MARUDHAR KESARI JAIN COLLEGE FOR WOMEN, VANIYAMBADI

PG & RESEARCH DEPARTMENT OF MATHEMATICS

CLASS : I B.Sc PHYSICS

SUBJECT CODE : MATHEMATICS I

SUBJECT NAME : 23UEMA10C

SYLLABUS

UNIT- II

Matrices

Symmetric – Skew-Symmetric – Hermitian– Skew – Hermitian – Orthogonal and Unitary matrices – Cayley - Hamilton theorem (without proof) – Verification - Computation of inverse of matrix using Cayley - Hamilton theorem.

8/8/23
Tuesday

Unit - II

Matrices

Kind of Matrices

We shall consider $n \times n$ square matrices. In these, the element in the i^{th} row and j^{th} column is denoted by a_{ij}. So the element in the leading diagonals are $a_{11}, a_{22}, a_{33}, a_{44}, \ldots, a_{nn}$.

Real Matrix :-

A matrix whose elements are real numbers is called a real matrix.

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}_{2\times 2} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$$

a Real number

Complex matrix :-

ib complex number

A matrix in which atleast one element is imaginary is called a complex number.

$$\begin{bmatrix} 5 & 10 \\ 5+i2 & 15 \end{bmatrix} \quad \begin{bmatrix} 45+i & 10 \\ 10 & i+2 \end{bmatrix}$$

Symmetric matrix :

$a = a^T$

A square matrix $[a_{ij}]$ is said to be symmetric Matrix of $a_{ij} = a_{ji}$ for all i and j, that is, the element in i^{th} row and j^{th} column is equal to the element in the j^{th} row and i^{th} column

Explanation:-

A Symmetric matrix is symmetrical about the diagnol. that is the image of an element in the reflection on the leading diagonal is the element it self. $(A = A^T)$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 3 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 1 & 3 \end{bmatrix} \quad A = A^T$$

$a_{11} = a_{11}$ $\quad a_{21} = a_{12}$ $\quad a_{31} = a_{13}$

$a_{12} = a_{21}$ $\quad a_{22} = a_{22}$ $\quad a_{32} = a_{23}$

$a_{13} = a_{31}$ $\quad a_{23} = a_{32}$ $\quad a_{33} = a_{33}$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$A = A^T$ (or) $a_{11} = a_{11}$

$a_{12} = a_{21}$

$a_{21} = a_{12}$

$a_{22} = a_{22}$

Sums

① Find the value of x, y, z, a which satisfies the matrix equation.

$$\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 3 & 2a \end{bmatrix}$$

Given:-

$x+3=0$, $2y+x=-7$, $z-1=3$, $4=2a$

$x=-3$ $\quad 2y-3=-7$ $\quad z=3+1$ $\quad a=2$

$z=4$

$2y=-4$

$y=-2$

$(A = -A^T)$

Skew Symmetric Matrix:-

A square $[a_{ij}]$ is said to be a skew symmetric matrix of $a_{ij} = -a_{ji}$ for all i and j, that is the element in the i^{th} row and j^{th} column is equal to the negative of the element in the j^{th} row and i^{th} column.

Explanation:-

In a skew symmetric matrix, the image of an element in the reflection on the leading diagonal is its negative.

11/08/23
Friday.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix},$$

$$-A^T = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$$

$A = -A^T$ skew symmetric.

RESULT:-

① Let the square matrix A is skew symmetric then $a_{ij} = -a_{ji}$. In this setting $j = i$, we get.

$$a_{ii} = -a_{ii}, \quad a_{ii} + a_{ii} = 0$$

$$\Rightarrow 2a_{ii} = 0$$

$\Rightarrow 2 \neq 0$ then $a_{ii} = 0$, so the leading diagonal elements are zero

② If A is any square matrix, then by using matrix algebra we have.

$$A = \frac{1}{2}(2A) = \frac{1}{2}(A + A)$$

$$= \frac{1}{2}(A - A^T + A^T + A)$$

$$= \frac{1}{2}[(A - A^T) + (A + A^T)]$$

$$= \frac{(A - A^T) + (A + A^T)}{2}$$

$$= \frac{(A - A^T)}{2} + \frac{(A + A^T)}{2}$$

In this $\frac{1}{2}(A+A^T)$ is Symmetric

$[(A+B)^T = A^T + B^T]$

$\rightarrow \frac{1}{2}(A+A^T) \Rightarrow \frac{1}{2}(A+A^T)^T = \frac{1}{2}(A^T+(A^T)^T)$

$= \frac{1}{2}(A^T+A)$

Also $\frac{1}{2}(A-A^T)$ is skew symmetric

$\frac{1}{2}(A-A^T) \Rightarrow \frac{1}{2}(A-A^T)^T = \frac{1}{2}(A^T-(A^T)^T)$

$= \frac{1}{2}(A^T-A) = -\frac{1}{2}(A-A^T)$

Thus, the matrix A has been expressed as the sum of the symmetric matrix $\frac{1}{2}(A+A^T)$ and the skew symmetric matrix $\frac{1}{2}(A-A^T)$

(1.) If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ prove that $A+A^T$ is symmetric and $A-A^T$ is skew-symmetric.

Given:-

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}, A^T \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$A+A^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & -1+3 & 2+1 \\ 3-1 & 0+0 & 1+-1 \\ 1+2 & -1+1 & 0+0 \end{bmatrix}$$

$$A+A^T = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

$$(A+A^T)^T = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}^T$$

$$(A+A)^T = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

$\therefore A^T$ is symmetric.

$$A-A^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1-1 & -1-3 & 2-1 \\ -3-1 & 0-0 & 1-1 \\ 1-2 & -1-1 & 0-0 \end{bmatrix}$$

$$A-A^T = \begin{bmatrix} 0 & -4 & 1 \\ 4 & 0 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

$$(A-A^T)^T = \begin{bmatrix} 0 & 4 & 1 \\ -4 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}$$

$\therefore A^T$ is skew symmetric.

Hermitian:-

A complex square matrix [aij] is said to be an hermitian matrix if aij = conjugate of aji or $a_{ij} = \overline{a_{ji}}$ for all i and j.

$$A = A^* = (\bar{A})^T$$

Conjugate of a matrix: element sign change

If A is a complex matrix, then the matrix obtained from A by replacing its elements by their conjugate is called conjugate of A and is denoted by $\bar{A}$.

$$A = \begin{bmatrix} 1+i & 2+2i \\ 3-3i & 4-4i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1-i & 2-2i \\ 3+3i & 4+4i \end{bmatrix}$$

$$(\bar{A})^T = \begin{bmatrix} 1-i & 3+3i \\ 2-2i & 4-4i \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3-i \\ 3+i & 4 \end{bmatrix}$$

$a_{11} = \overline{a_{11}}$
$a_{12} = 3+i$
$a_{21} = 3-i$
$\overline{a_{21}} = 3+i$
$a_{12} = \overline{a_{21}}$

$$\bar{A} = \begin{bmatrix} 1 & 3+i \\ 3-i & 4 \end{bmatrix}$$

$$A^* = (\bar{A})^T = \begin{bmatrix} 1 & 3-i \\ 3+i & 4 \end{bmatrix}$$

RESULT:-

(1) $(\overline{A+B}) = \overline{A}+\overline{B}$, $\overline{\overline{A}} = A$, $(\overline{A^T})$, $(\overline{A})^T$

(2) If A is hermitian Matrix, then $\overline{A}^T = A$,

th $a_{ij} = \overline{a_{ji}}$, Replacing j by i, we get $a_{ii} = \overline{a_{ii}}$ or a_{ii} is real. that is, in a hermitian matrix, the diagonal elements are all real.

$$a_{ij} = \overline{a_{ji}}$$

$$i=j \qquad a_{ii} = \overline{a_{ii}}$$

$$a_{ii} = a+ib \quad \& \quad a_{ii} = a$$

$$\overline{a_{ii}} = a-ib.$$

$$\cancel{a}+ib = \cancel{a}-ib$$

$$\cancel{i}b = -\cancel{i}b.$$

$$b = -b$$

$$b+b=0$$

$$2b=0$$

$$b=0$$

$\therefore$ A is a real number.

Note:- $(\overline{A}^T)^T) = A \quad [(\overline{A})^T]^T = A^T$

$\overline{A} = A^T$

(1) If A is a hermitian matrix, then $(\overline{A})^T = A$ and set $[(A)^T]^T = A^T$ (or)

$\overline{A} = A^T$

(16)

(2) Any hermitian matrix can be written as

$$A = \frac{1}{2}(A+\overline{A}) + \frac{1}{2}(A-\overline{A})$$

(or)

$= \frac{1}{2}(A+\bar{A}) + i\left(\frac{1}{2}(\bar{A}-A)\right] = R + iS$

In this $R = \frac{1}{2}(A+\bar{A})$

$S = \frac{1}{2}(\bar{A}-A)$

R is real and symmetric matrix

for real.

$R = \frac{1}{2}(A+\bar{A})$

$\bar{R} = \frac{1}{2}(\overline{A+A}) = \frac{1}{2}(\bar{A}+\bar{\bar{A}})$

$\bar{R} = \frac{1}{2}(A+\bar{A})$ R is real.

for symmetric

$R^T = \frac{1}{2}(A+\bar{A})^T$

$= \frac{1}{2}(A^T + (\bar{A})^T)$

$= \frac{1}{2}(A^T + A^*) = \frac{1}{2}(A^T + A)$

$= \frac{1}{2}(\bar{A}+A)$

$\boxed{R^T = R}$ R is a symmetric matrix and Real.

S is a real and Skew-symmetric

for real $(S = \bar{S})$

$S = \frac{i}{2}[\bar{A}-A)$

$\rightarrow \bar{S} = \frac{\bar{i}}{2}[\overline{\bar{A}-A}] = \frac{-i}{2}[\bar{\bar{A}}-\bar{A}]$

$= -\frac{i}{2}[A-\bar{A}]$

$= \frac{i}{2}[\bar{A} - A]$

$\boxed{\bar{S} = S}$. S is real for skew

skew symmetric $[S = -S^T]$

$S^T = i/2 [\bar{A} - A]^T$

$= i/2 [(\bar{A})^T - A^T]$

$= i/2 [A^* - \bar{A}]$

$= i/2 [A - \bar{A}] = -i/2 [\bar{A} - A]$

$\boxed{S^T = -S}$ (or) $\boxed{S = -S^T}$ S is a skew symmetric

∴ A Hermitian matrix can be written as sum of (real symmetric) and real, skew symmetric matrix.

③ If A is real and symmetric matrix then $\bar{A} = A$ and $A^T = A$. So $(\bar{A})^T = A^T = A$ that is a hermitian matrix, thus every symmetric matrix is a hermitian matrix. But the converse need not be the.

Eg:- $\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$ is a hermitian matrix but not a symmetric Matrix.

Skew Hermitian Matrix:-

$$[A = -A^* = -(\bar{A})^T]$$

A complex square matrix $[a_{ij}]$ is said to be a skew - Hermitian if

$$a_{ij} = -\overline{a_{ji}}.$$

Example:-

$$A = \begin{bmatrix} i & 3+i \\ -3+i & 2i \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} -i & 3-i \\ -3-i & -2i \end{bmatrix}$$

$$(\bar{A})^T = \begin{bmatrix} -i & -3-i \\ 3-i & -2i \end{bmatrix}, \quad -(\bar{A})^T = \begin{bmatrix} i & 3+i \\ -3+i & 2i \end{bmatrix}$$

$A = -(\bar{A})^T$ A is a skew Hermitian matrix

RESULT:-

① If A is a skew hermitian matrix then a_{ii} is purely imaginary

Skew Hermitian.

$$a_{ij} = -\overline{a_{ji}}$$

Replace $i = j \Rightarrow a_{ii} = -\overline{a_{ii}}$

$$a_{ii} = a + ib, \quad \overline{a_{ii}} = a - ib.$$

$$a + ib = -(a - ib)$$

$$a + \cancel{ib} = -a + \cancel{ib}$$

$$a = -a$$

$$2a = 0$$

$$a = 0$$

~~NOTE:-~~

~~If a = a+bi~~

$a_{ii} = ib$ then a_{ii} is purely imaginary

(or)

Diagonal elements are purely imaginary.

NOTE:-

① If A is a hermitian matrix, then iA is a skew-hermitian matrix.

Hermition.

$$A = (\bar{A})^T$$

Let $B = iA$, $\bar{B} = (\overline{iA}) = -i\bar{A}$

$$(\bar{B})^T = (-i\bar{A})^T = -i(\bar{A})^T$$

$$(\bar{B})^T = -iA \Rightarrow -(\bar{B})^T = -iA$$

$$-(\bar{B})^T = iA, \quad -(\bar{B})^T = B.$$

B is skew hermitian iA is skew Hermition.

② If A is real and skew symmetric matrix, then $\bar{A} = A$ and $A^T = -A$

Let real $A = \bar{A}$

Skew symmetric,

$$A = -A^T$$

$$\Rightarrow -A = A^T.$$

(3) If A is any square matrix, then by matrix Algebra, we have.

$$A = \tfrac{1}{2}(2A) = \tfrac{1}{2}(A + A) - \tfrac{1}{2}(A + (\bar{A})^T - (\bar{A})^T + A).$$

$$= \tfrac{1}{2}(A + (\bar{A})^T) + \tfrac{1}{2}(A - (\bar{A})^T)$$

In this matrix $\frac{1}{2}(A + (\bar{A})^T)$ is a hermitian matrix because we have.

To prove hermitian

$$B = \tfrac{1}{2}(A + (\bar{A})^T)$$

$$\bar{B} = \tfrac{1}{2}(\overline{A + (\bar{A})^T})$$

$$= \tfrac{1}{2}(\bar{A} + \overline{(\bar{A})^T})$$

$$= \tfrac{1}{2}(\bar{A} + (\bar{\bar{A}})^T).$$

$$= \tfrac{1}{2}(\bar{A} + A^T)$$

$$\Rightarrow (\bar{B})^T = \tfrac{1}{2}(\bar{A} + A^T)^T$$

$$= \tfrac{1}{2}((\bar{A})^T + (A^T)^T).$$

$$= \tfrac{1}{2}((\bar{A})^T + A)$$

$$(\bar{B})^T = B$$

$\Rightarrow \frac{1}{2}(A + (\bar{A})^T)$ is an hermitian Matrix.

And the Matrix

$\frac{1}{2}(A-(\bar{A})^T)$ is a skew-hermitian

Matrix because.

$$C = \frac{1}{2}(A-(\bar{A})^T)$$

$$\bar{C} = \frac{1}{2}\overline{(A-(\bar{A})^T)}$$

$$= \frac{1}{2}(\bar{A}-(\bar{\bar{A}})^T)$$

$$= \frac{1}{2}(\bar{A}-(\bar{\bar{A}})^T)$$

$$\bar{C} = \frac{1}{2}(\bar{A}-A^T)$$

$$\Rightarrow (\bar{C})^T = \frac{1}{2}(\bar{A}-A^T)^T$$

$$= \frac{1}{2}((\bar{A})^T-(A^T)^T)$$

$$= \frac{1}{2}((\bar{A})^T-A)$$

$$(\bar{C})^T = -\frac{1}{2}(A-(\bar{A})^T)$$

$$(\bar{C})^T = -C$$

$$\Rightarrow C = -(\bar{C})^T$$

$\Rightarrow \frac{1}{2}(A-(\bar{A})^T$ is an skew

hermitian matrix.

Any square matrix can be written

as sum of hermitian and skew-

hermitian matrix. 17

Orthogonal Matrix:·

A square matrix is called an orthogonal Matrix if $AA^T = A^TA = I$

Ex:-

① $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

② $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \rightarrow A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

③ $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

③ $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta + 0 & -\cos\theta\sin\theta + \cos\theta\sin\theta + 0 & 0+0+0 \\ -\cos\theta\sin\theta + \cos\theta\sin\theta + 0 & \sin^2\theta + \cos^2\theta + 0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta + 0 & \cos\theta\sin\theta - \cos\theta\sin\theta + 0 & 0+0+0 \\ \cos\theta\sin\theta - \cos\theta\sin\theta + 0 & \sin^2\theta + \cos^2\theta + 0 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $AA^T = A^TA = I$ is a orthogonal matrix.

(1) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$AA^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 1 + 0 = 1$$

$$A^TA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= 1 + 0 = 1$$

A is an orthogonal Matrix.

Properties of orthogonal matrix

* If A is an orthogonal matrix than $A^T = A^{-1}$.

* If A is orthoganal, then A^{-1} is also orthogonal for $(A^{-1})(A^{-1})^T = (A^{-1})^T A^{-1} = I$

* If A and B are orthogonal, then AB is also orthogonal. for $(AB)^T = B^T A^T = (AB)^{-1}$.

* If A is orthogonal then $|A| = \pm 1$ for

$$AA^T = I \Rightarrow |AA^T| = |I|$$

$$\Rightarrow |A||A^T| = |I|$$

$$|A||A| = |I|$$

$$|A|^2 = |I|$$

$$|A|^2 = 1$$

$$\boxed{|A| = \pm 1}$$

* The orthogonal matrix is called a proper orthogonal matrix if $|A| = 1$ and improper orthogonal matrix if $|A| = -1$.

Given:

* orthogonal matrix

$$AA^T = A^T A = I$$

choose 1^{st} condition.

$$AA^T = I$$

multiple inverse on both sides.

$$A^{-1} A A^T = A^{-1} I$$

$$I A^T = A^{-1}$$

$$\boxed{A^T = A^{-1}}$$

* $A^T = A^{-1}$

$$(A^{-1})^T = (A^{-1})^T = A$$

$$(A^{-1})^T A^{-1} = A^{-1} (A^{-1})^T = I$$

$$A A^{-1} = A^{-1} A = I$$

$$\boxed{I = I = I}$$

unitary matrix.

A square matrix A is called a unitary matrix it.

$$AA^* = A^* A = I$$

(or)

$$A(\bar{A})^T = (\bar{A})^T A.$$

Properties of unitary Matrix.

* If A is unitary matrix, then

$$A^{*} = A^{-1}$$

(or)

$$(\bar{A})^{T} = A^{-1}.$$

* If A and B are unitary then AB is also unitary.

$$(AB)^{-1} = B^{-1}A^{-1} = (\bar{B})^{T}(\bar{A})^{T} = (\bar{A}\bar{B})^{T} = (\overline{AB})^{T}$$

(1.) Express $\begin{bmatrix} 2 & 4 & 8 \\ 6 & 2 & 8 \\ 2 & 2 & 2 \end{bmatrix}$ as the sum of a symmetric and skew-symmetric matrix

$$A = \tfrac{1}{2}(A + A^{T})$$

Given:-

$$A = \begin{bmatrix} 2 & 4 & 8 \\ 6 & 2 & 8 \\ 2 & 2 & 2 \end{bmatrix}$$

$$A^{T} = \begin{bmatrix} 2 & 6 & 2 \\ 4 & 2 & 2 \\ 8 & 8 & 2 \end{bmatrix}$$

To prove $(A+A^{T})$ is symmetric.

$$A + A^{T} = \begin{bmatrix} 2 & 4 & 8 \\ 6 & 2 & 8 \\ 2 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 6 & 2 \\ 4 & 2 & 2 \\ 8 & 8 & 2 \end{bmatrix}$$

Let $B = A + A^T$

$$B = \begin{bmatrix} 4 & 10 & 10 \\ 10 & 4 & 10 \\ 10 & 10 & 4 \end{bmatrix}, \quad B^T = \begin{bmatrix} 4 & 10 & 10 \\ 10 & 4 & 10 \\ 10 & 10 & 4 \end{bmatrix}$$

$B = B^T$ symmetric

Then $(A + A^T)$ is symmetric.

To prove, $(A - A^T)$ is skew-symmetric

$$A - A^T = \begin{bmatrix} 2 & 4 & 8 \\ 6 & 2 & 8 \\ 2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 6 & 2 \\ 4 & 2 & 2 \\ 8 & 8 & 2 \end{bmatrix}$$

Let $C = A - A^T$.

$$C = \begin{bmatrix} 0 & -2 & 6 \\ 2 & 0 & 6 \\ -6 & -6 & 0 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0 & 2 & -6 \\ -2 & 0 & -6 \\ 6 & 6 & 0 \end{bmatrix}$$

$$-C^T = \begin{bmatrix} 0 & -2 & 6 \\ 2 & 0 & 6 \\ -6 & -6 & 0 \end{bmatrix}$$

$C = -C^T$, skew symmetric.

To prove.

$$A = \tfrac{1}{2}(A + A^T) + \tfrac{1}{2}(A - A^T)$$

RHS

$$\tfrac{1}{2}(A + A^T) + \tfrac{1}{2}(A - A^T) = \tfrac{1}{2}\begin{bmatrix} 4 & 10 & 10 \\ 10 & 4 & 10 \\ 10 & 10 & 4 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} 0 & -2 & 6 \\ 2 & 0 & 6 \\ -6 & -6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 5 \\ 5 & 2 & 5 \\ 5 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 3 \\ -3 & -3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 8 \\ 6 & 2 & 8 \\ 2 & 2 & 2 \end{bmatrix} = A = \frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T)$$

(2) show that the matrix $\frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$ is orthogonal.

$AA^T = A^TA = I$ (or) $A^T = A^{-1}$

$$A = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

$$A^T = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$AA^T = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 4+4+1 & -4+2+2 & 2-4+2 \\ -4+2+2 & 4+1+4 & -2-2+4 \\ 2-4+2 & -2-2+4 & 1+4+4 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A^TA = \frac{1}{3}\begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 4+4+1 & 4-2-2 & 2-4+2 \\ 4-2-2 & 4+1+4 & 2+2-4 \\ 2-4+2 & 2+2-4 & 1+4+4 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$AA^T = A^T A = I$$

Hence, A is orthogonal

(or)

$$A = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$|A| = \frac{2}{3}\left[\frac{2}{9} + \frac{4}{9}\right] - \frac{2}{3}\left[-\frac{4}{9} - \frac{2}{9}\right] + \frac{1}{3}\left[\frac{4}{9} - \frac{1}{9}\right]$$

$$= \frac{2}{3}\left[\frac{6}{9}\right] - \frac{2}{3}\left[-\frac{6}{9}\right] + \frac{1}{3}\left[\frac{3}{9}\right]$$

$$= \frac{12}{27} + \frac{12}{27} + \frac{3}{27} = \frac{27}{27} = 1$$

$$\boxed{|A| = 1}$$

$$\text{Cof}(A) = \begin{bmatrix} \left(\frac{2}{9} + \frac{4}{9}\right) & -\left(\frac{-4}{9} - \frac{2}{9}\right) & \left(\frac{4}{9} - \frac{1}{9}\right) \\ -\left(\frac{4}{9} + \frac{2}{9}\right) & \left(\frac{4}{9} - \frac{1}{9}\right) & -\left(\frac{-4}{9} - \frac{2}{9}\right) \\ \left(\frac{4}{9} - \frac{1}{9}\right) & -\left(\frac{4}{9} + \frac{2}{9}\right) & \left(\frac{2}{9} + \frac{4}{9}\right) \end{bmatrix}$$

$$\text{Cof}(A) = \begin{bmatrix} \frac{6}{9} & \frac{6}{9} & \frac{3}{9} \\ \frac{-6}{9} & \frac{3}{9} & \frac{6}{9} \\ \frac{3}{9} & \frac{-6}{9} & \frac{6}{9} \end{bmatrix}$$

$$\text{Cof}(A) = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{-2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \end{bmatrix}$$

$$\text{Adj}(A) = \begin{bmatrix} \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$A^{-1} = \frac{1}{1} \begin{bmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

$$A^{-1} = \frac{1}{3} \begin{bmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{bmatrix}$$

$$A^T = A^{-1}$$

Hence, A is orthogonal.

③ prove that the fallowing matrix is unitary.

$$\begin{bmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

Given:-

$$A = \begin{bmatrix} \frac{1+i}{2} & -\frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$AA^* = A^*A = I$$

$$A^* = (\bar{A})$$

$$\bar{A} = \begin{bmatrix} \frac{1-i}{2} & -\frac{1-i}{2} \\ \frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$(\bar{A})^T = \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ -\frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$AA^* = \begin{bmatrix} \frac{1+i}{2} & -\frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \begin{bmatrix} \frac{1-i}{2} & \frac{1-i}{2} \\ -\frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(1+i)(1-i)}{4} + \frac{(-1+i)(-1-i)}{4} & \frac{(1+i)(1-i)}{4} + \frac{(-1+i)(1+i)}{4} \\ \frac{(1+i)(1-i)}{4} + \frac{(1-i)(-1-i)}{4} & \frac{(1+i)(1-i)}{4} + \frac{(1-i)(1+i)}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1-\cancel{i}+\cancel{i}-i^2+1+\cancel{i}-\cancel{i}-i^2}{4} & \frac{\cancel{1}-\cancel{i}+\cancel{i}-i^2-\cancel{1}-i+i+i^2}{4} \\ \frac{\cancel{1}-\cancel{i}+\cancel{i}-i^2-\cancel{1}-i+i+i^2}{4} & \frac{1-\cancel{i}+\cancel{i}-i^2+1+\cancel{i}-\cancel{i}-i^2}{4} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1+1+1+1}{4} & 0 \\ 0 & \frac{1+1+1+1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^*A = \begin{bmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ -\frac{1-i}{2} & \frac{1+i}{2} \end{bmatrix} \begin{bmatrix} \frac{1+i}{2} & -\frac{1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(1-i)(1+i)}{4} + \frac{(1-i)(1+i)}{4} & \frac{(1-i)(-1+i)}{4} + \frac{(1-i)(1-i)}{4} \\ \frac{(-1-i)(1+i)}{4} + \frac{(1+i)(1+i)}{4} & \frac{(-1-i)(-1+i)}{4} + \frac{(1+i)(1-i)}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+i-i-i^2+1+i-i-i^2}{4} & \frac{-1+i+i-i^2+1-i-i+i^2}{4} \\ \frac{-1-i-i-i^2+1+i+i+i^2}{4} & \frac{1-i+i-i^2+1-i+i-i^2}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{4} & 0 \\ 0 & \frac{4}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$AA^* = A^*A = I$$

$\therefore$ A is unitary.

Cayley - Hamilton theorem:-

Statement

If (A is a square matrix), then the characteristic equation of A satisfied by A. $|A_{n\times n} - \lambda I_n| = 0$

Explanation:-

If A is 3×3 square matrix with the characteristic equation $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$. then $A^3 + a_1A^2 + a_2A + a_3I = 0$

where I is 3×3 unit matrix and 0 is the 3×3 null matrix.

Example:-

Find the characteristic equation of

(7) $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$.

Given:-

$$A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 0 \\ 3 & 2-\lambda \end{bmatrix}$$

$$0 = |A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 \\ 3 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(2-\lambda) - 0 = (2-\lambda)(2-\lambda)$$

$$= (4 - 2\lambda - 2\lambda + \lambda^2) = 4 - 4\lambda + \lambda^2 =$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$\Rightarrow \lambda = A$$

$$A^2 - 4A + 4I = 0$$

$$A^2 = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 12 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 12 & 4 \end{bmatrix} - 4\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 12 & 4 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 12 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 0 \\ 10 & -4 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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$$A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$$|A| = 4 \quad \frac{1}{4}\begin{bmatrix} 2 & 0 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -3/4 & 1/2 \end{bmatrix}$$

$$A^2 - 4A + 4I = 0$$

$$A^{-1}(A^2 - 4A + 4I) = 0$$

$$A^{-1}A^2 - 4A^{-1}A + 4A^{-1}I = 0$$

$$A^{-1}AA - 4I + 4A^{-1} = 0$$

$$A - 4I + 4A^{-1} = 0$$

$$4A^{-1} = 4I = A$$

$$4A^{-1} = 4I - A$$

$$= 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$$

$$4A^{-1} = \begin{bmatrix} 2 & 0 \\ -3 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -3/4 & 1/2 \end{bmatrix}$$

1. verity cayley's ~~[illegible]~~ Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{bmatrix}$$

$$0 = |A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 2 \\ -2 & 1-\lambda & 3 \\ 3 & 2 & -3-\lambda \end{vmatrix}$$

$$0=(1-\lambda)[(1-\lambda)(-3-\lambda)-6]+1[-2(-3-\lambda)-9]+2[-4-3(1-\lambda)]$$

$$0=(1-\lambda)[-3-\lambda+3\lambda+\lambda^2-6]+1[6+2\lambda-9]+2[-4-3+3\lambda]$$

$$0=(1-\lambda)[-9+2\lambda+\lambda^2]+1[-3+2\lambda]+2[-7+3\lambda]$$

$$0=-9+2\lambda+\lambda^2+9\lambda-2\lambda^2-\lambda^3-3+2\lambda-14+6\lambda$$

$$0=-\lambda^3-1\lambda^2+19\lambda-26$$

The characteristic equation of A is.

$$\lambda^3+\lambda^2-19\lambda+26=0$$

$$\lambda=A$$

$$A^3+A^2-19A+26=0$$

$$A^2=\begin{bmatrix} 9 & 2 & -7 \\ 5 & 9 & -10 \\ -10 & -7 & 21 \end{bmatrix}$$

$$A^3=\begin{bmatrix} -16 & -21 & 45 \\ -43 & -16 & 67 \\ 67 & 45 & -104 \end{bmatrix}$$

$$A^3+A^2-19A+26=0$$

$$A^2+A-19I+26A^{-1}=0$$

$$26A^{-1}=-A^2-A+19I$$

$$=\begin{bmatrix} 9 & 2 & -7 \\ 5 & 9 & -10 \\ -10 & -7 & 21 \end{bmatrix}-\begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{bmatrix}+\begin{bmatrix} 19 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & -2 & 7 \\ -5 & -9 & 10 \\ 10 & 7 & -21 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 3 \\ 3 & 2 & -3 \end{bmatrix} + \begin{bmatrix} 19 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -1 & 5 \\ -3 & -10 & 7 \\ 7 & 5 & -18 \end{bmatrix} + \begin{bmatrix} 19 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 19 \end{bmatrix} = \begin{bmatrix} 9 & -1 & 5 \\ -3 & 9 & 7 \\ 7 & 5 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{20} \begin{bmatrix} 9 & -1 & 5 \\ -3 & 9 & 7 \\ 7 & 5 & 1 \end{bmatrix}$$

(17) Find A^{-1} by cayley Hamilton theorem

when $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Given:

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{bmatrix} 1-\lambda & 0 & 3 \\ 2 & 1-\lambda & -1 \\ 1-\lambda & \lambda-1 & -1-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(1-\lambda)(1-\lambda)-1\right] - 0 + 3\left[-2-1+\lambda\right] = 0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)+\lambda^2-1\right] + 3\left[3+\lambda\right] = 0$$

$$\Rightarrow (1-\lambda)\left[1-2-\lambda+\lambda^2\right] - 9 + 3\lambda = 0$$

$$= (1-\lambda)\left[-2\lambda+\lambda^2+2\lambda^2-\lambda^3-9+3\lambda\right] = 0$$

$$= -2\lambda - \lambda^2 + 2\lambda - \lambda^3 - 9 + 3\lambda = 0$$

$$= -\lambda^3 - 3\lambda^2 + 3\lambda + 9 = 0$$

$$-A^3 + 3A^2 + A - 9I = 0$$

$$A^3 - 3A^2 - A + 9I = 0$$

$$A^2 - 3A - I + 9A^{-1} = 0$$

$$9A^{-1} = A^2 + 3A - I.$$

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$3A = 3 \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix}$$

$$9A^{-1} = -A^2 + 3A + I$$

$$9A^{-1} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ -0 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 9 \\ 6 & 3 & -3 \\ 3 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad A^{-1} = \frac{1}{9} \begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 & -6 \\ -3 & -2 & -4 \\ -0 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 9 \\ 6 & 4 & -3 \\ 3 & -3 & 4 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

(1) using cayley hamilton theorem, find A^4 given that $A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

Given:-

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{bmatrix} 2-\lambda & -2 & 1 \\ 0 & 1-\lambda & 2 \\ 1 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)\left[(1-\lambda)(1-\lambda) - 0\right] + 2\left[0-2\right] + 1\left[1(1-\lambda)\right] = 0$$

$$(2-\lambda)\left[1 - 2\lambda + \lambda^2\right] - 4 - 1 + \lambda = 0$$

$$2 - 2\lambda + 2\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 - 5 + \lambda = 0$$

$$-\lambda^3 + 4\lambda^2 - 4\lambda - 3 = 0$$

$$-A^3 + 4A^2 - 4A - 3I = 0$$

$$A^3 - 4A^2 - 4A + 3I = 0 \longrightarrow (1)$$

(x) A

$$A^4 - 4A^3 - 4A^2 + 3A = 0 \longrightarrow (2)$$

$4 \times (1)$

$$4A^3 - 16A^2 + 16A + 12I = 0 \longrightarrow (3)$$

(2) $A^4 - 4A^3 + 4A^2 + 3A$

(3) + (4)

$$A^4 - 12A^2 + 19 + 12I = 0$$

$$A^4 = 12A^2 - 19A - 12I.$$

$$A^4 = 12A^2 - 19A - 12I.$$

$$A^4 = 12\begin{bmatrix} 5 & -6 & -1 \\ 2 & 1 & 4 \\ 3 & -2 & 2 \end{bmatrix} - 19\begin{bmatrix} 2 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 60 & -72 & -12 \\ 24 & 12 & 48 \\ 36 & -24 & 24 \end{bmatrix} - \begin{bmatrix} 38 & -38 & 19 \\ 0 & 19 & 38 \\ 19 & 0 & 19 \end{bmatrix} - \begin{bmatrix} 12 & 0 \\ 0 & 12 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 82 & 34 & -31 \\ 24 & -7 & 10 \\ 17 & -24 & 5 \end{bmatrix} - \begin{bmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{bmatrix} = \begin{bmatrix} 10 & \\ 24 & \\ 17 & \end{bmatrix}$$

(2) If $A = \begin{bmatrix} 5 & 3 \\ 1 & 3 \end{bmatrix}$, find A^n in the form $aA + bI$ where a and b are scalars and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{bmatrix} 5-\lambda & 3 \\ 1 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (5-\lambda)(3-\lambda) - 3 = 0$$

$$\cancel{15} - 5\lambda - 3\lambda + \lambda^2 - 3 = 0$$

$$\lambda^2 - 8\lambda + 12 = 0$$

$$(\lambda-6)(\lambda-2) = 0$$

$$\therefore \lambda = 6, \lambda = 2.$$

$aA + bI$ where a and b.

$$A^n = aA + bI$$

$$\lambda^n = a\lambda + b$$

$$\lambda = 6$$

$$\Rightarrow 6^n = 6a + b \longrightarrow (1)$$

$$\lambda = 2$$

$$2^n = 2a + b \longrightarrow (2)$$

$$① \Rightarrow 6^n = 6a + b\cancel{I}$$

$$② \Rightarrow 2^n = 2a + b\cancel{I}$$

$$(-) \quad (-) \quad (-)$$

$$6^n - 2^n = 6a - 2a$$

$$6^n - 2^n = 4a.$$

$$a = \frac{6^n - 2^n}{4}$$

sub ⓐ in ② ⇒

$$2^n = \cancel{2}\left(\frac{6^n - 2^n}{\cancel{4}\,2}\right) + b$$

$$2^n = \frac{6^n - 2^n}{2} + b.$$

$$2^n = \frac{6^n - 2^n + 2b}{2}$$

$$2 \cdot 2^n = 6^n - 2^n + 2b.$$

$$2^{n+1} = 6^n - 2^n + 2b$$

$$2^{n+1} - 6^n + 2^n = 2b$$

$$b = \frac{2^{n+1} - 6^n + 2^n}{2}$$

$$A^n = \left(\frac{6^n - 2^n}{4}\right)A + \left(\frac{2^{n+1} - 6^n + 2^n}{2}\right)I$$

(or)

$$2^n - \left(\frac{6^n - 2^n}{2}\right) = b$$

$$2^n - \frac{6^n}{2} + \frac{2^n}{2} = b$$

$$2^n + \frac{2^n}{2} - \frac{6^n}{2} = b$$

$$2^n\left(1 + \frac{1}{2}\right) - \frac{6^n}{2} = b$$

$$\frac{3 \times 2^n}{2} - \frac{6^n}{2} = b$$

$$\frac{3 \times 2^n - 6^n}{2} = b$$