MARUDHAR KESARI JAIN COLLEGE FOR WOMEN, VANIYAMBADI

PG & RESEARCH DEPARTMENT OF MATHEMATICS

CLASS : I B.Sc PHYSICS

SUBJECT CODE : MATHEMATICS I

SUBJECT NAME : 23UEMA10C

SYLLABUS

UNIT- III

Numerical Methods

Newton's method to find a root approximately. Finite Differences: Interpolation: Operators ,, ∇,E, difference tables. Interpolation formulae: Newton's forward and backward interpolation formulae for equal intervals, Lagrange's interpolation formula.

23/09/2023

Unit: 3

Numerical methods:-

Newton's Method to find a root approximation:

$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$

Newton's method to find a root of approximation. $\frac{dx^n}{dx} = dx^{n-1}$.

This method depends on taylor's expansion of function.

Suppose we want a root of $f(x)=0$ (here $f(x)$ need not be a polynomial)

Let us - have the following assumption

α : A root of $f(x)=0$.

x_0 : A known number close to α.

h : A small number such that $\alpha = x_0+h$.

Then h measures the gap between the root α and x_0 known h is not known if h is known it means α, the root is known since α is a root of the equation $f(x)=0$.

$$f(x)=0.$$
$$\therefore f(x_0+h)=0.$$

By the taylor's series.

$$f(x_0+h) = f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0)$$

Therefore, using (1)

$$0 = f(x_0) + \frac{h}{1!} f'(x_0) \frac{h}{2!} f''(x_0+$$

$\therefore$ The equation gives the actual value of h which cannot be obtained easily however, neglecting h_1^2, h_1^3 terms, we get an approximate value for h boom.

$$0 = f(x_0) + \frac{h}{1!} f'(x_0) \text{ as } h = \frac{-f(x_0)}{f'(x_0)}$$

which this value of h, $x_0 + h = \alpha$ but this $x_0 + h$ is closer to α than x_0'

Denote this $x_0 + h$ by x, then,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Thus x_1 is a better approxmation for α then x_0 similarly the number x_2 is given by

$$x_2 = x_1 \frac{-f(x_1)}{f'(x_1)}$$

is a better approximation for α than x_1 etc thus in conclusion.

x_0 is not an approximation for α.

x_1 is a better approximation for α

x_2 is a still better approximation for α

the approximation x_n is given by

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

Key points — Justify Derive the rook of Newton's approximation!

$\alpha = x_0 + h$

$f(\alpha) = 0$

$f(x_0 + h) = 0$

$$\boxed{h = -\frac{f(x_0)}{f'(x_0)} \qquad \alpha = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

By taylor's series

$$f(x_0 + h) = f(x_0) + \frac{h f'(x_0)}{1!} + \frac{h^2 f''(x_0)}{2!} + \dots = 0$$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$f(x_0) + \frac{h f'(x_0)}{1!} = 0$$

$$h f'(x_0) = -f(x_0)$$

① Find by Newton's method the real root of $x^3+3x-1=0$, correct to a decimal part.

Given:

$$f(x) = x^3+3x-1$$

$$f(0) = 0+0-1 = -1$$

$$f(1) = 1+3-1 = 3$$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$x_0 = 0 \quad , f'(x) = 3x^2+3$$

$n=1$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0 - \frac{x_0^3+3x_0-1}{3x_0^2+3}$$

$$= -\frac{0^3+3(0)-1}{3(0)+3}$$

$$= -\left(\frac{-1}{3}\right)$$

$$x_1 = 1/3 = 0.3333$$

$n=2$,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= \frac{1}{3} - \frac{f(1/3)}{f'(1/3)}$$

$$= 1/3 - \frac{(1/3)^3 + 3(1/3) - 1}{3(1/3)+3}$$

$$= \frac{1}{3} - \frac{\left[\frac{1}{27}+1-1\right]}{3\left(\frac{1}{9}\right)+3}$$

$$= \frac{1}{3} - \frac{\left(\frac{1}{27}\right)}{\frac{1}{3}+3}$$

$$= \frac{1}{3} - \frac{\left(\frac{1}{27}\right)}{\frac{10}{3}}$$

$$= \frac{1}{3} - \left(\frac{1}{\cancel{27}_{9}} \times \frac{\cancel{3}}{10}\right)$$

$$= \frac{1}{3} - \frac{1}{9 \times 10} = \frac{1}{3} - \frac{1}{90}$$

Stop or continuous

$$= \frac{90-3}{270}$$

$$= \frac{87}{270}$$

$x_2 = 0.322$

$n = 3,$

$$x_3 = 0.322 - \frac{(0.322)^3 + 3(0.322) - 1}{3(0.322)^2 + 3}$$

$$= 0.322 - \frac{[0.0333 + 0.966 - 1]}{0.3110 + 3}$$

$$= 0.322 - \frac{0.0007}{3.3110}$$

$= 0.322 - 0.0021$

$= 0.3199 = 0.32.$

② using Newton's method to find the Smallest positive root of the equation.

$x^3 - 2x + 0.5 = 0$

x	-2	-1	0	1	2
$f(x)$	-3.5	1.5	0.5	-0.5	4.5

$-2 < x < -1$ $0 < x < 1$ $1 < x < 2.$

$f(x) = x^3 - 2x + 0.5$

$f'(x) = 3x^2 - 2$

$x_0 = 0$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$n = 1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)}$$

$$= \frac{-0.5}{-2} = \frac{0.5}{2} = 0.25$$

$n = 2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.25 - \frac{f(0.25)}{f'(0.25)}$$

$$= 0.25 - \frac{(0.25)^3 - 2(0.25) + 0.5}{3(0.25)^2 - 2}$$

$$= 0.25 - \frac{0.0156 - 0.5 + 0.5}{0.1875 - 2}$$

$$= 0.25 - \frac{0.0156}{-1.8125}$$

$$= 0.25 + \frac{0.0156}{1.8125}$$

$$= 0.25 + 0.0060$$

$$= 0.2586.$$

(3) Find the positive root of the equation $x^3 - 2x^2 - 3x - 4 = 0$ correct to 2 decimal

Given:-

$f(x) = x^3 - 2x^2 - 3x - 4 = 0$

$f'(x) = 3x^2 - 4x - 3$

x	-4	-3	-2	-1	0	1	2	3	4
f(x)		-40	-14	-4	-4	-7	-10	-4	16

$x_0 = 3$

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$n=1$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{3^3 - 2(3)^2 - 3(3) - 4}{3(3)^2 - 4(3) - 3}$$

$$= 3 - \frac{(-4)}{12}$$

$$= 3 + \frac{1}{3}$$

$$\boxed{x_1 = 3.3333}$$

$n=2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 3.333 - \frac{f(3.333)}{f'(3.333)}$$

$$= 3.333 - \frac{(3.333)^3 - 2(3.333)^2 - 3(3.333}{3(3.333)^2 - 4(3.333) - 3}$$

$$= 3.333 - \frac{0.8482}{16.99}$$

$$= 3.333 - 0.049$$

$$= 3.293.$$

03/10/2023

FINITE DIFFERENCES:-

Interpolation:-

Suppose, in a experiment, corresponding to the n+1 values

$$x_0, x_1, x_2 \ldots x_n \longrightarrow (1)$$

of a quantity x, the observed n+1 values of another quantity y are

$$y_0, y_1, y_2, \ldots y_n \longrightarrow (2)$$

There may be a relationship between x and y. when this relationship is not known explicity, a function $f(x)$ is found based on the values (1) & (2) such that the equation

$$y = f(x) \longrightarrow (3)$$

will give an approximate value for y corresponding to an x other than (1). If x lies within the range of (1) then this method finding y is called Interpolation. If x lies outside the range of (1), then the method is called extrapolation.

Nomenclature:

$x_0, x_1, x_2, \ldots x_n$ are called arguments

$y_0, y_1, y_2, \ldots y_n$ are called entries.

$y = f(x)$ is called a formula of interpolation or extrapolation.

Forward differences (f.d.'s) . suppose the x values are in the increasing order and are equally spaced, that

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1},$$

Then the n numbers.

$$y_1 - y_0,\ y_2 - y_1,\ y_3 - y_2, \cdots y_n - y_{n-1}$$

are defined to be the first order f.d's of the given $n+1$ entries.

$$y_0, y_1, y_2, \cdots, y_n.$$

and, using Δ (delta), are denoted

$$\Delta y_0, \Delta y_1, \Delta y_2 \cdots \Delta y_{n-1}.$$

i.e, $\Delta y_i = y_{i+1} - y_i$ or $\Delta y = (\text{next } y) - y$

Note that Δy_n is not known because y_{n+1} is not given.

Then $n-1$ first order f.d's of $\Delta y_0, \Delta y_1, \cdots, \Delta y_{n-1}$, namely.

$$(\Delta y_1) - (\Delta y_0), (\Delta y_2) - (\Delta y_1) \cdots$$

$$(\Delta y_{n-1}) - (\Delta y_{n-2}) \cdots$$

are defined to be the second order f.d's of the same $n+1$ values of y, namely $y_0, y_1, \cdots, y_n$ and are denoted by,

$$\Delta^2 y_0, \Delta^2 y_1, \cdots \Delta^2 y_{n-2}$$

Similarly, for the same $n+1$ values $y_0, y_1, \cdots, y_n$ of y, $\Delta^3 y_0, \Delta^3 y_1, \cdots,$

$\Delta^3 y_{n-3}$. are called the third order f.d's.

Forward difference table. The f.d.'s of all orders can be displayed in a tabular form. the respective table is called the f.d table.

For example, the following table is the f.d tables for 4 pairs of values of x and y.

x	y	Δ	Δ^2	Δ^3
x_0	y_0			
x_1	y_1	$\Delta y_0 = y_1 - y_0$		
x_2	y_2	$\Delta y_1 = y_2 - y_1$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$
x_3	y_3	$\Delta y_2 = y_3 - y_2$	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$	

In this case, to interpalate a value by Newton's by forward formula, we will require.

$$y_0, \Delta y_0, \Delta^2 y_0, \Delta^3 y_0.$$

which are the top elements of columns 2, 3, 4, 5.

Backward differences (b.d's). Then numbers

$$y_1 - y_0, \; y_2 - y_1, \; y_3 - y_2 \cdots, y_n - y_{n-1}$$

are defined as the first order bd's of $y_0, y_1, y_2 \cdots y_n$.

and using ∇ (del or nabla) are denoted by

$\nabla y_1 = y_1 - y_{i-1}$ i.e., $\nabla y = y -$ (previous)

Note that, ∇y_0 is not known because y_{-1} is not given.

The first order b.d's of $\nabla y_1, \nabla y_2 \ldots \nabla y_n$ namley

$(\nabla y_2) - (\nabla y_1); (\nabla y_3) - (\nabla y_2) \ldots (\nabla y_n) - (\nabla y_{n-1})$

are called the second order b.d's of $y_0, y_1, y_2 \ldots y$. and denoted by

$\nabla^2 y_2, \nabla^2 y_3 \ldots \nabla^2 y_n$

similarly the third order b.d's are

$\nabla^3 y_3, \nabla^3 y_4, \ldots \nabla^3 y_n$.

Backward difference table. As an example, the b.d table for 4 pairs of values of x & y is given below.

x	y	∇	∇^2	∇^3
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_2	y_2			
		$\nabla y_3 = y_3 - y_2$	$\nabla^2 y_3 = \Delta y_3 - \nabla y_2$	
x_3	y_3			

In this case to interpolate a value by Newton's backward formula, we will require,

$$y_3, \nabla y_3, \nabla^2 y_3, \nabla^3 y_3,$$

which are the bottom elements of coloumns 2, 3, 4, 5.

(Operator Δ). In general, for any value of x, the operator Δ is defined by. forward

$$\boxed{\Delta f(x) = f(x+h) - f(x)}.$$

In particular,

$$\Delta y_0 = \Delta f(x) = f(x_0+h) - f(x_0) = y_1 - y_0,$$

$$\Delta^2 y_0 = \Delta[\Delta y_0] = \Delta[y_1 - y_0] = \Delta y_1 - \Delta y_0.$$

$$= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0,$$

$$\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0$$

Note, The coefficients in $\Delta^3 y_0$ are the coefficients of $(1-x)^3$, and the coefficients in $\Delta^4 y_0$ will be the coefficients of $(1-x)^4$, etc.

Operator (∇) backward.

- In general, for any value of x, the operator ∇ is defined by

$$\boxed{\nabla f(x) = f(x) - f(x-h)}$$

In particular,

$\nabla y_n = \nabla f(x_n) = f(x_n) - f(x_{n-1}) = y_n - y_{n-1}$

$\nabla^2 y_n = \nabla[\nabla y_n] = \nabla[y_n - y_{n-1}] = \nabla y_n - \nabla y_{n-1}$

$= (y_n - y_{n-1}) - (y_{n-1} - y_{n-2}) = y_n - 2y_{n-1} + y_{n-2}$

$\nabla^3 y_n = y_n - 3y_{n-1} + 3y_{n-2} - y_{n-3}$.

Operator (E): the operator E is defined such that its operator on the y value at x yields the y value at x+h.

Thus, in general,

$$\boxed{E f(x) = f(x+h).}$$

In particular,

$E(y_0) = E f(x_0) = f(x_0 + h) = y_1$

$E(y_1) = y_2,\ E(y_2) = y_3 \cdots E(y_{n-1}) = y_n$

Also,

$E^2(y_0) = E[E(y_0)] = E(y_1) = y_2$

$E^3(y_0) = y_3,\ E^4(y_0) = y_4, \ldots E^n(y_0) = y_n$

Example 1. Find the missing y_x value in the table.

y_x	0	–	–	–	–	–
Δy_x	0	1	2	4	7	11

(2004/05)

with usual notations, the 2nd column in the following table gives first differences but we require the values of y_x, namely,

$y_1, y_2, y_3, \ldots y_6$

Given that $y_0 = 0$.

1/10/2023

(11) Find the missing yx value in the table.

yx	0	y1	y2	y3	y4	y5
Δyx	Δy0 0	Δy1 1	Δy2 2	Δy3 4	Δy4 7	Δy5 11

Soln:-

$$\Delta y_i = y_{i+1} - y_i$$

From the table.

$y_0 = 0$, $y_1 = ?$, $y_2 = ?$, $y_3 = ?$, $y_4 = ?$, $y_5 = ?$

$\Delta y_0 = 0$, $\Delta y_1 = 1$, $\Delta y_2 = 2$, $\Delta y_3 = 4$, $\Delta y_4 = 7$, $\Delta y_5 = 11$.

$i = 0$

$$\Delta y_0 = y_1 - y_0.$$

$$0 = y_1 - 0$$

$$\boxed{y_1 = 0}.$$

$i = 1$,

$$\Delta y_1 = y_2 - y_1$$

$$1 = y_2 - 0$$

$$\boxed{y_2 = 1}$$

$i = 2$,

$$\Delta y_2 = y_3 - y_2$$

$$2 = y_3 - 1$$

$$\boxed{y_3 = 3}$$

$i = 3$.

$$\Delta y_3 = y_4 - y_3$$

$$4 = y_4 - 3$$

$$\boxed{y_4 = 7}$$

$i = 4$.

$\Delta y_4 = y_5 - y_4$

$7 = y_5 - 7$

$\boxed{y_5 = 14}$.

y_x	0	0	1	3	7	14
Δy_x	0	1	2	4	7	11.

(2) Given the following values of x and y

x	0	1	2	3	4	5
y	3	12	81	200	100	8

Find $\Delta^5 y_0$.

Soln:-

x	y_1	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$.
0	$3 = y_0$	$\Delta y_0 = y_1 - y_0$ $= 12 - 3$ $= 9$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$ $= 69 - 9$ $= 60$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$ $= 50 - 60$ $= -10$.	$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$ $= -269 + 10$ $= -259$.	$\Delta^5 y_0 = \Delta^4 y_1 - \Delta^4 y_0$ $= 496 + 259$ $= 755$.
1	$12 = y_1$	$\Delta y_1 = y_2 - y_1$ $= 81 - 12$ $= 69$.	$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$ $= 119 - 69$ $= 50$	$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$ $= -219 - 50$ $= -269$.	$\Delta^4 y_1 = \Delta^3 y_2 - \Delta^3 y_1$ $= 227 + 269$ $= 496$	$\Delta^5 y_0 = \cancel{755}$
2	$81 = y_2$	$\Delta y_2 = y_3 - y_2$ $= 200 - 81$ $= 119$.	$\Delta^2 y_2 = \Delta y_3 - \Delta y_2$ $= -100 - 119$ $= -219$.	$\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$ $= -\cancel{8} + 219$ $= 227$		
3	$200 = y_3$	$\Delta y_3 = y_4 - y_3$ $= 100 - 200$ $= -100$	$\Delta^2 y_3 = \Delta y_4 - \Delta y_3$ $= -92 - 100$ $= \cancel{[illegible]}\ 8$			
4	$100 = y_4$	$\Delta y_4 = y_5 - y_4$ $= 8 - 100$ $= -92$				
5	$8 = y_5$					

11/10/23 Derivation of Newton's forward formula

(4) Let the value x and y be

$$x_0, x_1, x_2, \ldots, x_n$$

$$y_0, y_1, y_2, \ldots, y_n$$

where x values are increasing order

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 \ldots = x_n - x_{n-1} = h$$

(positive quantity)

Let $f(x)$ be a polynomial of degree n

then $f(x_0) = y_0, f(x_1) = y_1, \ldots, f(x_n) = y_n$

Now, $f(x)$ can be written as:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

In the setting $x = x_0$ and $f(x_0) = y_0$

$$f(x_0) = a_0.$$

$$y_0 = a_0$$

$$\Rightarrow \boxed{a_0 = y_0}$$

$$f(x) = y_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots a_n(x - x_0)\ldots(x_n - x_{n-1})$$

Let $x = x_1$, $f(x_1) = y_1$.

$$f(x_1) = y_0 + a_1(x_1 - x_0).$$

$$y_1 = y_0 + a_1(x_1 - x_0)$$

$$y_1 - y_0 = a_1(x_1 - x_0)$$

$$a_1 = \frac{y_1 - y_0}{(x_1 - x_0)} = \frac{\Delta y_0}{h}.$$

$$f(x) = y_0 + \left(\frac{\Delta y_0}{h}\right)(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

Let $x = x_2, f(x_2) = y_2$.

$$f(x_2) = y_0 + \left(\frac{\Delta y_0}{h}\right)(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$y_2 = y_0 + \left(\frac{\Delta y_0}{h}\right)(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$y_2 - y_0 = \left(\frac{\Delta y_0}{h}\right)(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$y_2 - y_0 = (x_2 - x_0)\left[\frac{\Delta y_0}{h} + a_2(x_2 - x_1)\right]$$

$$\frac{y_2 - y_0}{x_2 - x_0} = \frac{\Delta y_0}{h} + a_2 h.$$

$$\frac{y_2 - y_0}{x_2 - x_0} = \frac{\Delta y_0 + a_2 h^2}{h}$$

$$\frac{y_2 - y_0}{x_2 - x_1 + x_1 - x_0} = \frac{\Delta y_0 + a_2 h^2}{h}$$

$$\frac{y_2 - y_0}{h + h} = \frac{\Delta y_0 + a_2 h^2}{h}.$$

$$\frac{y_2 - y_0}{2\cancel{h}} = \frac{\Delta y_0 + a_2 h^2}{\cancel{h}}$$

$$y_2 - y_0 = 2(\Delta y_0 + a_2 h^2).$$

$$y_2 - y_0 = 2\Delta y_0 + 2a_2 h^2.$$

$\Delta y_0 = y_1 - y_0$ $\quad y_2 - y_0 - 2\Delta y_0 = 2a_2 h^2$

$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$ $\quad y_2 - y_0 - 2(y_1 - y_0) = 2a_2 h^2$.

$= y_2 - y_1 - (y_1 - y_0)$ $\quad y_2 - y_0 - 2y_1 + 2y_0 = 2a_2 h^2$.

$= y_2 - y_1 - y_1 + y_0$ $\quad y_2 - 2y_1 + y_0 = 2a_2 h^2$

$= y_2 - 2y_1 + y_0.$

$$\Delta^2 y_0 = 2a_2 h^2$$

$$a_2 = \frac{\Delta^2 y_0}{2h^2}.$$

$$\text{lly} \quad a_3 = \frac{\Delta^3 y_0}{3!\,h^3}$$

$$a_4 = \frac{\Delta^4 y_0}{4!\,h^4}.$$

$$a_n = \frac{\Delta^n y_0}{n!\, h^n}.$$

$$\therefore f(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!\, h^2}(x - x_0)(x - x_1) + \dots$$

$$+ \frac{\Delta^n y_0}{n!\, h^n}(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

Further denoting $\frac{x - x_0}{h} = u$.

$$\Rightarrow (x - x_0) = hu.$$

$$x - x_1 = x - x_0 + x_0 - x_1$$

$$= (x - x_0) - (x_1 - x_0)$$

$$= hu - h$$

$$= h(u - 1)$$

$$x - x_2 = x - x_0 + x_0 - x_2$$

$$= (x - x_0) - (x_2 - x_0)$$

$$= hu - [x_2 - x_1 + x_1 - x_0]$$

$$= hu - [(x_2 - x_1) + (x_1 - x_0)]$$

$$= hu - [h + h) = hu - 2h.$$

$$= h(u - 2)$$

IIIly

$$x - x_{n-1} = h[u - (n-1)]$$

$\therefore$ The Newton's forward difference

formula

$$f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \dots + \frac{u(u-1)\dots(u-(n-1))}{n!}$$

Where $u = \frac{x - x_0}{h}.$

(2) Derivation of Newton's backward formula.

12/10/23 Thursday

Let the function:

$$f(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \dots + a_n(x - x_n)(x - x_{n-1})\dots(x - x_1)$$

$$f(x_0) = y_0, \; f(x_1) = y_1, \dots f(x_n) = y_n$$

Let $x = x_n$

$f(x_n) = a_0$

$\Rightarrow y_n = a_0$

$\boxed{a_0 = y_n}$

$$f(x) = y_n + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \dots + a_n(x - x_n)(x - x_{n-1})(x - x_1)$$

$x = x_{n-1}$

$$f(x_{n-1}) = y_n + a_1(x_{n-1} - x_n)$$

$$y_{n-1} = y_n + a(-h)$$

$$y_{n-1} - y_n = a(-h)$$

$$-\nabla y_n = a(-h)$$

$$a_1 = \frac{\nabla y_n}{h}$$

$$f(x) = y_n + \frac{\nabla y_n}{h}(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \dots + a_n(x - x_n)(x - x_{n-1})\dots(x - x_1)$$

$x = x_{n-2}$

$$f(x_{n-2}) = y_n + \frac{\nabla y_n}{h}(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$y_{n-2} - y_n = (x_{n-2} - x_n)\left[\frac{\nabla y_n}{h} + a_2(-h)\right]$$

$$\frac{y_{n-2} - y_n}{x_{n-2} - x_n} = \frac{\nabla y_n}{h} + a_2(-h)$$

$$\frac{y_{n-2} - y_n}{(x_{n-2} - x_{n-1}) + (x_{n-1} - x_n)} = \frac{\nabla y_n}{h} + a_2(-h)$$

$$\frac{y_{n-2} - y_n}{(-h)+(-h)} - \frac{\nabla y_n}{h} = a_2(-h)$$

$$\frac{y_{n-2} - y_n}{-2h} - \frac{y_n - y_{n-1}}{h} = a_2(-h)$$

$$\frac{1}{-h}\left[\frac{y_{n-2} - y_n + 2y_n - 2y_{n-1}}{2}\right] = a_2(-h)$$

$$-\frac{1}{h}\left[\frac{y_{n-2} - y_n}{2} + \frac{y_n - y_{n-1}}{1}\right]$$

$$-\frac{1}{2h}\left[y_{n-2} + y_n - 2y_{n-1}\right] = a_2(+h)$$

$$\frac{1}{2h}\nabla^2 y_n = a_2 h$$

$$a_2 = \frac{\nabla^2 y_n}{2!\,h^2}$$

III^ly

$$a_3 = \frac{\nabla^3 y_n}{3!\,h^3}$$

$$a_n = \frac{\nabla^n y_n}{n!\,h^n}$$

$$\therefore f(x) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!\,h^2}(x - x_n)(x - x_{n-1}) + \cdots + \frac{\nabla^n y_n}{n!\,h^n}(x - x_n)(x - x_{n-1})\cdots(x - x_1)$$

Further, denoting

$$\frac{x - x_n}{h} = u$$

$$\Rightarrow x - x_n = hu$$

Now $x - x_{n-1} = x - x_n + x_n - x_{n-1}$

$= (x - x_n) + (x_n - x_{n-1})$

$= hu + h = h(u+1)$.

$x - x_{n-2} = x - x_n + x_n - x_{n-2}$

$= (x - x_n) + (x_n - x_{n-2})$

$= hu + (x_n - x_{n-1} + x_{n-1} - x_{n-2})$.

$= hu + (h+h) = hu + 2h$.

$x - x_{n-2} = h(u+2)$

|||ly $x - x_{n-3} = h(u+3)$.

$x - x_1 = h[u + (n-1)]$.

formula

$$\therefore f(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y + \dots$$

$$\dots + \frac{u(u+1) \quad (u+(n-1))}{n!}\nabla^n y_n$$

$\therefore$ where $u = \dfrac{x - x_n}{h}$.

13/10/2023

(3) Derivative of Lagrange's formula.

If $x_0, x_1, \dots \dots x_n$ are not equally spaced, then Langrange's interpolation formula given an n^{th} degree polynomial equation $y = f(x)$ which is satisfied by all the pairs of x and y values

Here $f(x)$ can be assumed as

$$f(x) = a_0 (x - x_1)(x - x_2) \dots (x - x_n)$$

$$+ a_1 (x - x_0)(x - x_2) \dots (x - x_n).$$

$$\vdots$$

$$+ a_n (x - x_0)(x - x_1) \dots (x - x_{n-1}).$$

let $x_1 = x_0$ and $f(x_0) = y_0$ then.

$$f(x_0) = a_0 (x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)$$

$$y_0 = a_0 (x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)$$

$$a_0 = \frac{y_0}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)}$$

let $x = x_1$, $f(x_1) = y_1$ then.

$$f(x_1) = a_1 (x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)$$

$$y_1 = a (x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)$$

$$a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)}$$

|||ly

$$a_2 \quad \frac{y_2}{(x_2 - x_0)(x_2 - x_1) \cdots (x_2 - x_n)}$$

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})}$$

Thus,

$$f(x) = \frac{y_0 (x - x_1)(x - x_2) \cdots (x - x_n)}{(x_0 - x_1)(x_0 - x_2) \cdots (x_0 - x_n)}$$

$$+ \frac{y_1 (x - x_0)(x - x_2) \cdots (x - x_n)}{(x_1 - x_0)(x_1 - x_2) \cdots (x_1 - x_n)}$$

$$\vdots$$

$$+ \frac{y_n (x - x_0)(x - x_1) \cdots (x - x_{n-1})}{(x_n - x_0)(x_n - x_1) \cdots (x_n - x_{n-1})}$$

Then

–	x	x_0	x_1		x_n	Product
x	–	$x-x_0$	$x-x_1$		$x-x_n$	p
x_0	x_0-x	—	x_0-x_1	—	x_0-x_n	p_0
x_1	x_1-x	x_1-x_0	—	—	x_1-x_n	p_1
				—		
x_n	x_n-x	x_n-x_0	x_n-x_1		—	p_n

$$f(x) = -\left[y_0\frac{p}{p_0} + y_1\frac{p}{p_1} + y_2\frac{p}{p_2} + \cdots + y_n\frac{p}{p_n}\right]$$

formula

$$\left(= -p\left[\frac{y_0}{p_0} + \frac{y_1}{p_1} + \frac{y_2}{p_2} + \cdots + \frac{y_n}{p_n}\right]\right)$$

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(1) It is given that

x	40_{x_0}	50_{x_1}	60_{x_2}	70_{x_3}	80_{x_4}
y	37	59	63	8	10.2

Find the value of y corresponding to $x = 45$, using Newton's forward formula.

$$f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots +$$

$$\frac{u(u-1)\cdots[u-(n-1)]}{n!}\Delta^n y_0$$

$$f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 +$$

$$\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

16/10/23
Monday

1) Apply Newton's backward difference formula to find a polynomial of degree 3, using the table given below.

x	3	4	5	6
y	6	24	60	120

$$f(x) = y_n + \frac{u}{1!}\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \dots + \frac{u(u+1)\dots[u+(n-1)]}{n!}\cdot\nabla^n y_n$$

$$f(x) = y_3 + \frac{u}{1!}\nabla y_3 + \frac{u(u+1)}{2!}\nabla^2 y_3 + \frac{u(u+1)(u+2)}{3}\nabla^3 y_3.$$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
3	6			
		$\nabla y_1 = 24-6 = 18$		
4	24		$\nabla^2 y_2 = 36+8 = 18$	
		$\nabla y_2 = 60-24 = 36$		$\nabla^3 y_3 = 24-18 = 6$
5	60		$\nabla^2 y_3 = 60-36 = 24$	
		$\nabla y_3 = 120-60 = 60$		
6	120			

$$f(x) = 120 + u(60) + \frac{u(u+1)}{2!} + \frac{u(u+1)}{2!}24 + \frac{u(u+1)(u+2)}{3!}(6).$$

$$u = \frac{x-x_n}{h} = x - x_3 - x - 6.$$

$$\boxed{u = x - 6}.$$

$$f(x) = 120 + (x-6)(60) + \frac{(x-6)(x-6+1)}{2!}(24) + \frac{(x-6)(x-6+1)(x-6+2)}{2!}(6)$$

$= 120 + 60x - 360 + \frac{(x-6)(x-5)}{2}(24)$

$+ (x-6)\frac{(x-5)(x-4)}{6}(6)$

$= -240 + 60x + [x^2 - 5x - 6x + 30](12)$

$+ (x^2 - 5x - 6x + 30)$

$+ [x^2 - 5x - 6x + 30][x-4]$

$= 240 + 60x + [x^2 - 11x + 30][12]$

$+ [x^2 - 11x + 30][x-4]$

$f(x) = -240 + 60x + 12x^2 - 132x + 360 +$

$[x^3 - 4x^2 - 11x^2 + 44x + 30x - 120]$

$= 120 - 72x + 12x^2 + [x^3 - 15x^2 +$

$74x - 120]$

$= \cancel{120} - 72x + 12x^2 + x^3 - 15x^2 + 74x - \cancel{120}$

$f(x) = x^3 - 3x^2 + 2x.$

(2) It is given that:

x	40	50	60	70	80
y	3.7	4.9	6.3	8	10.2

Find the value of corresponding to $x = 45$, using Newton's formulas.

<u>Soln:-</u>

$f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \dots +$

$\frac{u(n-1)(u-(n-1))}{n!}\Delta^n y_0$

$u = \frac{x - x_0}{h} = \frac{45 - 40}{10} = \frac{5}{10} = \frac{1}{2} = 0.5$

$f(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 +$

$$\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$f(x) = y_0 + \frac{0.5}{1!}\Delta y_0 + \frac{(0.5)(0.5-1)}{2!}\Delta^2 y_0 +$$

$$\frac{(0.5)(0.5-1)(0.5-2)}{1\times 2\times 3!}\Delta^3 y_0 + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}\Delta^4 y_0$$

$$f(x) = y_0 + 0.5\,\Delta y_0 + \frac{(0.5)(-0.5)}{2!}\Delta^2 y_0$$

$$+ \frac{(0.5)(-0.5)(-1.5)}{3!}\Delta^3 y_0 + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{4!}$$

$$1\times 2\times 3\times 4 \quad \Delta^4 y_0$$

$$= y_0 + (0.5)\Delta y_0 + \left(\frac{(-0.25)}{2}\right)\Delta^2 y_0 + \frac{0.375}{6}\Delta^3 y_0$$

$$+ \frac{(-0.9375)}{24}\Delta^4 y_0$$

$$= y_0 + (0.5)\Delta y_0 - 0.125\,\Delta^2 y_0 + 0.0625\,\Delta^3 y_0$$

$$- 0.0390\,\Delta^4 y_0.$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	3.7	$y_1 - y_0$			
		$\Delta y_0 = 1.2$			
50	4.9		$\Delta^2 y_0 = 0.2$		
		$\Delta y_1 = 1.4$		$\Delta^3 y_0 = 0.1$	
60	6.3		$\Delta^2 y_1 = 0.3$		$\Delta^4 y_0 = 0.1$
		$\Delta y_2 = 1.7$		$\Delta^3 y_1 = 0.2$	
			$\Delta^2 y_2 = 0.5$		
70	8	$\Delta y_3 = 2.2$			
80	10.2				

$$f(45) = 3.7 + (0.5)(1.2) - (0.125)(0.2)$$

$$+ (0.0625)(0.1) - (0.0390)(0.1)$$

$$= 3.7 + 0.6 - 0.025$$

$+0.00625 - 0.0390.$

$f(45) = 4.242$

18/10/23

(3) using newton's formula, find the values of y when $x=27$, from the following data.

x	10	15	20	25	30
y	35.4	32.2	29.1	26.6	23.1.

backward.

Given:-

$$f(x) = y_4 + \frac{u}{1!}\nabla y_4 + \frac{u(u+1)}{2!}\nabla^2 y_4 + \frac{u(u+1)(u+2)}{3!}\nabla^3 y + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_4$$

$$u = \frac{x - x_n}{h}$$

$x_n = 30$

$x = 27$

$h = 5$

$$u = \frac{27-30}{5}$$

$$= \frac{-3}{5}$$

$\boxed{u = -0.6}$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	35.4				
		-3.2			
15	32.2		0.1		
		-3.1		-0.1	0.3
20	29.1		0		
		-3.1		0.2	
25	26.6		0.2		
		-2.9			
30	23.1				

$f(27) = 28.1 + \frac{(-0.6)}{1!}(-2.934) + \frac{(-0.6)(-0.6+1)}{2!}(0.2) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(0.3)$

$= 28.1 + 1.74 - 0.024 - 0.112 - 0.01008$

$= 24.79472$

(4) Using Lagrange's formula, find $\log_{10} 301$ from the following table when x and $\log x$ values are given by

x	300	304	305	307
$\log_{10} x$	2.4771	2.4829	2.4843	2.4871

Given:-

$x = 301$, $x_0 = 300$, $x_1 = 304$, $x_2 = 305$, $x_3 = 307$, $f(x) = -p\left[\frac{y_0}{p_0} + \frac{y_1}{p_1} + \frac{y_2}{p_2} + \frac{y_3}{p_3}\right]$

–	x	x_0	x_1	x_2	x_3	product
–						p
x	–	$x-x_0$	$x-x_1$	$x-x_2$	$x-x_3$	p_0
x_0	x_0-x	–	x_0-x_1	x_0-x_2	x_0-x_3	p_1
x_1	x_1-x	x_1-x_0	–	x_1-x_2	x_1-x_3	p_2
x_2	x_2-x	x_2-x_0	x_2-x_1	–	x_2-x_3	p_3
x_3	x_3-x	x_3-x_0	x_3-x_1	x_3-x_2	–	

—	301	300	304	305	307	product
301	2	1	-3	-4	-6	P-72
300	[illegible]	[illegible]	-4	-5	-7	$P_0 = 140$
304	3	4	—	-1	-3	$P_1 = 36$
305	4	5	1	—	-2	$P_2 = -40$
307	6	7	3	2	—	$P_3 = 252$

$$\log_{10} 301 = -(-72)\left[\frac{2.4771}{140} + \frac{2.4829}{36} - \frac{2.4843}{40} + \frac{2.4871}{252}\right]$$

$$= 72(0.0176935 + 0.0689694 - 0.0621075 + 0.0098694)$$

$$= 72(0.0344248)$$

$$\log_{10} 301 = 2.4785856$$

Hence proved.

i

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

$2^1 \times 2^2$

4^2

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1