

MARUDHAR KESARI JAIN COLLEGE FOR WOMEN, VANIYAMBADI
PG & RESEARCH DEPARTMENT OF MATHEMATICS

CLASS : I B.Sc PHYSICS
SUBJECT CODE : MATHEMATICS I
SUBJECT NAME : 23UEMA10C

SYLLABUS

UNIT- IV

Trigonometry

Expansions of $\sin\theta$, $\cos\theta$ in a series of powers of $\sin\theta$ and $\cos\theta$ - Expansions of $\sin(n\theta)$ and $\cos(n\theta)$ in a series sines and cosines of multiples of “ θ ” - Expansions of $\sin\theta$, $\cos\theta$ and $\tan\theta$ in a series of powers of “ θ ” – Hyperbolic and inverse hyperbolic functions .

UNIT-4.

TRIGONOMETRY.

Expansions of $\sin^n \theta$, $\cos^n \theta$, $\sin n\theta$, $\cos n\theta$, $\tan n\theta$ - Expansions of $\sin \theta$, $\cos \theta$, $\tan \theta$ in terms of θ .

$$\text{Let, } x = \cos \theta + i \sin \theta \quad \text{--- (1)}$$

$$\frac{1}{x} = \cos \theta - i \sin \theta \quad \text{--- (2)}$$

$$\text{(1) + (2)} \Rightarrow x + \frac{1}{x} = 2 \cos \theta.$$

$$\text{(1) - (2)} \Rightarrow x - \frac{1}{x} = 2i \sin \theta.$$

Also,

$$x^n = \cos n\theta + i \sin n\theta \quad \text{--- (3)}$$

$$\frac{1}{x^n} = \cos n\theta - i \sin n\theta \quad \text{--- (4)}$$

$$\text{(3) + (4)} \Rightarrow x^n + \frac{1}{x^n} = 2 \cos n\theta.$$

$$\text{(3) - (4)} \Rightarrow x^n - \frac{1}{x^n} = 2i \sin n\theta.$$

NOTE:

$$1. (x+a)^n = nC_0 x^n a^0 + nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \dots + nC_n x^0 a^n.$$

$$2. (x-a)^n = nC_0 x^n a^0 - nC_1 x^{n-1} a^1 + nC_2 x^{n-2} a^2 + \dots + nC_n x^0 a^n.$$

$$3. (1+x)^n = 1 + \frac{nx}{1} + \frac{n(n-1)x^2}{2} + \frac{n(n-1)(n-2)x^3}{6} + \dots$$

$$4. (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$5. (1+x)^{-1} = 1 - x + x^2 - \dots + (-1)^n x^n + \dots$$

$$6. e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$7. e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} + \dots$$

$$8. \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$9. \frac{e^x - e^{-x}}{2} = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$10. nCr = \frac{n!}{r!(n-r)!} = \frac{n!}{r! (n-r)!}$$

$$11. nC_0 = 1, nC_n = 1; nC_1 = n; nC_{n-1} = n$$

$$12. \sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots$$

Examples:

$$1. \text{ State that } 2^5 \cos^6 \theta = \cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10$$

Sol:

$$\text{Let } x = \cos \theta + i \sin \theta$$

$$\frac{1}{x} = \cos \theta - i \sin \theta$$

$$\Rightarrow x + \frac{1}{x} = 2 \cos \theta \text{ and } x^n + \frac{1}{x^n} = 2 \cos n\theta$$

Consider,

$$2^5 \cos^6 \theta = \frac{1}{2} (2^6 \cos^6 \theta)$$

$$= \frac{1}{2} (2 \cos \theta)^6$$

$$= \frac{1}{2} \left(x + \frac{1}{x} \right)^6 \Rightarrow (x+a)^n$$

$$2^5 \cos^6 \theta = \frac{1}{2} \left[6C_0 x^6 \left(\frac{1}{x} \right)^0 + 6C_1 x^{6-1} \left(\frac{1}{x} \right)^1 + 6C_2 x^{6-2} \left(\frac{1}{x} \right)^2 + 6C_3 x^{6-3} \left(\frac{1}{x} \right)^3 + 6C_4 x^{6-4} \left(\frac{1}{x} \right)^4 + 6C_5 x^{6-5} \left(\frac{1}{x} \right)^5 + 6C_6 x^{6-6} \left(\frac{1}{x} \right)^6 \right]$$

$$\frac{3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 1} \quad \frac{2 \cdot 1 \cdot 0}{1 \cdot 2 \cdot 1}$$

$$= \frac{1}{2} \left[x^6 + 6x^5 \left(\frac{1}{x} \right) + 15x^4 \left(\frac{1}{x^2} \right) + 20x^3 \left(\frac{1}{x^3} \right) + 15x^2 \left(\frac{1}{x^4} \right) + 6x \left(\frac{1}{x^5} \right) + \frac{1}{x^6} \right]$$

$$\frac{3 \cdot 2 \cdot 1 \cdot 0 \cdot 0}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$$

$$= \frac{1}{2} \left[x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6} \right]$$

$$= \frac{1}{2} \left[\left(x^6 + \frac{1}{x^6} \right) + 6 \left(x^4 + \frac{1}{x^4} \right) + 15 \left(x^2 + \frac{1}{x^2} \right) + 20 \right]$$

$$= \frac{1}{2} \left[2 \cos 6\theta + 6(2 \cos 4\theta) + 15(2 \cos 2\theta) + 20 \right]$$

$$= \frac{2}{2} [\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10]$$

$$2^5 \cos^6 \theta = \cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10$$

$$2. \quad 2^6 \cos^7 \theta = \cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos \theta$$

state that.

Sol:

$$\text{Let } x = \cos \theta + i \sin \theta$$

$$\frac{1}{x} = \cos \theta - i \sin \theta$$

$$\Rightarrow x + \frac{1}{x} = 2 \cos \theta \text{ and } x^n + \frac{1}{x^n} = 2 \cos n\theta$$

Consider,

$$2^6 \cos^7 \theta = \frac{1}{2} [2^7 \cos^7 \theta].$$

$$= \frac{1}{2} [2 \cos \theta]^7 \Rightarrow \frac{1}{2} \left[x + \frac{1}{x} \right]^7.$$

$$2^6 \cos^7 \theta = \frac{1}{2} \left[7C_0 x^7 \left(\frac{1}{x} \right)^0 + 7C_1 x^7 \left(\frac{1}{x} \right)^1 + 7C_2 x^7 \left(\frac{1}{x} \right)^2 + 7C_3 x^{7-3} \left(\frac{1}{x} \right)^3 + 7C_4 x^{7-4} \left(\frac{1}{x} \right)^4 + 7C_5 x^{7-5} \left(\frac{1}{x} \right)^5 + 7C_6 x^{7-6} \left(\frac{1}{x} \right)^6 + 7C_7 x^{7-7} \left(\frac{1}{x} \right)^7 \right].$$

$$2^6 \cos^7 \theta = \frac{1}{2} \left[x^7 + 7x^6 \left(\frac{1}{x} \right) + 21x^5 \left(\frac{1}{x^2} \right) + 35x^4 \left(\frac{1}{x^3} \right) + 35x^3 \left(\frac{1}{x^4} \right) + 21x^2 \left(\frac{1}{x^5} \right) + 7x \left(\frac{1}{x^6} \right) + \frac{1}{x^7} \right]$$

$$= \frac{1}{2} \left[\left(x^7 + \frac{1}{x^7} \right) + 7x^5 + 21x^3 + 35x + \frac{35}{x} + \frac{21}{x^3} + \frac{7}{x^5} \right]$$

$$= \frac{1}{2} \left[\left(x^7 + \frac{1}{x^7} \right) + 7 \left(x^5 + \frac{1}{x^5} \right) + 21 \left(x^3 + \frac{1}{x^3} \right) + 35 \left(x + \frac{1}{x} \right) \right].$$

$$= \frac{1}{2} [2 \cos 7\theta + 7(2 \cos 5\theta) + 21(2 \cos 3\theta) + 35(2 \cos \theta)].$$

$$= \frac{2}{2} [\cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos \theta]$$

$$2^6 \cos^7 \theta = [\cos 7\theta + 7 \cos 5\theta + 21 \cos 3\theta + 35 \cos \theta].$$

Expansion of $\cos n\theta$, $\sin n\theta$ in powers of $\sin \theta$, $\cos \theta$.

If n is a positive integer

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta \quad \text{--- (1)}$$

By Binomial Theorem,

$$(x+y)^n = x^n + nC_1 x^{n-1} y + nC_2 x^{n-2} y^2 + \dots + y^n$$

Here $x = \cos \theta$, $y = i \sin \theta$.

$$(\cos \theta + i \sin \theta)^n = \cos^n \theta + nC_1 \cos^{n-1} \theta i \sin \theta + nC_2 \cos^{n-2} \theta i^2 \sin^2 \theta + \dots + i^n \sin^n \theta.$$

$$(\cos \theta + i \sin \theta)^n = \cos^n \theta + i nC_1 \cos^{n-1} \theta \sin \theta - nC_2 \cos^{n-2} \theta \sin^2 \theta + \dots + i^n \sin^n \theta.$$

Applying eqn (1).

$$(\cos n\theta + i \sin n\theta) = \cos^n \theta + i nC_1 \cos^{n-1} \theta \sin \theta - nC_2 \cos^{n-2} \theta \sin^2 \theta + \dots + i^n \sin^n \theta.$$

Equating real and imaginary Part.

$$\cos n\theta = \cos^n \theta - nC_2 \cos^{n-2} \theta \sin^2 \theta + nC_4 \cos^{n-4} \theta \sin^4 \theta + \dots$$

$$\sin n\theta = nC_1 \cos^{n-1} \theta \sin \theta - nC_3 \cos^{n-3} \theta \sin^3 \theta + \dots$$

The above equation can written as.

$$\cos n\theta = C^n - nC_2 C^{n-2} S^2 + \dots$$

$$\sin n\theta = nC_1 C^{n-1} S - nC_3 C^{n-3} S^3 + \dots$$

(or)

$$\cos n\theta = \cos^n \theta - \frac{n(n-1)}{2!} \cos^{n-2} \theta \sin^2 \theta + \dots$$

$$\sin n\theta = n \cos^{n-1} \theta \sin \theta - \frac{n(n-1)(n-2)}{3!} \cos^{n-3} \theta \sin^3 \theta + \dots$$

* Expansion of $\tan^n \theta$ in powers of $\tan \theta$.

Sol:

$$\tan^n \theta = \frac{\sin^n \theta}{\cos^n \theta}$$

$$\tan^n \theta = \frac{nC_1 \cos^{n-1} \theta \sin \theta - nC_3 \cos^{n-3} \theta \sin^3 \theta + \dots}{\cos^n \theta - nC_2 \cos^{n-2} \theta \sin^2 \theta + nC_4 \cos^{n-4} \theta \sin^4 \theta - \dots}$$

In R.H.S Dividing $\cos^n \theta$ in numerator and denominator.

$$\tan^n \theta = \frac{\frac{nC_1 \cos^{n-1} \theta \sin \theta}{\cos^n \theta} - \frac{nC_3 \cos^{n-3} \theta \sin^3 \theta}{\cos^n \theta} + \dots}{\frac{\cos^n \theta}{\cos^n \theta} - \frac{nC_2 \cos^{n-2} \theta \sin^2 \theta}{\cos^n \theta} + \frac{nC_4 \cos^{n-4} \theta \sin^4 \theta}{\cos^n \theta} + \dots}$$

$$\tan^n \theta = \frac{nC_1 \cos^{n-1} \theta \sin \theta \cos^{-n} \theta - nC_3 \cos^{n-3} \theta \sin^3 \theta \cos^{-n} \theta + \dots}{1 - nC_2 \cos^{n-2} \theta \sin^2 \theta \cos^{-n} \theta + nC_4 \cos^{n-4} \theta \sin^4 \theta \cos^{-n} \theta - \dots}$$

$$\tan^n \theta = \frac{nC_1 \cos^{-1} \theta \sin \theta - nC_3 \cos^{-3} \theta \sin^3 \theta + \dots}{1 - nC_2 \cos^{-2} \theta \sin^2 \theta + nC_4 \cos^{-4} \theta \sin^4 \theta + \dots}$$

$$\tan^n \theta = nC_1 \frac{\sin \theta}{\cos \theta} - nC_3 \frac{\sin^3 \theta}{\cos^3 \theta} + \dots$$

$$1 - nC_2 \frac{\sin^2 \theta}{\cos^2 \theta} + nC_4 \frac{\sin^4 \theta}{\cos^4 \theta} + \dots$$

$$\tan^n \theta = nC_1 \tan \theta - nC_3 \tan^3 \theta + \dots$$

$$1 - nC_2 \tan^2 \theta + nC_4 \tan^4 \theta + \dots$$

Formula:

$${}^nC_r = \frac{n!}{(n-r)!r!}$$

Problems:

1. Expand $\cos 6\theta$ in terms of $\sin \theta$.

Sol:

By De-Moivre's theorem.

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

put, $n=6$.

$$(\cos \theta + i \sin \theta)^6 = \cos 6\theta + i \sin 6\theta \quad \text{--- (1)}$$

By Binomial Theorem:

$$(x+y)^n = x^n + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \dots + y^n$$

$$\begin{aligned} (\cos \theta + i \sin \theta)^6 &= \cos^6 \theta + 6C_1 \cos^5 \theta i \sin \theta + 6C_2 \cos^4 \theta i^2 \sin^2 \theta + 6C_3 \cos^3 \theta i^3 \sin^3 \theta + 6C_4 \cos^2 \theta i^4 \sin^4 \theta + \\ &6C_5 \cos \theta i^5 \sin^5 \theta + 6C_6 \cos^0 \theta i^6 \sin^6 \theta. \\ &= \cos^6 \theta + i 6 \cos^5 \theta \sin \theta - 15 \cos^4 \theta \sin^2 \theta - \\ &20 \cos^3 \theta \sin^3 \theta + 15 \cos^2 \theta \sin^4 \theta + i 6 \cos \theta \sin^5 \theta - \sin^6 \theta \quad \text{--- (2)} \end{aligned}$$

Sub eqn (1) in eqn (2).

$$\begin{aligned} \cos 6\theta + i \sin 6\theta &= \cos^6 \theta + i 6 \cos^5 \theta \sin \theta - \\ &15 \cos^4 \theta \sin^2 \theta - 20 \cos^3 \theta \sin^3 \theta + \\ &15 \cos^2 \theta \sin^4 \theta + i 6 \cos \theta \sin^5 \theta - \sin^6 \theta. \end{aligned}$$

Formula:

$$\cos^2 \theta = 1 - \sin^2 \theta.$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

$$(a-b)^2 = a^2 + b^2 - 2ab.$$

Equating the real part. $a^3 - 3a^2b + 3ab^2 - b^3$

$$\cos 6\theta = \cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta \sin^4 \theta -$$

$$(a-b)^3 = a^3 + b^3 - 3ab(a+b)$$

$$\cos 6\theta = (\cos^2 \theta)^3 - 15 (1 - \sin^2 \theta)^2 \sin^2 \theta + 15 (1 - \sin^2 \theta) \sin^4 \theta - \sin^6 \theta.$$

$$= (1 - \sin^2 \theta)^3 - 15 (1 - \sin^2 \theta)^2 \sin^2 \theta + 15 (1 - \sin^2 \theta) \sin^4 \theta - \sin^6 \theta.$$

$$= (1 - 3\sin^2 \theta + 3\sin^4 \theta - \sin^6 \theta) - 15\sin^2 \theta.$$

$$(1 + \sin^4 \theta - 2\sin^2 \theta) + 15 \sin^4 \theta - 15 \sin^6 \theta - \sin^6 \theta.$$

$$= 1 - 3\sin^2 \theta + 3\sin^4 \theta - \sin^6 \theta - 15\sin^2 \theta -$$

$$15 \sin^6 \theta + 30 \sin^4 \theta + 15 \sin^4 \theta -$$

$$15 \sin^6 \theta - \sin^6 \theta.$$

$$\cos 6\theta = 1 - 18 \sin^2 \theta + 48 \sin^4 \theta - 32 \sin^6 \theta.$$

2. Prove that $\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1.$

Sol:

W.K.T.

$$\cos n\theta = \cos^n \theta - nC_2 \cos^{n-2} \theta \sin^2 \theta + nC_4 \cos^{n-4} \theta$$

$$\sin^4 \theta - nC_6 \cos^{n-6} \theta \sin^6 \theta$$

Here $n=6$.

$$\cos 6\theta = \cos^6 \theta - 6C_2 \cos^4 \theta \sin^2 \theta + 6C_4 \cos^2 \theta \sin^4 \theta - 6C_6 \cos^0 \theta \sin^6 \theta.$$

$$= \cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta \sin^4 \theta - \sin^6 \theta.$$

$$= \cos^6 \theta - 15 \cos^4 \theta \sin^2 \theta + 15 \cos^2 \theta (\sin^2 \theta)^2 - (\sin^2 \theta)^3$$

$$\cos 6\theta = \cos^6 \theta - 15 \cos^4 \theta (1 - \cos^2 \theta) + 15 \cos^2 \theta (1 - \cos^2 \theta)^2 - (1 - \cos^2 \theta)^3$$

$$= \cos^6 \theta - 15 \cos^4 \theta + 15 \cos^6 \theta + 15 \cos^2 \theta (1 + \cos^4 \theta - 2 \cos^2 \theta) - (1 - 3 \cos^2 \theta + 3 \cos^4 \theta - \cos^6 \theta)$$

$$= \cos^6 \theta - 15 \cos^4 \theta + 15 \cos^6 \theta + 15 \cos^2 \theta + 15 \cos^6 \theta - 30 \cos^4 \theta - 1 + 3 \cos^2 \theta - 3 \cos^4 \theta + \cos^6 \theta$$

$$\cos 6\theta = 32 \cos^6 \theta - 48 \cos^4 \theta + 18 \cos^2 \theta - 1$$

Hence Proved.

Ex. Prove that $\cos 4\theta = 8 \sin^4 \theta - 8 \sin^2 \theta + 1$.

Sol:
W.K.T.

$$\cos n\theta = \cos^n \theta - n C_2 \cos^{n-2} \theta \sin^2 \theta + n C_4 \cos^{n-4} \theta \sin^4 \theta$$

Here $n=4$.

$$\cos 4\theta = \cos^4 \theta - 4 C_2 \cos^2 \theta \sin^2 \theta + 4 C_4 \cos^0 \theta \sin^4 \theta$$

$$= \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$$

$$= (1 - \sin^2 \theta)^2 - 6(1 - \sin^2 \theta) \sin^2 \theta + \sin^4 \theta$$

$$= 1 + \sin^4 \theta - 2 \sin^2 \theta - 6 \sin^2 \theta + 6 \sin^4 \theta + \sin^4 \theta$$

$$\cos 4\theta = 8 \sin^4 \theta - 8 \sin^2 \theta + 1$$

Hence Proved.

4. Prove that $\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$.

Sol:

W.K.T

$$\sin n\theta = nC_1 \cos^{n-1}\theta \sin\theta - nC_3 \cos^{n-3}\theta \sin^3\theta + nC_5 \cos^{n-5}\theta \sin^5\theta - nC_7 \cos^{n-7}\theta \sin^7\theta$$

$n=7$

$$\begin{aligned} 7\theta &= 7C_1 \cos^6\theta \sin\theta - 7C_3 \cos^4\theta \sin^3\theta + 7C_5 \cos^2\theta \sin^5\theta - 7C_7 \cos^0\theta \sin^7\theta \\ &= 7\cos^6\theta \sin\theta - 35\cos^4\theta \sin^3\theta + 21\cos^2\theta \sin^5\theta - \sin^7\theta \end{aligned}$$

$$= 7(\cos^2\theta)^3 \sin\theta - 35(\cos^2\theta)^2 \sin^3\theta + 21\cos^2\theta \sin^5\theta - \sin^7\theta$$

$$= 7\sin\theta(1-\sin^2\theta)^3 - 35\sin^3\theta(1-\sin^2\theta)^2 + 21\sin^5\theta(1-\sin^2\theta) - \sin^7\theta$$

$$\begin{aligned} &= [7\sin\theta(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta)] - [35\sin^3\theta(1-2\sin^2\theta+\sin^4\theta)] + 21\sin^5\theta - 21\sin^7\theta - \sin^7\theta \end{aligned}$$

$$\begin{aligned} &= 7\sin\theta - 21\sin^3\theta + 21\sin^5\theta - 7\sin^7\theta - 35\sin^3\theta + 70\sin^5\theta - 35\sin^7\theta + 21\sin^5\theta - 21\sin^7\theta - \sin^7\theta \end{aligned}$$

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$$

Hence proved.

5. Expand $\tan 7\theta$ in terms of $\tan \theta$.

Sol:
W.K.T

$$\tan n\theta = \frac{nC_1 \tan \theta - nC_3 \tan^3 \theta + \dots}{1 - nC_2 \tan^2 \theta + nC_4 \tan^4 \theta + \dots}$$

Here, $n = 7$

$$\tan 7\theta = \frac{7C_1 \tan \theta - 7C_3 \tan^3 \theta + 7C_5 \tan^5 \theta - 7C_7 \tan^7 \theta}{1 - 7C_2 \tan^2 \theta + 7C_4 \tan^4 \theta - 7C_6 \tan^6 \theta}$$

$$\tan 7\theta = \frac{7 \tan \theta - 35 \tan^3 \theta + 21 \tan^5 \theta - \tan^7 \theta}{1 - 21 \tan^2 \theta + 35 \tan^4 \theta - 7 \tan^6 \theta}$$

6. Expand $\frac{\cos 5\theta}{\cos \theta}$ in terms of $\cos \theta$.

Sol:
W.K.T.

$$\cos n\theta = \cos^n \theta - nC_2 \cos^{n-2} \theta \sin^2 \theta + nC_4 \cos^{n-4} \theta \sin^4 \theta - \dots$$

$n = 5$,

$$\cos 5\theta = \cos^5 \theta - 5C_2 \cos^3 \theta \sin^2 \theta + 5C_4 \cos \theta \sin^4 \theta - \dots$$

$$= \cos^5 \theta - 10 \cos^3 \theta \sin^2 \theta + 5 \cos \theta \sin^4 \theta \quad \text{--- (1)}$$

Divide $\cos \theta$ in above eqn.

$$\frac{\cos 5\theta}{\cos \theta} = \cos^4 \theta - 10 \cos^2 \theta \sin^2 \theta + 5 \sin^4 \theta$$

$$= \cos^4 \theta - 10 \cos^2 \theta (1 - \cos^2 \theta) + 5(1 - \cos^2 \theta)^2$$

$$= \cos^4 \theta - 10 \cos^2 \theta + 10 \cos^4 \theta + 5(1 - 2\cos^2 \theta + \cos^2 \theta)$$

$$= \cos^4 \theta - 10 \cos^2 \theta + 10 \cos^4 \theta + 5 - 10 \cos^2 \theta + 5 \cos^2 \theta$$

$$\frac{\cos 5\theta}{\cos \theta} = 16 \cos^4 \theta - 20 \cos^2 \theta + 5$$

7. Expand $\tan 5\theta$.

Sol:

W.K.T.

$$\tan n\theta = \frac{nC_1 \tan \theta - nC_3 \tan^3 \theta + \dots}{1 - nC_2 \tan^2 \theta + nC_4 \tan^4 \theta + \dots}$$

$$n=5$$

$$\tan 5\theta = \frac{5C_1 \tan \theta - 5C_3 \tan^3 \theta + 5C_5 \tan^5 \theta}{1 - 5C_2 \tan^2 \theta + 5C_4 \tan^4 \theta + \dots}$$

$$\tan 5\theta = \frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}$$

8. Prove that $\cos 8\theta = 1 - 32 \sin^2 \theta + 160 \sin^4 \theta - 256 \sin^6 \theta + 128 \sin^8 \theta$.

Sol:

W.K.T.

$$\cos n\theta = \cos^n \theta - nC_2 \cos^{n-2} \theta \sin^2 \theta + nC_4 \cos^{n-4} \theta \sin^4 \theta - \dots$$

$$n=8,$$

$$\cos 8\theta = \cos^8 \theta - 8C_2 \cos^6 \theta \sin^2 \theta + 8C_4 \cos^4 \theta \sin^4 \theta - 8C_6 \cos^2 \theta \sin^6 \theta + 8C_8 \cos^0 \theta \sin^8 \theta.$$

$$= \cos^8 \theta - 28 \cos^6 \theta \sin^2 \theta + 70 \cos^4 \theta \sin^4 \theta - 28 \cos^2 \theta \sin^6 \theta + \sin^8 \theta.$$

$$= (\cos^2 \theta)^4 - 28(\cos^2 \theta)^3 \sin^2 \theta + 70(\cos^2 \theta)^2 \sin^4 \theta - 28 \cos^2 \theta \sin^6 \theta + \sin^8 \theta.$$

$$= [(1 - \sin^2 \theta)]^4 - [28(1 - \sin^2 \theta)^3 \sin^2 \theta] + [70(1 - \sin^2 \theta)^2 \sin^4 \theta] - 28(1 - \sin^2 \theta) \sin^6 \theta + \sin^8 \theta.$$

$$\begin{aligned}
&= [1 - 4 \sin^2 \theta + 6 \sin^4 \theta - 4 \sin^6 \theta + \sin^8 \theta] - \\
&28 \sin^2 \theta (1 - 3 \sin^2 \theta + 3 \sin^4 \theta - \sin^6 \theta) + [70 \sin^4 \theta \\
&\quad (1 - 2 \sin^2 \theta + \sin^4 \theta)] - 28 \sin^6 \theta + 28 \sin^8 \theta + \sin^8 \theta. \\
&= 1 - 4 \sin^2 \theta + 6 \sin^4 \theta - 4 \sin^6 \theta + \sin^8 \theta - 28 \sin^2 \theta + \\
&84 \sin^4 \theta - 84 \sin^6 \theta - 28 \sin^8 \theta + 70 \sin^4 \theta - \\
&140 \sin^6 \theta + 70 \sin^8 \theta - 28 \sin^6 \theta + 28 \sin^8 \theta \\
&+ \sin^8 \theta.
\end{aligned}$$

$$\cos 8\theta = 1 - 32 \sin^2 \theta + 160 \sin^4 \theta - 256 \sin^6 \theta + 128 \sin^8 \theta.$$

Hence Proved.

Expand $\cos 7\theta$ in terms of $\cos \theta$.

Sol:
W.K.T.

$$\cos n\theta = \cos^n \theta - n C_2 \cos^{n-2} \theta \sin^2 \theta + n C_4 \cos^{n-4} \theta \sin^4 \theta - \dots$$

Here $n=7$.

$$\begin{aligned}
\cos 7\theta &= \cos^7 \theta - 7 C_2 \cos^5 \theta \sin^2 \theta + 7 C_4 \cos^3 \theta \sin^4 \theta \\
&\quad - 7 C_6 \cos \theta \sin^6 \theta.
\end{aligned}$$

$$= \cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta$$

$$\begin{aligned}
&= \cos^7 \theta - [21 \cos^5 \theta (1 - \cos^2 \theta)] + [35 \cos^3 \theta (1 - \cos^2 \theta)^2] \\
&\quad - [7 \cos \theta (1 - \cos^2 \theta)^3]
\end{aligned}$$

$$\begin{aligned}
&= \cos^7 \theta - 21 \cos^5 \theta - 21 \cos^7 \theta + 35 \cos^3 \theta - 35 \cos^5 \theta - \\
&70 \cos^5 \theta + 35 \cos^7 \theta - 7 \cos \theta + 21 \cos^3 \theta - \\
&21 \cos^5 \theta + 7 \cos^7 \theta.
\end{aligned}$$

$$\cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta.$$

10. Expand $\cos 9\theta$ as a series in powers of $\cos \theta$.

Sol:

W.K.T.

$$\cos n\theta = \cos^n \theta - nC_2 \cos^{n-2} \theta \sin^2 \theta + nC_4 \cos^{n-4} \theta \sin^4 \theta - \dots$$

$$n=9,$$

$$\cos 9\theta = \cos^9 \theta - 9C_2 \cos^7 \theta \sin^2 \theta + 9C_4 \cos^5 \theta \sin^4 \theta$$

$$- 9C_6 \cos^3 \theta \sin^6 \theta + 9C_8 \cos \theta \sin^8 \theta.$$

$$= \cos^9 \theta - 36 \cos^7 \theta \sin^2 \theta + 126 \cos^5 \theta \sin^4 \theta$$

$$- 84 \cos^3 \theta \sin^6 \theta + 9 \cos \theta \sin^8 \theta.$$

$$= \cos^9 \theta - 36 \cos^7 \theta (1 - \cos^2 \theta) + 126 \cos^5 \theta (1 - \cos^2 \theta)^2$$

$$- 84 \cos^3 \theta (1 - \cos^2 \theta)^3 + 9 \cos \theta (1 - \cos^2 \theta)^4.$$

$$= \cos^9 \theta - 36 \cos^7 \theta + 36 \cos^9 \theta + [126 \cos^5 \theta (1 - 2\cos^2 \theta$$

$$+ \cos^4 \theta)] - [84 \cos^3 \theta (1 - 3\cos^2 \theta + 3\cos^4 \theta - \cos^6 \theta)] +$$

$$9 \cos \theta (1 - 4\cos^2 \theta + 6\cos^4 \theta - 4\cos^6 \theta + \cos^8 \theta).]$$

$$= \cos^9 \theta - 36 \cos^7 \theta + 36 \cos^9 \theta + 126 \cos^5 \theta -$$

$$252 \cos^7 \theta + 126 \cos^9 \theta - 84 \cos^3 \theta + 252 \cos^5 \theta$$

$$- 252 \cos^7 \theta + 84 \cos^9 \theta + 9 \cos \theta - 36 \cos^3 \theta$$

$$+ 54 \cos^5 \theta - 36 \cos^7 \theta + 9 \cos^9 \theta.$$

$$\cos 9\theta = 256 \cos^9 \theta - 576 \cos^7 \theta + 432 \cos^5 \theta -$$

$$120 \cos^3 \theta + 9 \cos \theta.$$

Using $\frac{\cos 7\alpha}{\cos \alpha} = 64 \cos^6 \alpha - 112 \cos^4 \alpha + 56 \cos^2 \alpha - 7$

Prove that $\frac{1 + \cos 7\theta}{1 + \cos \theta} = (x^3 - x^2 - 2x + 1)^2$ where $x = 2 \cos \theta$.

Sol:

$$\frac{1 + \cos 7\theta}{1 + \cos \theta} = \frac{2 \cos^2 \frac{7\theta}{2}}{2 \cos^2 \frac{\theta}{2}}$$

Let $\frac{\theta}{2} = \alpha$.

$$\frac{1 + \cos 7\theta}{1 + \cos \theta} = \frac{2 \cos^2 7\alpha}{2 \cos^2 \alpha}$$

$$= \frac{\cos^2 7\alpha}{\cos^2 \alpha} \Rightarrow \left[\frac{\cos 7\alpha}{\cos \alpha} \right]^2 \text{ --- (1)}$$

$\therefore x = 2 \cos \theta$

$$x = 2 \left[\frac{2 \cos^2 \frac{\theta}{2}}{2} - 1 \right]$$

$$x = 4 \cos^2 \frac{\theta}{2} - 2$$

$$x + 2 = 4 \cos^2 \alpha \text{ --- (2)}$$

$$\begin{aligned} \left[\frac{\cos 7\alpha}{\cos \alpha} \right]^2 &= (64 \cos^6 \alpha - 112 \cos^4 \alpha + 56 \cos^2 \alpha - 7)^2 \\ &= [(4 \cos^2 \alpha)^3 - 7(4 \cos^2 \alpha)^2 + 14(4 \cos^2 \alpha) - 7]^2 \end{aligned}$$

$$\text{(1)} \Rightarrow \frac{1 + \cos 7\theta}{1 + \cos \theta} = \left[\frac{\cos 7\alpha}{\cos \alpha} \right]^2$$

$$= (64 \cos^6 \alpha - 112 \cos^4 \alpha + 56 \cos^2 \alpha - 7)^2.$$

$$= [(x+2)^3 - 7(x+2)^2 + 14(x+2) - 7]^2.$$

$$= [x^3 + 6x^2 + 12x + 8 - 7x^2 - 28x - 28 + 14x + 28 - 7]^2.$$

$$= [x^3 - x^2 - 2x + 1]^2$$

$$\therefore \frac{1 + \cos 7\theta}{1 + \cos \theta} = (x^3 - x^2 - 2x + 1)^2.$$

2. Expansion for $\cos(\theta_1 + \theta_2 + \dots + \theta_n)$.

w.k.t.

$$\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i \sin(\theta_1 + \theta_2 + \dots + \theta_n).$$

$$= \cos \theta_1 + \cos \theta_2 + \dots + \cos \theta_n + i \sin \theta_1 + i \sin \theta_2 + \dots$$

$$+ i \sin \theta_n.$$

$$= (\cos \theta_1 + i \sin \theta_1) + (\cos \theta_2 + i \sin \theta_2) + \dots + (\cos \theta_n + i \sin \theta_n).$$

$$= \cos \theta_1 \left[\frac{1 + i \sin \theta_1}{\cos \theta_1} \right] + \cos \theta_2 \left[\frac{1 + i \sin \theta_2}{\cos \theta_2} \right] + \dots + \cos \theta_n \left[\frac{1 + i \sin \theta_n}{\cos \theta_n} \right]$$

$$= \cos \theta_1 (1 + i \tan \theta_1) + \cos \theta_2 (1 + i \tan \theta_2) + \dots + \cos \theta_n (1 + i \tan \theta_n).$$

$$= \cos \theta_1 \cos \theta_2 \dots \cos \theta_n [1 + i \tan \theta_1][1 + i \tan \theta_2] \dots [1 + i \tan \theta_n].$$

$$= \cos \theta_1 \cos \theta_2 \dots \cos \theta_n [1 + i \sum \tan \theta_1 + i \sum^2 \tan \theta_1 \tan \theta_2 + i \sum^3 \tan \theta_1 \tan \theta_2 \tan \theta_3]$$

Let,

$$S_1 = \tan \theta_1 + \tan \theta_2 + \dots + \tan \theta_n = \sum \tan \theta_1.$$

$$S_2 = \tan \theta_1 \tan \theta_2 + \tan \theta_2 \tan \theta_3 + \dots = \sum_1 \tan \theta_1 \tan \theta_2.$$

$$S_3 = \tan \theta_1 \tan \theta_2 \tan \theta_3 + \tan \theta_3 \tan \theta_4 \tan \theta_5 + \dots =$$

$$\sum_1 \tan \theta_1 \tan \theta_2 \tan \theta_3.$$

$$\cos(\theta_1 + \theta_2 + \dots + \theta_n) + i \sin(\theta_1 + \theta_2 + \dots + \theta_n) = \cos\theta_1 (\cos\theta_2 \cos\theta_3 \dots \cos\theta_n [1 + i s_1 - s_2 - i s_3 + \dots])$$

Equating real and imaginary part.

$$\cos(\theta_1 + \theta_2 + \dots + \theta_n) = \cos\theta_1 \cos\theta_2 \cos\theta_3 \dots \cos\theta_n [1 - s_2 + s_4 - \dots] \rightarrow (1)$$

$$\sin(\theta_1 + \theta_2 + \dots + \theta_n) = \cos\theta_1 \cos\theta_2 \cos\theta_3 \dots \cos\theta_n [s_1 - s_3 + s_5 - \dots] \rightarrow (2)$$

divide (2) / (1)

$$\tan(\theta_1 + \theta_2 + \dots + \theta_n) = \frac{s_1 - s_3 + s_5 - s_7 + \dots}{1 - s_2 + s_4 - s_6 + \dots}$$

Result

If the given eqn is $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$

$$(\div a_0) x^n + \frac{a_1}{a_0} x^{n-1} + \frac{a_2}{a_0} x^{n-2} + \dots + \frac{a_{n-1}}{a_0} x + \frac{a_n}{a_0} = 0$$

$$s_1 = \frac{a_1}{a_0}, s_2 = \frac{a_2}{a_0}, \dots, s_n = \frac{a_n}{a_0}$$

If α, β are the roots of eqn $ax^2 + bx + c = 0$

$$\text{Then } s_1 = \alpha + \beta = -\frac{b}{a}$$

$$s_2 = \alpha\beta = \frac{c}{a}$$

If α, β, γ are the roots of eqn $ax^3 + bx^2 + cx + d = 0$

Then

$$s_1 = \alpha + \beta + \gamma = -\frac{b}{a}$$

$$s_2 = \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$$

$$s_3 = \alpha\beta\gamma = -\frac{d}{a}$$

4. If $\alpha, \beta, \gamma, \delta$ are the roots of the eqn $ax^4 + bx^3 + cx^2 + dx + e = 0$ then

$$S_1 = \alpha + \beta + \gamma + \delta = -\frac{b}{a}$$

$$S_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -\frac{c}{a}$$

$$S_3 = \alpha\beta\gamma + \beta\gamma\delta + \alpha\beta\delta + \alpha\gamma\delta = -\frac{d}{a}$$

$$S_4 = \alpha\beta\gamma\delta = \frac{e}{a}$$

5. If $\alpha, \beta, \gamma, \delta$ are the roots of eqn $x^4 + px^3 + qx^2 + rx + d = 0$

Then

$$S_1 = \alpha + \beta + \gamma + \delta = -p$$

$$S_2 = \sum \alpha\beta = q$$

$$S_3 = \sum \alpha\beta\gamma = -r$$

$$S_4 = \alpha\beta\gamma\delta = d$$

problems

1. If α, β, γ are the roots of eqn $x^3 + px^2 + qx + p = 0$
 p.T. $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ (radian)
 except when $q = 1$

Sol

W.T. α, β, γ are the roots of eqn

$$x^3 + px^2 + qx + p = 0$$

$$[a=1, b=p, c=q, d=p]$$

$$S_1 = \alpha + \beta + \gamma = -p$$

$$S_2 = \alpha\beta + \beta\gamma + \gamma\alpha = q$$

$$S_3 = \alpha\beta\gamma = -p$$

$$\text{let } \tan^{-1}\alpha = \theta_1 \quad \tan^{-1}\beta = \theta_2 \quad \tan^{-1}\gamma = \theta_3$$

$$\alpha = \tan\theta_1 \quad \beta = \tan\theta_2 \quad \gamma = \tan\theta_3$$

W.K.T

$$\tan(\theta_1 + \theta_2 + \dots + \theta_n) = \frac{s_1 + s_3 + s_5 + \dots}{1 - s_2 + s_4 - \dots}$$

$$= \frac{-p + p}{1 - q} = 0$$

$$\theta_1 + \theta_2 + \dots + \theta_n = \tan^{-1}(0)$$

$$\Rightarrow \theta_1 + \theta_2 + \theta_3 = \tan^{-1}(0)$$

$$\boxed{\tan^{-1}d + \tan^{-1}p + \tan^{-1}r = n\pi}$$

If $\tan\theta_1, \tan\theta_2, \tan\theta_3, \tan\theta_4$ are the roots of eqn $x^4 + px^3 + qx^2 + rx + s = 0$ show that

$$\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \frac{r-p}{1-q+s}$$

Sol

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$a=1, b=p, c=q, d=r, e=s$$

$$s_1 = d + p + r + s = -p$$

$$s_2 = dp + pr + rd + ds = q$$

$$s_3 = dpr + prs + rds = -r$$

$$s_4 = dprs = s$$

W.K.T

$$\tan(\theta_1 + \theta_2 + \theta_3 + \dots + \theta_n) = \frac{s_1 - s_3}{1 - s_2 + s_4}$$

$$= \frac{-p + r}{1 - q + s}$$

$$\boxed{\tan(\theta_1 + \theta_2 + \theta_3 + \theta_4) = \frac{r-p}{1+s-q}}$$

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2$ ST
 $xy + yz + zx = 1$

Sol

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2 \rightarrow \text{---} \text{---}$$

Let

$$\tan^{-1}x = \theta_1 \rightarrow x = \tan\theta_1$$

$$\tan^{-1} y = \theta_2 \Rightarrow y = \tan \theta_2$$

$$\tan^{-1} z = \theta_3 \Rightarrow z = \tan \theta_3$$

$$0 \Rightarrow \theta_1 + \theta_2 + \theta_3 = \pi/2$$

Multiply by $\tan$ on b.s

$$\tan(\theta_1 + \theta_2 + \theta_3) = \tan \pi/2$$

$$\tan(\theta_1 + \theta_2 + \theta_3) = \infty$$

$$\frac{s_1 - s_3}{1 - s_2} = \infty$$

$$\frac{s_1 - s_3}{1 - s_2} = \frac{1}{0}$$

$$0 = 1 - s_2$$

$$1 - s_2 = 0$$

$$s_2 = 1$$

$$\tan \theta_1 \tan \theta_2 + \tan \theta_2 \tan \theta_3 + \tan \theta_3 \tan \theta_1 = 1$$

$$xy + yz + zx = 1$$

4. Prove the eqn. $\frac{a \cos \theta}{\cos \theta} - \frac{b \sin \theta}{\sin \theta} = a^2 - b^2$ has four roots and that sum of four values of θ with satisfied its equal to odd multiples of π (radians)

given

$$\frac{a \cos \theta}{\cos \theta} - \frac{b \sin \theta}{\sin \theta} = a^2 - b^2 \quad \text{--- (1)}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$= 2 \tan \theta/2$$

$$\frac{1 + \tan^2 \theta/2}{1 - \tan^2 \theta/2}$$

$$\frac{1 + \tan^2 \theta/2}{1 - \tan^2 \theta/2}$$

$$\tan \theta = \frac{2 \tan \theta/2}{1 - \tan^2 \theta/2}$$

put $t = \tan \theta/2$

$$\sin \theta = \frac{2t}{1+t^2}$$

$$\cos \theta = \frac{1-t^2}{1+t^2}$$

$$0 = \frac{ah}{\frac{1-t^2}{1+t^2}} - \frac{bk}{\frac{2t}{1+t^2}} = a^2 - b^2$$

$$\frac{ah(1+t^2)}{1-t^2} - \frac{bk(1+t^2)}{2t} = a^2 - b^2$$

$$2ah(1+t^2) - \frac{bk(1+t^2)}{2t} = a^2 - b^2$$

$$2ah + 2ah t^2 - bk + bk t^2 = (a^2 - b^2)(2t - 2t^3)$$

$$2ah + 2ah t^2 - bk + bk t^2 = 2a^2 t - 2a^2 t^3 - 2t^2 + t + 2b^2 t^3$$

$$bk t^2 + 2ah t^3 + 2ah - bk - 2a^2 t + 2a^2 t^3 + 2b^2 t^3 = 0$$

$$bk t^2 + t^3 (2ah + 2a^2 - 2b^2) + 0 t^2 + t (2ah - 2a^2 + 2b^2) - bk = 0$$

Compare this eqn to

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

$$a = bk, b = (2ah + 2a^2 - 2b^2), c = 0$$

$$d = (2ah - 2a^2 + 2b^2), e = -bk$$

$$S_1 = \frac{-b}{a} = \frac{-(2ah + 2a^2 - 2b^2)}{bk}$$

$$S_2 = \frac{c}{a} = 0$$

$$S_3 = \frac{-d}{a} = \frac{-(2ah - 2a^2 + 2b^2)}{bk}$$

$$S_4 = \frac{e}{a} = \frac{-bk}{bk} = -1$$

$$\tan \left(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2} \right) = \frac{s_1 - s_3}{1 - s_2 + s_4}$$

$$= \frac{-(2ah + 2a^2 - 2b^2)}{bk} + \frac{(2ah - 2a^2 + 2b^2)}{bk}$$

$$\tan \frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2} = \frac{1-0-1}{0} = \infty$$

$$\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2} = \tan^{-1} \infty$$

$$\frac{\theta_1 + \theta_2 + \theta_3 + \theta_4}{2} = n\pi + \pi/2.$$

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 = \pi(2n+1)$$

Sum of the Value of θ is equal to the odd Multiples of π (radians)

Powers of sines and cosines of θ in terms of θ .

Formulae:

$$1. (a+b)^n = a^n + nC_1 a^{n-1} b + nC_2 a^{n-2} b^2 + \dots + b^n$$

$$2. (x+a)^n = x^n + nC_1 x^{n-1} a + nC_2 x^{n-2} a^2 + \dots + a^n$$

$$3. (x-a)^n = x^n - nC_1 x^{n-1} a + nC_2 x^{n-2} a^2 + \dots + (-1)^n a^n$$

$$4. \left(x + \frac{1}{x}\right)^n = x^n + nC_1 x^{n-1} \left(\frac{1}{x}\right) + nC_2 x^{n-2} \left(\frac{1}{x}\right)^2 + \dots + \left(\frac{1}{x}\right)^n$$

$$5. \left(x - \frac{1}{x}\right)^n = x^n - nC_1 x^{n-1} \left(\frac{1}{x}\right) + nC_2 x^{n-2} \left(\frac{1}{x}\right)^2 + \dots + (-1)^n \left(\frac{1}{x}\right)^n$$

$$6. \text{ Let } x = \cos \theta + i \sin \theta$$

$$\frac{1}{x} = \cos \theta - i \sin \theta$$

$$\left(x + \frac{1}{x}\right) = \cos \theta + i \sin \theta + \cos \theta - i \sin \theta \\ = 2 \cos \theta$$

$$\left(x - \frac{1}{x}\right) = \cos \theta + i \sin \theta - \cos \theta + i \sin \theta \\ = 2i \sin \theta$$

$$+ \text{ Let } x^n = (\cos \theta + i \sin \theta)^n$$

$$x^n = \cos n\theta + i \sin n\theta$$

$$\frac{1}{x^n} = \cos n\theta - i \sin n\theta$$

$$x^n + \frac{1}{x^n} = 2 \cos n\theta, \quad x^n - \frac{1}{x^n} = 2i \sin n\theta$$

$$8. 1 \text{ Radian} = 57^\circ 17' 44.8''$$

Problems:

1. Expand $\sin^7 \theta$ in terms of $\sin \theta$ (or) Show that
 $64 \sin^7 \theta = \sin 7\theta - 7 \sin 5\theta + 21 \sin 3\theta - 35 \sin \theta$

Sol:

$$x - \frac{1}{x} = 2i \sin \theta, \quad x^n - \frac{1}{x^n} = 2i \sin n\theta$$

$$\left(x - \frac{1}{x}\right)^n = x^n - n C_1 x^{n-1} \left(\frac{1}{x}\right) + n C_2 x^{n-2} \left(\frac{1}{x^2}\right) + \dots + (-1)^n \left(\frac{1}{x}\right)^n$$

Here $n=7$.

$$\left(x - \frac{1}{x}\right)^7 = x^7 - 7 C_1 x^{7-1} \left(\frac{1}{x}\right) + 7 C_2 x^{7-2} \left(\frac{1}{x^2}\right) - 7 C_3 x^{7-3} \left(\frac{1}{x^3}\right) \\ + 7 C_4 x^{7-4} \left(\frac{1}{x^4}\right) - 7 C_5 x^{7-5} \left(\frac{1}{x^5}\right) + 7 C_6 x^{7-6} \left(\frac{1}{x^6}\right) \\ - 7 C_7 x^{7-7} \left(\frac{1}{x^7}\right)$$

$$= x^7 - 7x^6\left(\frac{1}{x}\right) + 21x^5\left(\frac{1}{x^2}\right) - 35x^4\left(\frac{1}{x^3}\right) + 35x^3\left(\frac{1}{x^4}\right) - 21x^2\left(\frac{1}{x^5}\right) + 7x\left(\frac{1}{x^6}\right) - \left(\frac{1}{x^7}\right).$$

$$= x^7 - 7x^5 + 21x^3 - 35x + \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}.$$

$$\left(x - \frac{1}{x}\right)^7 = \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right).$$

$$(2i \sin \theta)^7 = 2i \sin 7\theta - 7(2i \sin 5\theta) + 21(2i \sin 3\theta) - 35(2i \sin \theta)$$

$$(2i)^7 (\sin \theta)^7 = 2i (\sin 7\theta - 7 \sin 5\theta + 21 \sin 3\theta - 35 \sin \theta)$$

$$-64 \sin^7 \theta = \sin 7\theta - 7 \sin 5\theta + 21 \sin 3\theta - 35 \sin \theta.$$

2. Prove that $2^8 \cos^9 \theta = \cos 9\theta + 9 \cos 7\theta + 36 \cos 5\theta + 84 \cos 3\theta + 126 \cos \theta.$

Sol:

W.K.T.

$$x + \frac{1}{x} = 2 \cos \theta, \quad x^n + \frac{1}{x^n} = 2^n \cos n\theta.$$

$$\left(x + \frac{1}{x}\right)^n = x^n + nC_1 x^{n-1} \left(\frac{1}{x}\right) + nC_2 x^{n-2} \left(\frac{1}{x}\right)^2 + \dots$$

Put $n=9$,

$$\left(x + \frac{1}{x}\right)^9 = x^9 + 9C_1 x^8 \left(\frac{1}{x}\right) + 9C_2 x^7 \left(\frac{1}{x}\right)^2 + 9C_3 x^6 \left(\frac{1}{x}\right)^3 +$$

$$9C_4 x^5 \left(\frac{1}{x}\right)^4 + 9C_5 x^4 \left(\frac{1}{x}\right)^5 + 9C_6 x^3 \left(\frac{1}{x}\right)^6 +$$

$$9C_7 x^2 \left(\frac{1}{x}\right)^7 + 9C_8 x \left(\frac{1}{x}\right)^8 + 9C_9 x^0 \left(\frac{1}{x}\right)^9.$$

$$= x^9 + 9x^8 \left(\frac{1}{x}\right) + 36x^7 \left(\frac{1}{x^2}\right) + 84x^6 \left(\frac{1}{x^3}\right) +$$

$$126x^5 \left(\frac{1}{x^4}\right) + 126x^4 \left(\frac{1}{x^5}\right) + 84x^3 \left(\frac{1}{x^6}\right) + 36x^2 \left(\frac{1}{x^7}\right) +$$

$$9x \left(\frac{1}{x^8}\right) + \frac{1}{x^9}.$$

$$= 2^9 + 9x^7 + 36x^5 + 84x^3 + 126x + 126\frac{1}{x} + 84\frac{1}{x^3} + 36\frac{1}{x^5} + 9\frac{1}{x^7} + \frac{1}{x^9}$$

$$\left(2 + \frac{1}{x}\right)^9 = \left(x^9 + \frac{1}{x^9}\right) + 9\left(x^7 + \frac{1}{x^7}\right) + 36\left(x^5 + \frac{1}{x^5}\right) + 84\left(x^3 + \frac{1}{x^3}\right) + 126\left(x + \frac{1}{x}\right)$$

$$2^9 \cos^9 \theta = 2 \cos^9 \theta + 9(2 \cos^7 \theta) + 36(2 \cos^5 \theta) + 84(2 \cos^3 \theta) + 126(2 \cos \theta)$$

$$2^9 \cos^9 \theta = 2 [\cos^9 \theta + 9 \cos^7 \theta + 36 \cos^5 \theta + 84 \cos^3 \theta + 126 \cos \theta]$$

$$2^8 \cos^9 \theta = \cos^9 \theta + 9 \cos^7 \theta + 36 \cos^5 \theta + 84 \cos^3 \theta + 126 \cos \theta$$

3. Prove that: $16 \sin^5 \theta = \sin^5 \theta - 5 \sin^3 \theta + 10 \sin \theta$

Sol:

N.K.T

$$x - \frac{1}{x} = 2i \sin \theta, \quad x^9 - \frac{1}{x^9} = 2i \sin^9 \theta$$

$$\left(x - \frac{1}{x}\right)^9 = x^9 - 9C_1 x^7 \left(\frac{1}{x}\right) + 9C_2 x^5 \left(\frac{1}{x}\right)^2 + \dots$$

Put $n=5$

$$\left(x - \frac{1}{x}\right)^5 = x^5 - 5C_1 x^4 \left(\frac{1}{x}\right) + 5C_2 x^3 \left(\frac{1}{x}\right)^2 - 5C_3 x^2 \left(\frac{1}{x}\right)^3 + 5C_4 x \left(\frac{1}{x}\right)^4 + 5C_5 x^0 \left(\frac{1}{x}\right)^5$$

$$= x^5 - 5x^3 + 10x - \frac{10}{x} + \frac{5}{x^3} - \frac{1}{x^5}$$

$$\left(x - \frac{1}{x}\right)^5 = \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$(2i \sin^5 \theta)^5 = (2i \sin^5 \theta) - 5(2i \sin^3 \theta) + 10(2i \sin \theta)$$

$$(2i)^5 \sin^5 \theta = 2i [\sin 5\theta - 5\sin 3\theta + 10\sin \theta]$$

$$16i^4 \sin^5 \theta = \sin 5\theta - 5\sin 3\theta + 10\sin \theta$$

$$16 \sin^5 \theta = \sin 5\theta - 5\sin 3\theta + 10\sin \theta$$

Hence Proved.

4. Expand $\cos^7 \theta \sin^3 \theta$ in terms of $\sin \theta$.

Sol:

W.K.T.

$$x + \frac{1}{x} = 2 \cos \theta, \quad x - \frac{1}{x} = 2i \sin \theta$$

$$\left(x + \frac{1}{x}\right)^7 \left(x - \frac{1}{x}\right)^3 = \left(x + \frac{1}{x}\right)^3 \left(x + \frac{1}{x}\right)^4 \left(x - \frac{1}{x}\right)^3$$

$$= \left(x^2 - \frac{1}{x^2}\right)^3 \left(x + \frac{1}{x}\right)^4$$

$$= \left[(x^2)^3 - 3C_1(x^2)^2 \left(\frac{1}{x^2}\right) + 3C_2(x^2) \left(\frac{1}{x^2}\right)^2 - \right.$$

$$\left. 3C_3 \left(\frac{1}{x^2}\right)^3 \right] \left(x + \frac{1}{x}\right)^4$$

$$= \left[x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6} \right] \left[x^4 + 4x^2 + 6 + \frac{4}{x^2} + \frac{1}{x^4} \right]$$

$$= x^{10} + 4x^8 + 6x^6 + 4x^4 + x^2 - 3x^6 - 12x^4 - 18x^2$$

$$- 12 - \frac{3}{x^2} + 3x^2 + 12 + \frac{24}{x^2} + \frac{12}{x^2} + \frac{3}{x^6} - \frac{1}{x^2}$$

$$- \frac{4}{x^4} - \frac{6}{x^6} - \frac{4}{x^8} - \frac{1}{x^{10}}$$

$$= \left(x^{10} - \frac{1}{x^{10}}\right) + 4 \left(x^8 - \frac{1}{x^8}\right) + 3 \left(x^6 - \frac{1}{x^6}\right) - 8 \left(x^4 - \frac{1}{x^4}\right)$$

$$- 14 \left(x^2 - \frac{1}{x^2}\right)$$

$$2^7 \cos^7 \theta \cdot 2^3 i^3 \sin^3 \theta = 2i \sin 10\theta + 4(2i \sin 8\theta) + 3(2i \sin 6\theta) - 8(2i \sin 4\theta) - 14(2i \sin 2\theta)$$

$$2^{10} i^3 \cos^7 \theta \sin^3 \theta = 2i [\sin 10\theta + 4 \sin 8\theta + 3 \sin 6\theta - 8 \sin 4\theta - 14 \sin 2\theta]$$

$$\cos^7 \theta \sin^3 \theta = \frac{-1}{2^9} [\sin 10\theta + 4 \sin 8\theta + 3 \sin 6\theta - 8 \sin 4\theta - 14 \sin 2\theta]$$

5. Prove that $\sin^3 \theta \cos^4 \theta = \frac{1}{2^6} [\sin 7\theta + \sin 5\theta - 3 \sin 3\theta - 3 \sin \theta]$.

Sol:

$$x + \frac{1}{x} = 2 \cos \theta; \quad x - \frac{1}{x} = 2i \sin \theta$$

$$\left(x + \frac{1}{x}\right)^4 \left(x - \frac{1}{x}\right)^3 = \left(x + \frac{1}{x}\right) \left(x + \frac{1}{x}\right)^3 \left(x - \frac{1}{x}\right)^3$$

$$= \left(x^2 - \frac{1}{x^2}\right)^3 \left(x + \frac{1}{x}\right)$$

$$= \left[(x^2)^3 - 3C_1 (x^2)^2 \left(\frac{1}{x^2}\right) + 3C_2 (x^2) \left(\frac{1}{x^2}\right)^2 - 3C_3 (x^2)^0 \left(\frac{1}{x^2}\right)^3 \right] \left(x + \frac{1}{x}\right)$$

$$= \left[x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6} \right] \left[x + \frac{1}{x} \right]$$

$$= x^7 + x^5 - 3x^3 - 3x + \frac{3}{x} + \frac{3}{x^3} - \frac{1}{x^5} - \frac{1}{x^7}$$

$$\left(x + \frac{1}{x}\right)^4 \left(x - \frac{1}{x}\right)^3 = \left(x^7 - \frac{1}{x^7}\right) + \left(x^5 - \frac{1}{x^5}\right) - 3\left(x^3 - \frac{1}{x^3}\right) - 3\left(x - \frac{1}{x}\right)$$

$$2^4 \cos^4 \theta \cdot 2^3 i^3 \sin^3 \theta = 2i \sin 7\theta + 2i \sin 5\theta - 3(2i \sin 3\theta) - 3(2i \sin \theta)$$

$$2^7 \sin^3 \theta \cos^4 \theta = 2i [\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin \theta]$$

$$-\sin^3 \theta \cos^4 \theta = \frac{1}{2^6} [\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin \theta]$$

hence proved.

Formula.

$$\tan \theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \dots$$

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

1. Evaluate $\lim_{x \rightarrow 0} \frac{\tan 2x - 2 \tan x}{x^3}$

Sol:

$$\theta = 2x$$

$$\tan \theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \dots$$

$$\frac{\tan 2x - 2 \tan x}{x^3} = \frac{2x + \frac{(2x)^3}{3} + \frac{2(2x)^5}{15} + \dots}{x^3} -$$

$$\frac{2x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots}{x^3}$$

$$= \frac{2x + \frac{8x^3}{3} + \frac{64x^5}{15} + \dots}{x^3} - \frac{2x + \frac{2x^3}{3} + \frac{4x^5}{15} + \dots}{x^3}$$

$$x^3$$

$$\frac{\frac{x^3}{3} + \frac{64x^5}{15} + \dots - 2x - \frac{2x^3}{3} - \frac{4x^5}{15} + \dots}{x^3}$$

$$= \frac{\frac{6x^3}{3} + \frac{60x^5}{15}}{x^3} \Rightarrow x^3 \left(\frac{6}{x} + \frac{60x^2}{15} \right)$$

$$\frac{\tan 2x - 2 \tan x}{x^3} = \frac{6}{3} + \frac{60x^2}{15}$$

$$\lim_{x \rightarrow 0} \frac{\tan 2x - 2 \tan x}{x^3} = \frac{6}{3} + \frac{60(0)}{15} = 2$$

$$\lim_{x \rightarrow 0} \frac{\tan 2x - 2 \tan x}{x^3} = 2$$

2. Evaluate $\lim_{x \rightarrow 0} \frac{\sin x - x \cos x}{x^3}$.

Sol:

$$\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$$

$$\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots$$

$$\frac{\sin x - x \cos x}{x^3} = \frac{\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right] - x \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right]}{x^3}$$

$$= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + x + \frac{x^3}{2!} - \frac{x^5}{4!}}{x^3}$$

$$= \frac{-\frac{x^5}{6} + \frac{x^5}{120} + \frac{x^3}{2} - \frac{x^5}{24}}{x^3}$$

$$= \frac{x^3}{x^3} \left(\frac{-1}{6} + \frac{x^2}{120} - \dots + \frac{1}{2} - \frac{x^2}{24} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{-1}{6} + \frac{x^2}{120} - \dots + \frac{1}{2} - \frac{x^2}{24} \right)$$

$$= \frac{-1}{6} + \frac{1}{2} = \frac{1}{3}$$

3. Evaluate $\lim_{x \rightarrow 0} \frac{\sin x - x + \frac{x^6}{5}}{x^5}$

Sol:

W.K.T

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\frac{\sin x - x + \frac{x^6}{5}}{x^5} = \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) - x + \frac{x^6}{5}}{x^5}$$

$$= \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \dots - x + \frac{x^6}{5}}{x^5}$$

$$= \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - x + \frac{x^6}{5}}{x^5} = \frac{x^5 \left(\frac{-1}{6x^2} + \frac{1}{120} + \frac{x}{5} \right)}{x^5}$$

$$= \frac{-\frac{1}{6x^2} + \frac{1}{120} + \frac{x}{5}}{x^5}$$

$$\lim_{x \rightarrow 0} \frac{\sin x - x + \frac{x^6}{5}}{x^5} = \frac{-1}{6(0)^2} + \frac{1}{120} + \frac{0}{5}$$

Ans = $\frac{1}{120}$

4. Evaluate $\lim_{x \rightarrow 0} \frac{\sin 2x - 2 \sin x}{x^3}$

Q1: W.K.T. $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$

$$\frac{\sin 2x - 2 \sin x}{x^3} = \frac{\left[2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \dots \right] - 2 \left[x - \frac{x^3}{6} + \frac{x^5}{120} \right]}{x^3}$$

$$= \frac{2x - \frac{8x^3}{6} + \frac{32x^5}{120} - \dots - 2x + \frac{2x^3}{6} - \frac{2x^5}{120}}{x^3}$$

$$= x^3 \left(\frac{8}{6} + \frac{32x^2}{120} + \frac{2}{6} - \frac{2x^2}{120} \right) = \frac{-8}{6} + \frac{32x^2}{120} + \frac{2}{6} - \frac{2x^2}{120}$$

At $x \rightarrow 0$ $\frac{\sin 2x - 2 \sin x}{x^3} = \frac{-8}{6} + \frac{32(0)}{120} + \frac{2}{6} - \frac{2(0)}{120}$

Ans: -1

5. Evaluate $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$

Sol:

W.K.T. $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$

$$\frac{x - \sin x}{x^3} = \frac{x - x + \frac{x^3}{6} - \frac{x^5}{120} + \dots}{x^3} = \frac{x^3 \left(\frac{1}{6} - \frac{x^2}{120} \right)}{x^3}$$

At $x \rightarrow 0$ $\frac{x - \sin x}{x^3} = \frac{1}{6} - \frac{0}{120}$

Ans: $\frac{1}{6}$

6. Evaluate $\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^2}$

Sol:

W.K.T. $\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots$

$\tan \theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \dots$

$$\frac{\sin x - \tan x}{x^2} = \frac{\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots \right) - \left(x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots \right)}{x^2}$$

$$= 2 - \frac{2^3}{6} + \frac{2^5}{120} - \dots - 2 - \frac{2^3}{3} - \frac{22^5}{15} - \dots$$

$$= 2^2 \left(\frac{-2}{6} + \frac{2^3}{120} - \frac{2}{3} - \frac{22^3}{15} \right)$$

$$\lim_{x \rightarrow 0} \frac{\sin x - \tan x}{x^2} = \frac{-0}{6} + \frac{0}{120} - \frac{0}{3} - \frac{2(0)}{15} = 0.$$

7. If $\frac{\tan \theta}{\theta} = \frac{2524}{2523}$ show that value of θ is $1^\circ 58'$

Sol:

$$\frac{\tan \theta}{\theta} = \frac{2524}{2523}$$

$$\frac{\theta + \frac{\theta^3}{3} + \frac{\theta^5}{15} + \dots}{\theta} = \frac{2524}{2523} \Rightarrow \theta \left[1 + \frac{\theta^2}{3} + \frac{2\theta^4}{15} + \dots \right] = \frac{2524}{2523}$$

$$\Rightarrow 1 + \frac{\theta^2}{3} = \frac{2524}{2523} \Rightarrow \frac{\theta^2}{3} = \frac{2524}{2523} - 1 \Rightarrow \frac{\theta^2}{3} = \frac{1}{2523}$$

$$\theta^2 = \frac{3}{2523} \Rightarrow \theta^2 = 0.00118 \Rightarrow \theta = \sqrt{0.00118}$$

$$\theta = 0.03435 \times 57' 17.448''$$

$$\theta = 1^\circ 58'$$

8. Find the approximate value of θ in radian if $\frac{\sin \theta}{\theta} = \frac{863}{864}$

Sol:

$$\frac{\sin \theta}{\theta} = \frac{863}{864} \Rightarrow \left[\frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots}{\theta} \right] = \frac{863}{864}$$

$$\theta \left[1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - \dots \right] = \frac{863}{864} \Rightarrow 1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} - \dots = \frac{863}{864}$$

Neglect higher power

$$1 - \frac{\theta^2}{6} = \frac{863}{864} \Rightarrow -\frac{\theta^2}{6} = \frac{863}{864} - 1$$

$$-\frac{\theta^2}{6} = \frac{-1}{864} \Rightarrow \theta^2 = \frac{6}{864}$$

$$\theta^2 = 0.006944$$

$$\theta = 0.08333$$

1. Express $\frac{\sin 6\theta}{\sin \theta \cos \theta}$ in terms of $\cos \theta$ and $\sin \theta$.

Sol: W.K.T

$$\sin n\theta = nC_1 \cos^{n-1} \theta \sin \theta - nC_3 \cos^{n-3} \theta \sin^3 \theta + \dots$$

$$n=6$$

$$\begin{aligned} \sin 6\theta &= 6C_1 \cos^5 \theta \sin \theta - 6C_3 \cos^3 \theta \sin^3 \theta + 6C_5 \cos \theta \sin^5 \theta \\ &= 6\cos^5 \theta \sin \theta - 20\cos^3 \theta \sin^3 \theta + 6\cos \theta \sin^5 \theta. \end{aligned}$$

Divide by $\sin \theta \cos \theta$.

$$\begin{aligned} \frac{\sin 6\theta}{\sin \theta \cos \theta} &= \frac{6\cos^5 \theta \sin \theta}{\sin \theta \cos \theta} - \frac{20\cos^3 \theta \sin^3 \theta}{\sin \theta \cos \theta} + \frac{6\cos \theta \sin^5 \theta}{\sin \theta \cos \theta} \\ &= 6\cos^4 \theta - 20\cos^2 \theta \sin^2 \theta + 6\sin^4 \theta \quad \text{--- (1)} \end{aligned}$$

In terms of $\cos \theta$.

$$\begin{aligned} \frac{\sin 6\theta}{\sin \theta \cos \theta} &= 6\cos^4 \theta - 20\cos^2 \theta (1 - \sin^2 \theta) + 6(1 - \cos^2 \theta)^2 \\ &= 6\cos^4 \theta - 20\cos^2 \theta (1 - \cos^2 \theta) + 6(1 - 2\cos^2 \theta + \cos^4 \theta) \\ &= 6\cos^4 \theta - 20\cos^2 \theta + 20\cos^4 \theta + 6 - 12\cos^2 \theta + 6\cos^4 \theta \\ \frac{\sin 6\theta}{\sin \theta \cos \theta} &= 32\cos^4 \theta - 32\cos^2 \theta + 6. \end{aligned}$$

In terms of $\sin \theta$.

$$\begin{aligned} \frac{\sin 6\theta}{\sin \theta \cos \theta} &= 6(1 - 2\sin^2 \theta + \sin^4 \theta) - 20\sin^2 \theta (1 - \sin^2 \theta) + 6\sin^4 \theta \\ &= 6 - 12\sin^2 \theta + 6\sin^4 \theta - 20\sin^2 \theta + 20\sin^4 \theta + 6\sin^4 \theta \\ \frac{\sin 6\theta}{\sin \theta \cos \theta} &= 6 - 32\sin^2 \theta + 32\sin^4 \theta. \end{aligned}$$

2. Express $\frac{\cos 5\theta}{\cos \theta}$ in terms of $\sin \theta$ and $\cos \theta$.

Sol:

W.K.T

$$\cos n\theta = \cos^n \theta - nC_2 \cos^{n-2} \theta \sin^2 \theta + nC_4 \cos^{n-4} \theta \sin^4 \theta - \dots$$

$$n=5$$

$$\cos 5\theta = \cos^5 \theta - 5C_2 \cos^3 \theta \sin^2 \theta + 5C_4 \cos \theta \sin^4 \theta.$$

$$\begin{aligned} \frac{\cos 5\theta}{\cos \theta} &= \frac{\cos^5 \theta}{\cos \theta} - \frac{10\cos^3 \theta \sin^2 \theta}{\cos \theta} + \frac{5\cos \theta \sin^4 \theta}{\cos \theta} \\ &= \cos^4 \theta - 10\cos^2 \theta \sin^2 \theta + 5\sin^4 \theta. \end{aligned}$$

In terms of $\sin \theta$.

$$\frac{\cos 5\theta}{\cos \theta} = (1 - \sin^2 \theta)^2 - 10(1 - \sin^2 \theta) \sin^2 \theta + 5\sin^4 \theta.$$

$$= 1 - 2\sin^2\theta + \sin^4\theta - 10\sin^2\theta + 10\sin^4\theta + 5\sin^6\theta$$

$$\frac{\cos 5\theta}{\cos \theta} = 16\sin^4\theta - 12\sin^2\theta + 1$$

$$\cos \theta$$

In terms of $\cos \theta$.

$$\frac{\cos 5\theta}{\cos \theta} = \cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + 5(1-\cos^2\theta)^2$$

$$\cos \theta$$

$$= \cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 5 - 10\cos^2\theta - 5\cos^4\theta$$

$$\frac{\cos 5\theta}{\cos \theta} = 16\cos^4\theta - 20\cos^2\theta + 5$$

3. Express $\frac{\sin 7\theta}{\sin \theta}$ in terms of $\sin \theta$ and $\cos \theta$.

Sol:

$$\text{W.K.T } \sin n\theta = nC_1 \cos^{n-1}\theta \sin \theta - nC_3 \cos^{n-3}\theta \sin^3\theta + \dots$$

$$n=7$$

$$\sin 7\theta = 7C_1 \cos^6\theta \sin \theta - 7C_3 \cos^4\theta \sin^3\theta + 7C_5 \cos^2\theta \sin^5\theta - 7C_7 \cos^0\theta \sin^7\theta$$

$$\sin 7\theta = 7 \cos^6\theta \sin \theta - 35 \cos^4\theta \sin^3\theta + 21 \cos^2\theta \sin^5\theta - \sin^7\theta$$

$$\frac{\sin 7\theta}{\sin \theta} = \frac{7 \cos^6\theta \sin \theta}{\sin \theta} - \frac{35 \cos^4\theta \sin^3\theta}{\sin \theta} + \frac{21 \cos^2\theta \sin^5\theta}{\sin \theta} - \frac{\sin^7\theta}{\sin \theta}$$

$$= 7 \cos^6\theta - 35 \cos^4\theta \sin^2\theta + 21 \cos^2\theta \sin^4\theta - \sin^6\theta$$

In terms of $\sin \theta$.

$$\frac{\sin 7\theta}{\sin \theta} = 7(1-\sin^2\theta)^3 - 35(1-\sin^2\theta)^2 \sin^2\theta + 21(1-\sin^2\theta) \sin^4\theta - \sin^6\theta$$

$$= 7 - 21\sin^2\theta + 21\sin^4\theta - 7\sin^6\theta - 35\sin^2\theta + 70\sin^4\theta - 35\sin^6\theta + 21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$$

$$\frac{\sin 7\theta}{\sin \theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

In terms of $\cos \theta$.

$$\frac{\cos 7\theta}{\cos \theta} = 7 \cos^6\theta - 35 \cos^4\theta (1-\cos^2\theta) + 21 \cos^2\theta (1-\cos^2\theta)^2 - (1-\cos^2\theta)^3$$

$$= 7 \cos^6\theta - 35 \cos^4\theta + 35 \cos^6\theta + 21 \cos^2\theta (1-2\cos^2\theta + \cos^4\theta) - (1-3\cos^2\theta + 3\cos^4\theta - \cos^6\theta)$$

$$= 7 \cos^6\theta - 35 \cos^4\theta + 35 \cos^6\theta + 21 \cos^2\theta - 42 \cos^4\theta + 21 \cos^6\theta - 1 + 3 \cos^2\theta - 3 \cos^4\theta + \cos^6\theta$$

$$\frac{\cos 7\theta}{\cos \theta} = 64 \cos^6\theta - 80 \cos^4\theta + 24 \cos^2\theta - 1$$