[1 - \frac{1}{2} + \frac{1}{3} + \dots\right]$$

$$= \frac{1}{\log_e 3}\log_e 2 = \frac{\log_e 2}{\log_e 3}$$

8\. P.T $\log\sqrt{12} = 1 + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3} + \cdots$

Soln:

R.H.S

$$= 1 + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3} + \cdots$$

$$= 1 + \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4^2} + \frac{1}{5}\cdot\frac{1}{4^2} + \frac{1}{6}\cdot\frac{1}{4^3} + \frac{1}{7}\cdot\frac{1}{4^3} + \cdots$$

$$= \left[1 + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{4^2} + \frac{1}{7}\cdot\frac{1}{4^3} + \cdots\right] + \left[\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4^2} + \frac{1}{6}\cdot\frac{1}{4^3} + \cdots\right]$$

$$= \left[1 + \frac{1}{3}(1/2)^2 + \frac{1}{5}(1/2)^4 + \frac{1}{7}(1/2)^6 + \cdots\right] + \left[\frac{1}{2}\left\{\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{4^2} + \frac{1}{3}\cdot\frac{1}{4^3} + \cdots\right\}\right]$$

Multiply and divide by 2 in first term

$$= 2\left[\frac{1}{2} + \frac{1}{3}(1/2)^3 + \frac{1}{5}(1/2)^5 + \frac{1}{7}(1/2)^7 + \cdots\right] + \frac{1}{2}\left[-\log\left(1-\frac{1}{4}\right)\right]$$

$$= 2\left[\frac{1}{2}\log\left(\frac{1+1/2}{1-1/2}\right)\right] + \frac{1}{2}\log\left(\frac{4-1}{4}\right)$$

$$= 2\left[\frac{1}{2}\log\left(\frac{3/2}{1/2}\right)\right] - 1/2\log\left(\frac{3}{4}\right)$$

$$= \log 3 - \frac{1}{2}[\log 3 - \log 4]$$

$$= \log 3 - \frac{1}{2}\log 3 + \log 4\cdot\frac{1}{2}$$

$$= \log 3\left(1-\frac{1}{2}\right) + \log 4\cdot\frac{1}{2}$$

$$= \log 3\left(1/2\right) + \log 4\cdot\frac{1}{2}$$

$$= \frac{1}{2}\log 3 + \frac{1}{2}\log 4$$

$= \frac{1}{2} \log(4 \times 3)$

$= \frac{1}{2} \log 12$

$= \log 12^{1/2}$

$= \log\sqrt{12} = L.H.S.$

R.H.S = L.H.S

9\. S.T $\left(\frac{1}{1}+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)\frac{1}{9}+\left(\frac{1}{5}+\frac{1}{6}\right)\frac{1}{9^2}+\dots = 9\log_3 - 12\log_2$

Soln.

L.H.S

$\left(\frac{1}{1}+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)\frac{1}{9}+\left(\frac{1}{5}+\frac{1}{6}\right)\frac{1}{9^2}+\dots$

$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}\cdot\frac{1}{9}+\frac{1}{4}\cdot\frac{1}{9}+\frac{1}{5}\cdot\frac{1}{9^2}+\frac{1}{6}\cdot\frac{1}{9^2}+\dots$

$\left[\frac{1}{1}+\frac{1}{3}\cdot\frac{1}{9}+\frac{1}{5}\cdot\frac{1}{9^2}+\dots\right]+\left[\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{9}+\frac{1}{6}\cdot\frac{1}{9^2}+\dots\right]$

Multiply and divide by 3 in the first term

multiply and divide by 9 in second term

$3\left[\frac{1}{3}+\frac{1}{3}(1/3)^3+\frac{1}{5}(1/3)^5+\dots\right]+$

$\frac{9}{2}\left[\frac{1}{9}+\frac{1}{2}(1/9)^2+\frac{1}{3}(1/9)^3+\dots\right]$

$= 3\left[\frac{1}{2}\log\left(\frac{1+1/3}{1-1/3}\right)\right]-\frac{9}{2}\log\left(\frac{1-1/9}{\quad}\right)$

$= 3\left[\frac{1}{2}\log\left(\frac{4/3}{2/3}\right)\right]-\frac{9}{2}\log(8/9)$

$= 3\left[\frac{1}{2}\log 2\right]-\frac{9}{2}\log\frac{8}{9}$

$8^{1/2} = 2\sqrt{2}$

$9^{1/2} = 3$

$= 3\log 2^{1/2} - \frac{9}{2}\log 8 + \frac{9}{2}\log 9$

$= 3\log\sqrt{2} - \log(2\sqrt{2})^9 + \log 3^9$

$= \log(\sqrt{2})^3 - \log(2\sqrt{2})^9 + \log 3^9$

$$= \log \frac{(\sqrt{2})^3}{(2\sqrt{2})^9} + 9\log 3$$

$$= \log \frac{2\sqrt{2}}{(2\sqrt{2})^9} + 9\log 3$$

$$= \log \frac{1}{(2\sqrt{2})^8} + 9\log 3$$

$$= \log (2\sqrt{2})^{-8} + 9\log 3$$

$$= -8\log 2\sqrt{2} + 9\log 3$$

$$= -8\log 2^1 \cdot 2^{1/2} + 9\log 3$$

$$= -8\log 2^{3/2} + 9\log 3$$

$$= -\cancel{8}^{4} \times \frac{3}{\cancel{2}} \log 2 + 9\log 3$$

$$= -12\log 2 + 9\log 3$$

$$= 9\log 3 - 12\log 2 = R.H.S$$

L.H.S = R.H.S

SUMMATION - Logarithmic series

1. S.T $\frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots = \log 2$

Soln:

Let t_n be the nth term

$$t_n = \frac{1}{(2n-1)(2n)}$$

$$\frac{1}{(2n-1)(2n)} = \frac{A}{2n-1} + \frac{B}{2n}$$

$$1 = A(2n) + B(2n-1)$$

Put $n=0$

$1 = A(0) + B(2(0)-1)$

$1 = -B$

$\boxed{B=-1}$

Equate the coeff of n^{th} term

$0 = 2A + 2B$

$0 = 2A - 2B$

$2A = 2(1)$

$A = \frac{\cancel{2}}{\cancel{2}}$

$\boxed{A=1}$

$t_n = \frac{1}{(2n-1)(2n)} = \frac{1}{2n-1} - \frac{1}{2n}$

Put $n = 1, 2, 3, \ldots$

$t_1 = \frac{1}{1} - \frac{1}{2}$

$t_2 = \frac{1}{3} - \frac{1}{4}$

$t_3 = \frac{1}{5} - \frac{1}{6}$

$\ldots$

$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

$\boxed{S = \log 2}$

2. $\frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \ldots = -\log(1-x) + \log\frac{(1-x)}{x}$

Soln:

L.H.S

Let $t_n = \frac{x^n}{n(n+1)} \rightarrow (1)$

$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} \rightarrow (2)$

$1 = A(n+1) + B n$; Put $n=0$

$1 = A(0+1) + B(0)$

$\boxed{1 = A}$

Equate n^{th} term

$0 = A + B$

$0 = 1 + B$

$\boxed{B = -1}$

Sub A & B value in (2)

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$(1) \Rightarrow t_n = \left[\frac{1}{n} - \frac{1}{n+1}\right] x^n$

$t_n = \frac{x^n}{n} - \frac{x^n}{n+1}$

Put $n = 1, 2, 3 \ldots$

$t_1 = \frac{x}{1} - \frac{x}{2} = x\left(\frac{1}{1} - \frac{1}{2}\right)$

$t_2 = \frac{x^2}{2} - \frac{x^2}{3} = x^2\left(\frac{1}{2} - \frac{1}{3}\right)$

$t_3 = \frac{x^3}{3} - \frac{x^3}{4} = x^3\left(\frac{1}{3} - \frac{1}{4}\right)$

. . . .

Adding these values

$$S = \frac{x}{1} - \frac{x}{2} + \frac{x^2}{2} - \frac{x^2}{3} + \frac{x^3}{3} - \frac{x^3}{4} + \cdots$$

$$= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) - \left(\frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots\right)$$

multiply & divide by x in second term

$$= -\log(1-x) - \frac{1}{x}\left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) - x$$

$$= -\log(1-x) - \frac{1}{x}\left[-\log(1-x) - x\right]$$

$$= -\log(1-x) + \frac{\log(1-x) + x}{x} = R.H.S$$

L.H.S = R.H.S

3. S.T $\frac{1}{2\cdot3\cdot4} + \frac{1}{4\cdot5\cdot6} + \frac{1}{6\cdot7\cdot8} + \cdots = \frac{3}{4} - \log 2$

Soln:

L.H.S

Let t_n be the nth term

$$t_n = \frac{1}{(2n)(2n+1)(2n+2)}$$

$$\frac{1}{2n(2n+1)(2n+2)} = \frac{A}{2n} + \frac{B}{2n+1} + \frac{C}{2n+2}$$

$$1 = A\underbrace{(2n+1)(2n+2)}_{4n^2+6n+2} + B\underbrace{2n(2n+2)}_{4n^2+4n} + C\underbrace{2n(2n+1)}_{4n^2+2n}$$

put $n = 0$

$$1 = A(1)(2) + B(0) + C(0)$$

$2n+1 = 0$

$$1 = 2A$$

$2n = -1$

$n = -1/2$

$$\boxed{A = \frac{1}{2}}$$

Put $n = -\frac{1}{2}$

$1 = A(2(-\frac{1}{2})+1)[2(-\frac{1}{2})+2] + B(2(-\frac{1}{2}))[2(-\frac{1}{2})+2] + C[2(-\frac{1}{2})(2(-\frac{1}{2})+1)]$

$1 = A(0) + B(-1)(1) + C(0)$

$1 = -B$

$\boxed{B = -1}$

Equate co-eff of n^{th} terms

$0 = 8A + 4B + 2C$

$0 = 8(\frac{1}{2}) + 4(-1) + 2C$

$= \frac{3}{2} - 4 + 2C$

$0 = -1 + 2C$

$2C = 1$

$\boxed{C = \frac{1}{2}}$

Sub A, B, C values

$t_n \quad \dfrac{1}{2n(2n+1)(2n+2)} = \dfrac{\frac{1}{2}}{2n} - \dfrac{1}{2n+1} + \dfrac{\frac{1}{2}}{2n+2}$

put $n = 1, 2, 3 \ldots$

$t_1 = \dfrac{\frac{1}{2}}{2} - \dfrac{1}{3} + \dfrac{\frac{1}{2}}{4}$

$t_2 = \dfrac{\frac{1}{2}}{4} - \dfrac{1}{5} + \dfrac{\frac{1}{2}}{6}$

$t_3 = \dfrac{\frac{1}{2}}{6} - \dfrac{1}{7} - \dfrac{\frac{1}{2}}{8}$

$\cdots$

$S = \frac{1}{4} - \frac{1}{3} + \frac{1}{8} + \frac{1}{8} - \frac{1}{5} + \frac{1}{12} + \frac{1}{12} - \frac{1}{7} - \frac{1}{16} + \ldots$

$= \frac{1}{4} - \frac{1}{3} + \frac{2}{8_4} - \frac{1}{5} + \frac{2}{12_6} - \frac{1}{7} + \ldots$

$$= \frac{1}{4} - \left[\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right]$$

Add & subtract $1 - \frac{1}{2}$ in R.H.S

$$= \frac{1}{4} - \left\{\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \cdots\right] - \left[1 - \frac{1}{2}\right]\right\}$$

$$= \frac{1}{4} - \left[\log 2 - \left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{4} - \log 2 + \frac{1}{2}$$

$$= \frac{1}{4} + \frac{1}{2} - \log 2$$

$$= \frac{4+2}{8} - \log 2$$

$$= \frac{\cancel{6}^{3}}{\cancel{8}_{4}} - \log 2$$

$$= \frac{3}{4} - \log 2 = \text{R.H.S}$$

L.H.S = R.H.S

4. S.T $\frac{1}{1\cdot2\cdot3} + \frac{5}{3\cdot4\cdot5} + \frac{9}{5\cdot6\cdot7} + \cdots = \frac{5}{2} - 3\log 2$

Soln:

L.H.S

Let t_n be the n^{th} term

$$t_n = \frac{4n-3}{(2n-1)(2n)(2n+1)}$$

$$\frac{4n-3}{(2n-1)(2n)(2n+1)} = \frac{A}{2n-1} + \frac{B}{2n} + \frac{C}{2n+1}$$

$$4n-3 = A\,\underbrace{2n(2n+1)}_{4n^2+2n} + B\,\underbrace{(2n-1)(2n+1)}_{4n^2-1} + C\,\underbrace{2n(2n-1)}_{4n^2-2n}$$

Put $n=0$

$-3 = A(0) + B(-1)(1) + C(0)$

$-3 = -B$

$\boxed{B=3}$

Put $n=\frac{1}{2}$

$4(\frac{1}{2})^2 - 3 = A\, 2(\frac{1}{2})(2(\frac{1}{2})+1) + B(2\cdot\frac{1}{2}-1)(2\cdot\frac{1}{2}+1) + C\, 2(\frac{1}{2})(2(\frac{1}{2})-1)$

$2-3 = A(1)(2) + B(0) + C(0)$

$-1 = 2A$

$\boxed{A = -\frac{1}{2}}$

Equate the co-eff of n term

$4 = 2A - 2C$

$4 = 2(-\frac{1}{2}) - 2C$

$4 = -1 - 2C$

$5 = -2C$

$\boxed{C = -\frac{5}{2}}$

sub A, B, C values

$t_n = \frac{-\frac{1}{2}}{2n-1} + \frac{3}{2n} - \frac{\frac{5}{2}}{2n+1}$

Put $n = 1, 2, 3 \ldots$

$t_1 = \frac{-\frac{1}{2}}{1} + \frac{3}{2} - \frac{\frac{5}{2}}{3}$ $\quad t_4 = \frac{-\frac{1}{2}}{7} + \frac{3}{8} - \frac{\frac{5}{2}}{9}$

$t_2 = \frac{-\frac{1}{2}}{3} + \frac{3}{4} - \frac{\frac{5}{2}}{5}$

$t_3 = -\frac{1}{2}/5 + \frac{3}{6} - \frac{\frac{5}{2}}{}$

Adding the values $-\frac{6}{2}(1/3)$

$$= -\frac{1}{2} + \frac{3}{2} - \frac{5}{2}(1/3) - \frac{1}{2}(1/3) + \frac{3}{4} - 5/2\,(4/3) \ldots$$

$$-(5/2 + 1/2)(1/3)\ 6/2\ \frac{3}{2}(1/3) - 1/2 \ldots$$

$$= -\frac{1}{2} + \frac{3}{2} - \frac{1}{3}(3) + \frac{3}{4} - \frac{3}{5} + \frac{3}{6} - \frac{3}{7} + \ldots$$

$$= -\frac{1}{2} - 3\left[-1/2 + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \ldots\right)$$

Add & sub 1 on R.H.S

$$= -\frac{1}{2} - 3\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots\right] - 1$$

$$= -\frac{1}{2} - 3\log 2 + 3$$

$$= -\frac{1}{2} + 3 - 3\log 2$$

$$= \frac{-1+6}{2} - 3\log 2$$

$$= \frac{5}{2} - 3\log 2 = R.H.S$$

L.H.S = R.H.S

5. S.T $\frac{1}{1\cdot 3} + \frac{1}{2\cdot 5} + \frac{1}{3\cdot 7} + \ldots = 2 - 2\log 2$

Soln:

$$t_n = \frac{2}{2n(2n+1)} = \frac{A}{2n} + \frac{B}{2n+1}$$

$$2 = A(2n+1) + B2n$$

Equate the constant term

$\boxed{2 = A} \Rightarrow \boxed{A = 2}$

Equate the co-eff of n^{th} term

$$0 = 2A + 2B$$

$0 = 2(2) + 2B$

$-4 = 2B$

$\boxed{B = -2}$

$t_n = \frac{2}{2n} - \frac{2}{2n+1}$

Put $n = 1, 2, 3 \ldots$

$t_1 = \frac{2}{2} - \frac{2}{3}$

$t_2 = \frac{2}{4} - \frac{2}{5}$

$t_3 = \frac{2}{6} - \frac{2}{7} \ldots$

Adding these values

$S = \frac{2}{2} - \frac{2}{3} + \frac{2}{4} - 2/5 + 2/6 - 2/7 + \ldots$

$= 2\left[\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - 1/7 + \ldots\right]$

$= -2\left[-\frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - 1/6 + \ldots\right]$

Add & Sub 1 on R.H.S

$= -2\{[1 - \frac{1}{2} + 1/3 - 1/4 + \ldots] - 1\}$

$= -2\{\log 2 - 1]$

$= -2\log 2 + 2$

$= 2 - 2\log 2 = R.H.S$

L.H.S = R.H.S