# MARUDHAR KESARI JAIN COLLEGE FOR WOMEN, VANIYAMBADI

## PG & RESEARCH DEPARTMENT OF MATHEMATICS

**CLASS : I B.Sc PHYSICS**

**SUBJECT CODE : MATHEMATICS I**

**SUBJECT NAME : 23UEMA10C**

## SYLLABUS

## UNIT- V

## Differential Calculus

Successive differentiation, n th derivatives, Leibnitz theorem (without proof) and applications, Jacobians, maxima and minima of functions of two variables- Simple problems

Formulas:-

1. $\frac{d}{dx}(c) = 0$

2. $\frac{d}{dx}(x^n) = nx^{n-1}$

3. $\frac{d}{dx}(x) = 1$

4. $\frac{d}{dx}(x^2) = 2x$

5. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

6. $\frac{d}{dx}(e^x) = e^x$

7. $\frac{d}{dx}(e^{mx}) = me^{mx}$

8. $\frac{d}{dx}(\sin x) = \cos x$

9. $\frac{d}{dx}(\cos x) = -\sin x$

10. $\frac{d}{dx}(\tan x) = \sec^2 x$

11. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

12. $\frac{d}{dx}(\sec x) = \sec x \tan x$

13. $\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$

14. $\frac{d}{dx}(\cos ax) = -a \sin ax$

15. $\frac{d}{dx}(\sin ax) = a \cos ax$

16. $\frac{d}{dx}(\log x) = \frac{1}{x}$

17. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

18. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

19. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

20. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

21. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

22. $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

23. $\frac{d}{dx} a^x = a^x \log a$

24. $\frac{d}{dx}(uv) = uv' + vu'$

25. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

26. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

27. $\frac{d}{dx}(u^v) = vu^{n-1} d(u) + u^v (\log u)\, d(v)$.

Successive difference $n^{th}$ derivative.

If $y$ is a function of $x$, its derivative $\frac{dy}{dx}$ will be some other function of $x$ and the differentiation of this function with respect to $x$ is called second derivative and is denoted by $\frac{d^2y}{dx^2}$.

i.e, $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

III'y the ~~dx~~ third derivative is denoted by

$$\frac{d^3y}{dx^3}.$$

# Unit - 1

2. If $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$

find $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\left(\frac{dt}{dx}\right)$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$\frac{dy}{dt} = [a[\cos t - (1)\cos t + t(\sin t)]]$$

$$= a[\cos t - \cos t + t\sin t]$$

$$= a\cancel{\cos t} - a\cancel{\cos t} + at\sin t = at\sin t$$

$$\frac{dx}{dt} = a[-\cancel{\sin t} + (1)\cancel{\sin t} + t\cos t]$$

$$= -a\sin t + a\sin t + at\cos t = at\cos t$$

$$\frac{dy}{dx} = \frac{\cancel{at}\sin t}{\cancel{at}\cos t} = \tan t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(\tan t)\frac{1}{at\cos t}$$

$$= \sec^2 t\,\frac{\sec t}{at}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^3 t}{at}$$

3. If $y = a\cos(\log x) + b\sin(\log x)$. S.T

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

$y = a\cos(\log x) + b\sin(\log x)$

Diff with r. to $x$.

$$\frac{dy}{dx} = \frac{-a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$$

$$x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$$

Again diff w.r to $x$.

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx}(1) = \frac{-a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = -\frac{1}{x}\left[a\cos(\log x) + b\sin(\log x)\right]$$

$$x^2\frac{d^2y}{dx^2} + \frac{dy}{dx} = -y$$

$$x^2\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

4. If $y = (x + \sqrt{1+x^2})^m$. S.T $(1+x^2)y_2 + xy_1 - m^2y = 0$

$$y = (x + (1+x^2)^{1/2})^m$$

Diff w. r to $x$.

$$\frac{dy}{dx} = m\left(x + (1+x^2)^{1/2}\right)^{m-1}\left[1 + (\cancel{2}x)\tfrac{1}{\cancel{2}}(1+x^2)^{-1/2}\right]$$

$$= m\left(x + (1+x^2)^{1/2}\right)^{m-1}\left(1 + \frac{x}{\sqrt{1+x^2}}\right)$$

$$= m\left(x + (1+x^2)^{1/2}\right)^{m-1}\left(\frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}}\right)$$

$$= \frac{m\left(x + (1+x^2)^{1/2}\right)^m}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \frac{my}{\sqrt{1+x^2}}$$

$$\sqrt{1+x^2}\,\frac{dy}{dx} = my$$

sq. on b.s

$$(1+x^2)\left(\frac{dy}{dx}\right)^2 = m^2y^2$$

diff w.r to $x$.

$$(1+x^2)\,2\,\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2(2x) = m^2\,2y\,\frac{dy}{dx}$$

$$2\,\cancel{\frac{dy}{dx}}\left[(1+x^2)\cdot\frac{d^2y}{dx^2} + \frac{dy}{dx}\,x\right] = m^2\,2y\,\cancel{\frac{dy}{dx}}$$

$$(1+x^2)\,\frac{d^2y}{dx^2} + x\,\frac{dy}{dx} = m^2y$$

$$(1+x^2)\,y_2 + x\,y_1 - m^2y = 0$$

5. If $y = e^{a\sin^{-1}x}$. S.T $(1-x^2)\,y_2 - xy_1 - a^2y = 0$

$$y = e^{a\sin^{-1}x}$$

diff w.r to $x$.

$$\frac{dy}{dx} = e^{a\sin^{-1}x}\cdot\frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,\frac{dy}{dx} = ay$$

sq. on b.s

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2$$

diff w.r to $x$.

$$(1-x^2)\,2\left(\frac{dy}{dx}\right)\cdot\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\cdot(-2x) = a^2\,2y\,\frac{dy}{dx}$$

$$\Rightarrow 2\frac{dy}{dx}\left[(1-x^2)\frac{d^2y}{dx^2}+\frac{dy}{dx}(-x)\right]=a^2 2y\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$$

6. If $y=\sin(m\sin^{-1}x)$. S.T $(1-x^2)y_2-xy_1+m^2y=0$

$$y=\sin(m\sin^{-1}x)$$

$$\sin^{-1}y=m\sin^{-1}x$$

diff w.r to x.

$$\frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx}=m\frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,\frac{dy}{dx}=m\sqrt{1-y^2}$$

sq. on b.s

$$(1-x^2)\left(\frac{dy}{dx}\right)^2=m^2(1-y^2)$$

diff w.r to x.

$$(1-x^2)\,2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2(-2x)=m^2(-2y)\frac{dy}{dx}$$

$$2\frac{dy}{dx}\left[(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}\right]=m^2(-2y)\frac{dy}{dx}$$

$$(1-x^2)y_2-xy_1+m^2y=0$$

7. If $2x=y^{1/m}+y^{-1/m}$ p.T $(x^2-1)y_2+xy_1-m^2y=0$

$$2x=y^{1/m}+y^{-1/m}$$

diff w.r to x

$$2=\frac{1}{m}y^{1/m-1}\frac{dy}{dx}+\left(\frac{-1}{m}\right)y^{\frac{-1}{m}-1}\frac{dy}{dx}$$

$$2=\frac{1}{m}y^{\frac{1-m}{m}}y_1-\frac{1}{m}y^{\frac{-1-m}{m}}y_1$$

$$2 = \frac{1}{m} y^{\frac{1}{m}} y^{-m/m} \cdot y_1 - \frac{1}{m} y^{-1/m} \cdot y^{-m/m} \cdot y_1$$

$$2 = \frac{1}{my} \cdot y_1 \left( y^{1/m} - y^{-1/m} \right)$$

$$2my = y_1 \left( y^{1/m} - y^{-1/m} \right)$$

$(a-b)^2$

Sq. on b.s. $\Rightarrow a^2+b^2-2ab$

$$4m^2y^2 = y_1^2 \left( y^{1/m} - y^{-1/m} \right)^2$$

$(a+b)^2 - 4ab$

$a^2+b^2+2ab-4a$

$$4m^2y^2 = y_1^2 \left[ \left( y^{1/m} + y^{-1/m} \right)^2 - 4 \right]$$

$$4m^2y^2 = y_1^2 (4x^2-4)$$

$$\cancel{4}m^2y^2 = y_1^2\, 4(x^2-1)$$

$$m^2y^2 = y_1^2 (x^2-1)$$

diff w.r to $x$.

$$m^2 2y \frac{dy}{dx} = 2(y_1)(y_2)(x^2-1) + y_1^2 (2x)$$

$$m^2 \cancel{2} y' y_1 = \cancel{2} y_1 \left[ y_2 (x^2-1) + y_1 x \right]$$

$$m^2 y = y_2 (x^2-1) + y_1 x$$

$$(x^2-1) y_2 + x y_1 - m^2 y = 0$$

8. If $y = \sin^m x$. P.T $\sin^2 x \frac{d^2y}{dx^2} = y(m^2\cos^2 x - m$

$$y = \sin^m x$$

diff w.r to $x$

$$\frac{dy}{dx} = m \sin^{m-1} x \cdot \cos x$$

diff w.r to $x$

$$\frac{d^2y}{dx^2} = m(m-1)\sin^{m-2}x \cdot \cos x \cdot \cos x + m\sin^{m-1}x\,(-\sin x)$$

$$= m(m-1)\sin^{m-2}x\cos^2 x - m\sin^m x$$

x'ly by $\sin^2 x$ on b.s

$$\sin^2 x\frac{d^2y}{dx^2} = m(m-1)\sin^{m-2}x \cdot \sin^2 x\cos^2 x - m\sin^m x\,\sin^2 x$$

$$= (m^2-m)y\sin^m x\cos^2 x - my\sin^2 x$$

$$= (m^2-m)y\cos^2 x - my\sin^2 x$$

$$= m^2y\cos^2 x - my\cos^2 x - my\sin^2 x$$

$$= m^2y\cos^2 x - my(\cos^2 x + \sin^2 x)$$

$$= m^2y\cos^2 x - my$$

$$\sin^2 x\frac{d^2y}{dx^2} = y(m^2\cos^2 x - m)$$

9. If $y = \frac{ax+b}{cx+d}$ find $\frac{d^2y}{dx^2}$ $\quad \frac{u}{v} = \frac{vdu - udv}{v^2}$

Diff w.r to $x$, $\quad vu' - uv'$

$$\frac{dy}{dx} = \frac{(cx+d)a - (ax+b)c}{(cx+d)^2}$$

$$= \frac{acx + ad - acx - bc}{(cx+d)^2}$$

$$\frac{dy}{dx} = \frac{ad - bc}{(cx+d)^2}$$

Again diff. w.r to $x$.

$$\frac{d^2y}{dx^2} = \frac{(cx+d)^2(0) - (ad-bc)2(cx+d).c}{(cx+d)^4}$$

$$= \frac{-(ad-bc)2c(cx+d)}{(cx+d)^4}$$

$$\frac{d^2y}{dx^2} = \frac{-(ad-bc)2c}{(cx+d)^3}$$

# Standard $n^{th}$ derivatives.

1. Find the $n^{th}$ derivatives of $e^{ax}$

Let, $y = e^{ax}$

$y_1 = e^{ax} \cdot a$

$y_2 = e^{ax} \cdot a^2$

$y_3 = e^{ax} \cdot a^3$

lll'y $y_n = a^n e^{ax}$

2. Find the $n^{th}$ derivatives of $\frac{1}{ax+b}$

Let $y = \frac{1}{ax+b} = (ax+b)^{-1}$

diff w.r to $x$.

$\frac{dy}{dx} = y_1 = (-1)(ax+b)^{-2} \cdot a$

$y_2 = (-1)(-2)(ax+b)^{-3} \cdot a^2$

$y_3 = (-1)(-2)(-3)(ax+b)^{-4} \cdot a^3$

$\vdots$

$y_n = (-1)(-2) \cdots (-n)(ax+b)^{-(n+1)} \cdot a^n$

$y_n = \frac{(-1)^n n! \, a^n}{(ax+b)^{n+1}}$

3. Find the $n^{th}$ derivatives of $\frac{1}{(ax+b)^2}$

Let, $y = \frac{1}{(ax+b)^2} = (ax+b)^{-2}$

$y_1 = (-2)(ax+b)^{-3} a$

$y_2 = (-2)(-3)(ax+b)^{-4} a^2$

$y_3 = (-2)(-3)(-4)(ax+b)^{-5} a^3$

$\vdots$

$$y_n = \frac{(-1)^n (n+1)!\, a^n}{(ax+b)^{n+2}}$$

4. Find the nth derivatives of $\log(ax+b)$

Let $y = \log(ax+b)$

$$y_1 = \frac{1}{ax+b} \cdot a = a(ax+b)^{-1}$$

$$y_2 = (-1)(ax+b)^{-2} a^2$$

$$y_3 = (-1)(-2)(ax+b)^{-3} a^3$$

$\vdots$ $(-1)(-2)\ldots(n-1)(ax+b)^{-n} a^n$

$$y_n = \frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$$

5. Find the nth derivatives of $\sin(ax+b)$

Let $y = \sin(ax+b)$

$y_1 = \cos(ax+b) \cdot a$ $\quad \sin(\theta + \pi/2) = \cos\theta$

$\Rightarrow y_1 = \sin[\pi/2 + (ax+b)] \cdot a$

$y_2 = \cos[\pi/2 + (ax+b)] a^2$

$\Rightarrow y_2 = \sin[\pi/2 + \pi/2 + (ax+b)] a^2$

$= \sin[2\pi/2 + (ax+b)] a^2$

$y_3 = \cos[2\pi/2 + (ax+b)] a^3$

$\Rightarrow y_3 = \sin[2\pi/2 + \pi/2 + (ax+b)] a^3$

$= \sin[3\pi/2 + (ax+b)] a^3$

$\vdots$

$y_n = \sin[n\pi/2 + (ax+b)] a^n$

6. Find $n^{th}$ derivatives of $\cos(ax+b)$

Let $y = \cos(ax+b)$

$\cos(90+\theta) = -\sin\theta$

$y_1 = -\sin(ax+b)\, a$

$= \cos[\pi/2 + (ax+b)]\, a$

$y_2 = -\sin[\pi/2 + (ax+b)]\, a^2$

$= \cos[\pi/2 + \pi/2 + (ax+b)]\, a^2$

$y_3 = -\sin[2\pi/2 + (ax+b)]\, a^2$

$= \cos[2\pi/2 + \pi/2 + (ax+b)]\, a^3$

$= \cos[3\pi/2 + (ax+b)]\, a^3$

$\vdots$

$y_n = \cos[n\pi/2 + (ax+b)]\, a^n$

7. Find the $n^{th}$ derivatives $e^{ax}\sin(bx+c)$

$u'v + v'u$

Let, $y = \overset{u}{e^{ax}}\, \overset{v}{\sin(bx+c)}$

$y_1 = e^{ax} \cdot a \sin(bx+c) + e^{ax} \cdot \cos(bx+c) \cdot b$

$= e^{ax}[a\sin(bx+c) + b \cdot \cos(bx+c)]$

Let, $a = r\cos\theta$, $b = r\sin\theta$

$r = \sqrt{a^2+b^2} = (a^2+b^2)^{1/2}$

$\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$

$\theta = \tan^{-1}(b/a)$

$y_1 = e^{ax}[r\cos\theta\sin(bx+c) + r\sin\theta\cos(bx+c)]$

$y_1 = r[e^{ax} \cdot [\sin(\theta + (bx+c))]]$

$\sin(A+B)$

$$y_2 = r\left[e^{ax} . a\left[\sin(\theta+(bx+c))\right] + \left[e^{ax}\cos(\theta+(bx+c)).b\right]\right]$$

$$= r\left[a e^{ax}\sin(\theta+(bx+c)) + b e^{ax}\cos(\theta+(bx+c))\right]$$

$$y_2 = r e^{ax}\left[a\sin(\theta+(bx+c)) + b\cos(\theta+(bx+c))\right]$$

$$y_2 = r e^{ax}\left[r\cos\theta\sin(\theta+(bx+c)) + r\sin\theta\cos(\theta+(bx+c))\right]$$

$$= r^2 e^{ax}\left[\sin\left[\theta+\theta+(bx+c)\right]\right]$$

$$y_n = r^n e^{ax}\left[\sin(n\theta+(bx+c))\right]$$

$$= (a^2+b^2)^{n/2} e^{ax} . \sin n\left[\left(\tan^{-1}(b/a)\right)+(bx+c)\right]$$

8. Find the $n^{th}$ derivatives of $e^{ax}\cos(bx+c)$

Let, $y = e^{ax}\cos(bx+c)$

$$y_1 = e^{ax} . a\cos(bx+c) + e^{ax}(-\sin(bx+c))b$$

$$y_1 = e^{ax}\left[a\cos(bx+c) - b\sin(bx+c)\right]$$

Let $a = r\cos\theta$, $b = r\sin\theta$ $\quad a+ib = x+iy$

$r = \sqrt{a^2+b^2} = (a^2+b^2)^{1/2}$ $\quad x = r(\cos\theta + i\sin\theta)$

$= r\cos\theta + ir\sin\theta$

$$\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

$$\theta = \tan^{-1}(b/a)$$

$$y_1 = e^{ax}\left[r\cos\theta\cos(bx+c) - r\sin\theta\sin(bx+c)\right]$$

$\cos(A+B)$

$= re^{ax} \cos(\theta + (bx+c))$

$y_2 = r e^{ax} \cdot a \cos(\theta + (bx+c)) + re^{ax} - \sin(\theta + (bx+c))$

$y_2 = r\left[e^{ax}\left[a \cos(\theta + (bx+c)) - b\sin(\theta + (bx+c))\right]\right]$

$= re^{ax}\left[r\cos\theta \cos(\theta + (bx+c)) - r\sin\theta \sin(\theta + (bx+c))\right]$

$= r^2 e^{ax}\left[\cos(2\theta + (bx+c))\right]$

$\vdots$

$y_n = r^n e^{ax}\left[\cos(n\theta + (bx+c))\right]$

$y_n = (a^2+b^2)^{n/2} e^{ax} \cos\left[n\left(\tan^{-1}(b/a)\right) + (bx+c)\right].$

9. Find the $n^{th}$ derivatives of $\dfrac{2x+1}{(2x-1)(2x+3)}$

By Using partial fraction Method,

$$\frac{2x+1}{(2x-1)(2x+3)} = \frac{A}{2x-1} + \frac{B}{2x+3} \quad \text{--- (1)}$$

$$\frac{2x+1}{(2x-1)(2x+3)} = \frac{A(2x+3) + B(2x-1)}{(2x-1)(2x+3)}$$

$2x+1 = A(2x+3) + B(2x-1)$

Put $x = -3/2$

$\cancel{2}\left(\frac{-3}{\cancel{2}}\right) + 1 = A\left((\cancel{2})\left(-\frac{3}{\cancel{2}}\right) + 3\right) + B\left(2\left(\frac{-3}{2}\right) - 1\right)$

$-2 = -4B$ $\qquad \boxed{B = 1/2}$

$\frac{-2}{-4} = B$

Put $x = \frac{1}{2}$

$$2\left(\frac{1}{2}\right)+1 = A\left(2\cdot\frac{1}{2}+3\right) + B\left(2\cdot\frac{1}{2}-1\right)$$

$$2 = 4A$$

$$A = \frac{1}{2}$$

sub, A & B in ①,

$$\frac{2x+1}{(2x-1)(2x+3)} = \frac{1/2}{2x-1} + \frac{1/2}{2x+3}$$

$$y = \frac{1/2}{2x-1} + \frac{1/2}{2x+3}$$

$$y = \frac{1}{2}\left[\frac{1}{2x-1} + \frac{1}{2x+3}\right]$$

$n^{th}$ derivative is,

$$y_n = \frac{1}{2}\left[\frac{(-1)^n n!\, 2^n}{(2x-1)^{n+1}} + \frac{(-1)^n n!\, 2^n}{(2x+3)^{n+1}}\right]$$

$$= \frac{(-1)^n n!\, 2^n}{2}\left[\frac{1}{(2x-1)^{n+1}} + \frac{1}{(2x+3)^{n+1}}\right]$$

$$= (-1)^n n!\, 2^n \cdot 2^{-1}\left[\frac{1}{(2x-1)^{n+1}} + \frac{1}{(2x+3)^{n+1}}\right]$$

$$= (-1)^n n!\, 2^{n-1}\left[\frac{1}{(2x-1)^{n+1}} + \frac{1}{(2x+3)^{n+1}}\right]$$

10. Find the $n^{th}$ derivatives of $\frac{x^2+1}{(2x-1)(2x+1)(2x+3)}$

By using partial fraction method

$$\frac{x^2+1}{(2x-1)(2x+1)(2x+3)} = \frac{A}{(2x-1)} + \frac{B}{(2x+1)} + \frac{C}{(2x+3)}$$

$$= \frac{A(2x+1)(2x+3) + B(2x-1)(2x+3) + C(2x-1)(2x+1)}{(2x-1)(2x+1)(2x+3)}$$

$$x^2+1 = A(2x+1)(2x+3) + B(2x-1)(2x+3) + C(2x-1)(2x+1)$$

put $x = 1/2$

$$(1/2)^2+1 = A(2(1/2)+1)(2(1/2)+3) + B(2(1/2)-1)(2(1/2)+3) + C(2(1/2)-1)(2(1/2)+1)$$

$$\frac{1+4}{4} = 8A$$

$$5/4 = 8A$$

$$\frac{5}{4\times 8} = A$$

$$A = 5/32$$

Put $x = -3/2$

$$(-3/2)^2+1 = A(0) + B(0) + C(2(-3/2)+1)(2(-3/2)+1)$$

$$\frac{9+4}{4} = 8C$$

$$\frac{13}{4} = 8C$$

$$C = \frac{13}{32}$$

Put $x = -1/2$

$$(-1/2)^2 + 1 = A(0) + B(2(-1/2) - 1)(2(-1/2) + 3) + C(0)$$

$$\frac{1+4}{4} = -B4$$

$$\frac{5}{4} = -4B$$

$$B = -\frac{5}{16}$$

Sub., A, B, C, values in ①

$$\frac{x^2+1}{(2x-1)(2x+1)(2x+3)} = \frac{5/32}{(2x-1)} + \frac{-5/16}{(2x+1)} + \frac{13/32}{(2x+3)}$$

$$y = \frac{5}{32}\left(\frac{1}{2x-1}\right) - \frac{5}{16}\left(\frac{1}{2x+1}\right) + \frac{13}{32}\left(\frac{1}{2x+3}\right)$$

$$y_n = \frac{5}{32}\left[\frac{(-1)^n n!\, 2^n}{(2x-1)^{n+1}}\right] - \frac{5}{16}\left[\frac{(-1)^n n!\, 2^n}{(2x+1)^{n+1}}\right] + \frac{13}{32}\left[\frac{(-1)^n n!\, 2^n}{(2x+3)^{n+1}}\right]$$

$$y_n = (-1)^n n!\, 2^n \left[\frac{5/32}{(2x-1)^{n+1}} - \frac{5/16}{(2x+1)^{n+1}} + \frac{13/32}{(2x+3)^{n+1}}\right]$$

11. Find the $n^{th}$ derivative of $\frac{1}{x^2(2x+3)}$ By Using partial fraction method.

$$\frac{1}{x^2(2x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{2x+3}$$

$$\frac{1}{x^2(2x+3)} = \frac{Ax(2x+3) + B(2x+3) + Cx^2}{x^2(2x+3)}$$

$$1 = A\underbrace{x(2x+3)}_{2x^2+3x} + B\underbrace{(2x+3)}_{2x+3} + \underbrace{C\cdot x^2}_{Cx^2}$$

Put $x=0$,

$$1 = A(0) + B(3) + C(0)$$

$$B = 1/3$$

Put $x = -3/2$

$$1 = A\left(\frac{-3}{2}\right)\left(\cancel{2}\left(\frac{3}{\cancel{2}}\right)+3\right) + B\left(\frac{-3}{2}\right)\left(\cancel{\frac{-3}{2}\times 2}+3\right)$$
$$+ C\left(\frac{-3}{2}\right)^2$$

$$1 = C\left(\frac{9}{4}\right)$$

$$C = \frac{4}{9}$$

coefficient of $x^2$

$0 = 2A + C$

$2A + \frac{4}{9}$

$-\frac{4}{9} = 2A$

$-\frac{4}{9 \times 2} = A$

$A = -\frac{2}{9}$

$$0 = 2A + C$$

$$= 2A + \frac{4}{9}$$

$$-\frac{4}{9} = 2A$$

$$A = -\frac{2}{9}$$

Sub A, B, C values in ①,

$$\frac{1}{x^2(2x+3)} = \frac{-2/9}{x} + \frac{1/3}{x^2} + \frac{4/9}{2x+3}$$

$$y = -\frac{2}{9}\left[\frac{1}{(x+0)}\right] + \frac{1}{3}\left[\frac{1}{(x+0)^2}\right] + \frac{4}{9}\left[\frac{1}{2x+3}\right]$$

$n^{th}$ derivative.

$$y_n = -\frac{2}{9}\left[\frac{(-1)^n n!\,(1)^n}{(x+0)^{n+1}}\right] + \frac{1}{3}\left[\frac{(-1)^n (n+1)!\,(1)^n}{(x+0)^{n+2}}\right] +$$

$$\frac{4}{9}\left[\frac{(-1)^n n!\,(2)^n}{(2x+3)^{n+1}}\right]$$

12 Find the $n^{th}$ derivative of $\sin^3 x \cdot \cos^2 x$

$y = \sin^3 x \cos^2 x$

$\sin(A+B) - \sin(A-B) = 2\sin A \cos B$

$\sin A \cos B + \cos A \sin B - \sin A \cos B + \cos A \sin B$

$$= \left[\frac{3\sin x - \sin 3x}{4}\right]\left[\frac{1+\cos 2x}{2}\right]$$

$$= \frac{1}{8}\left[(3\sin x - \sin 3x)(1+\cos 2x)\right]$$

$$= \frac{1}{8}\left[3\sin x - \sin 3x + 3\sin x \cos 2x - \sin 3x \cos 2x\right]$$

$\times^{ly}$ by '2' and $\div$ by '2'

$$= \frac{2}{16}\left[3\sin x - \sin 3x + 3\sin x \cos 2x - \sin 3x \cos 2x\right]$$

$$= \frac{1}{16}\left[6\sin x - 2\sin 3x + 3(2\sin x \cos 2x) - 2\sin 3x \cos 2x\right]$$

$$= \frac{1}{16}\left[6\sin x - 2\sin 3x + 3(\sin(3x) + \sin(-x)) - (\sin 5x + \sin x)\right]$$

$2\sin A \cos B = \sin(A+B)$

$$= \frac{1}{16}\left[6\sin x - 2\sin 3x + 3\sin 3x - 3\sin x - \sin 5x - \sin x\right]$$

$$= \frac{1}{16}\left[2\sin x + \sin 3x - \sin 5x\right]$$

$$= \frac{1}{16}\left[2\sin x + \sin 3x - \sin 5x\right]$$

$\sin^n(ax+c) = a^n \sin\left(ax+c+\frac{n\pi}{2}\right)$

$n^{th}$ derivative.

$$y_n = \frac{1}{16}\left[2(1)^n \sin\left(x+\frac{n\pi}{2}\right) + 3^n \sin\left(3x+\frac{n\pi}{2}\right) - 5^n \sin\left(5x+\frac{n\pi}{2}\right)\right].$$

13. If $y = \tan^{-1}(x/a)$ S.T $y_n = (-1)^n (n-1)!\, a^{-n} \sin^n\theta \sin n\theta$

where $\theta = \tan^{-1}(a/x)$

$$y = \tan^{-1}(x/a)$$

Diff. w.r to '$x$'

$$y_1 = \frac{1}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1}{a}$$

$$\theta = \tan^{-1}(a/x)$$

$$y_1 = \frac{1}{1+\frac{x^2}{a^2}} \cdot \frac{1}{a}$$

$$= \frac{1}{\frac{a^2+x^2}{a^2}} \cdot \frac{1}{a}$$

$$= \frac{a^2}{a^2+x^2} \cdot \frac{1}{a}$$

$$y_1 = \frac{a}{a^2+x^2}$$

$$y_1 = \frac{1}{2i}\left[\frac{1}{x-ai} - \frac{1}{x+ai}\right]$$

$n^{th}$ derivative.

$$y_n = \frac{1}{2i}\left[\frac{(-1)^n (n-1)!}{(x-ai)^n} - \frac{(-1)^n (n-1)! (1)^n}{(x+ai)^n}\right]$$

$$= \frac{1}{2i}(-1)^{n-1}(n-1)!\left[\frac{1}{(x-ai)^n} - \frac{1}{(x+ai)^n}\right]$$

Put $x = r\cos\theta$, $a = r\sin\theta$. $a = r\sin\theta$

$\frac{a}{\sin\theta} = r$

$$y_n = \frac{1}{2i}(-1)^{n-1}(n-1)!\left[\frac{1}{(r\cos\theta - ir\sin\theta)^n} - \frac{1}{(r\cos\theta + ir\sin\theta)^n}\right]$$

$$= \frac{1}{2i}(-1)^{n-1}(n-1)!\left[(r\cos\theta - ir\sin\theta)^{-n} - (r\cos\theta + ir\sin\theta)^{-n}\right]$$

$$= \frac{1}{2i}(-1)^{n-1}(n-1)!\, r^{-n}\left[\cancel{\cos\theta} - i\sin\theta - \cancel{\cos\theta} - i\sin\theta\right]^{-n}$$

$$= \frac{1}{\cancel{2i}}(-1)^{n-1}(n-1)!\, r^{-n}(-2i\sin\theta)^{-n}$$

$$= (-1)^{n-1}(n-1)!\, r^{-n}(-\sin n\theta)$$

$$= (-1)^{n-1}(n-1)!\, r^{-n}(-1)(+\sin n\theta)$$

$$= (-1)^n (n-1)!\, r^{-n}(\sin n\theta)$$

$$= (-1)^n (n-1)!\left(\frac{a}{\sin\theta}\right)^{-n}\sin n\theta$$

$= (-1)^n (n-1)!\, a^{-n} \left(\frac{1}{\sin\theta}\right)^{-n} \sin n\theta$

$= (-1)^n (n-1)!\, a^{-n} (\sin^{-1}\theta)^{-n} \sin n\theta$

$y_n = (-1)^n (n-1)!\, a^{-n} \sin^n\theta \sin n\theta.$

14. Find the n th derivative $e^{ax}\cos^2 x \sin x$.

$e^{ax}\sin(bx+c) = (a^2+b^2)^{n/2}\, e^{ax}\sin(bx+c) + n\tan^{-1}(b/a)]$

Let. $y = e^{ax}\cos^2 x \sin x.$

$= e^{ax}\left(\frac{1+\cos 2x}{2}\right)\sin x$

$= \frac{e^{ax}}{2}(1+\cos 2x)(\sin x)$

$= \frac{e^{ax}}{2}(\sin x + \cos 2x \sin x)$

$x^{ly}$ & $\div$ by 2

$= \frac{2e^{ax}}{2\times 2}(2\sin x + \cos 2x \sin x)$

$= \frac{e^{ax}}{4}\left(2\sin x + \underbrace{2\cos 2x \sin x}_{\substack{2\sin A\cos B \\ \sin(A+B)+\sin(A-B)}}\right)$

$= \frac{e^{ax}}{4}(2\sin x + \sin 3x - \sin x)$

$= \frac{e^{ax}}{4}(\sin x + \sin 3x)$

$y = \frac{1}{4}(e^{ax}\sin x + e^{ax}\sin 3x)$

$y_n = \frac{1}{4}\Big((a^2+1)^{n/2} e^{ax}\sin\left(x + n\tan^{-1}\left(\frac{1}{a}\right)\right) +$

$(a^2+3^2)^{n/2} e^{ax}\sin\left(3x + n\tan^{-1}\left(\frac{3}{a}\right)\right)$

$= \frac{e^{ax}}{4}\Big[(a^2+1)^{n/2}\sin(x + n\tan^{-1}(1/a)) +$

$(a^2+3^2)^{n/2}\sin(3x + n\tan^{-1}(3/a))\Big]$

15. Find the $n$th derivative of $\sin^3 2x$.

Let $y = \sin^3 2x$

W.K.T

$$\boxed{\sin 3x = 3\sin x - 4\sin^3 x}$$

$$4\sin^3 x = 3\sin x - \sin 3x$$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

Replace $x$ by $2x$

$$\sin^3 2x = \frac{3}{4}\sin 2x - \frac{1}{4}\sin 6x$$

$$y = \frac{3}{4}\sin(2x+0) - \frac{1}{4}(6x+0)$$

$n$th derivatives

$$y_n = \frac{3}{4}\left[2^n \sin\left(2x+0+\frac{n\pi}{2}\right)\right] - \frac{1}{4}\left[6^n \sin\left(6x+0+\frac{n\pi}{2}\right.\right.$$

16. Find the $n$th derivative of $\cos^4 x$

$$y = (\cos^2 x)^2$$

$$= \left(\frac{1+\cos 2x}{2}\right)^2$$

$$= \frac{1}{4}\left(1 + 2\cos 2x + \cos^2 2x\right)$$

$$= \frac{1}{4}\left[1 + 2\cos 2x + \left(\frac{1+\cos 2(2x)}{2}\right)\right]$$

$$= \frac{1}{4}\left[\frac{2 + 4\cos 2x + 1 + \cos 4x}{2}\right]$$

$$= \frac{1}{4\times 2}\left[3+4\cos 2x+\cos 4x\right],$$

$$y = \frac{1}{8}\left[3+4\cos(2x+0)+\cos(4x+0)\right]$$

$n^{th}$ derivative,

$$y_n = \frac{1}{8}\left[4\left(2^n\cos\left(2x+0+\frac{n\pi}{2}\right)\right)+4^n\cos\left(4x+0+\frac{n\pi}{2}\right)\right].$$

17. Find the $n^{th}$ derivative of $\sin 2x \sin 4x \sin 6x$

$$y = \sin 2x \sin 4x \sin 6x$$

$x^{ly}$ & $\div$ by 2,

$$y = \frac{2\sin 2x \sin 4x \sin 6x}{2}$$

$$= \frac{1}{2}\left[2\sin 2x\sin 4x\right]\sin 6x$$

$$= \frac{1}{2}\left[\cos(2x-4x)-\cos(2x+4x)\right]\sin 6x$$

$$= \frac{1}{2}\left[\cos(-2x)-\cos(6x)\right]\sin 6x.$$

$$= \frac{1}{2}\left[\cos(2x)-\cos(6x)\right]\sin 6x$$

$$= \frac{1}{2}\left[\cos 2x\sin 6x-\cos 6x\sin 6x\right]$$

$x^{ly}$ & $\div$ by 2.

$$= \frac{2}{4}\left[\cos 2x\sin 6x-\cos 6x\sin 6x\right]$$

$$= \frac{1}{4}\left[2\cos 2x\sin 6x-2\cos 6x\sin 6x\right]$$

$$= \frac{1}{4}\left[\sin(2x+6x) - \sin(2x-6x)\right] -$$

$$\left[\sin(6x+6x) - \sin(6x-6x)\right]$$

$$= \frac{1}{4}\left[\sin 8x - \sin(-4x) - \sin 12x + \sin(0)\right]$$

$\sin 8x + \sin 4x - \sin 12x + \sin 0$

$n^{th}$ derivatives.

$$y_n = \frac{1}{4}\left[8^n \sin\left(8x + \frac{n\pi}{2}\right) + \left(4^n \sin\left(4x + \frac{n\pi}{2}\right)\right) + \left(-12^n \sin\left(12x + \frac{n\pi}{2}\right)\right)\right]$$

17. Find the $n^{th}$ derivative of $e^{3x} \sin x \sin 2x \sin 3x$

$$y = e^{3x} \sin x \sin 2x \sin 3x.$$

$x'y$ & $\div$ by 2,

$$y_n = \frac{e^{3x}}{2}\left[2\sin x \sin 2x\right]\sin 3x$$

$$= \frac{e^{3x}}{2}\left[\cos(x-2x) - \cos(x+2x)\right]\sin 3x$$

$$= \frac{e^{3x}}{2}\left[\cos(-x) - \cos(3x)\right]\sin 3x$$

$$= \frac{e^{3x}}{2}\left[\cos x - \cos 3x\right]\sin 3x$$

$$= \frac{e^{3x}}{2}\left[\cos x \sin 3x - \cos 3x \sin 3x\right]$$

$x'y$ $\div$ by 2,

$$= \frac{e^{3x}}{4}\left[2\cos x \sin 3x - 2\cos 3x \sin 3x\right]$$

$$= \frac{e^{3x}}{4}\left[\sin(4x) - \sin(-2x) - \sin(6x) - \sin(0)\right]$$

$$= \frac{e^{3x}}{4}\left[\sin 4x + \sin 2x - \sin 6x\right]$$

$$= \frac{1}{4}\left[e^{3x}\sin 4x + e^{3x}\sin 2x - e^{3x}\sin 6x\right]$$

$n^{th}$ derivatives is

$$y_n = \frac{1}{4}\Big[(3^2+4^2)^{n/2} e^{3x}\sin(4x + n\tan^{-1}4/3) + (3^2+4^2)^{n/2} e^{3x}\sin(2x + n\tan^{-1}2/3)\Big] - (3^2+6^2)^{n/2} e^{3x}\{6x + n\tan^{-1}6/3$$

Leibinitz's theorem.

If $u$ and $v$ are the positive integer of $x$ and $n$, then $D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \dots + nc_r u_{n-r} v_r + \dots + u v_n$ but $D^n(u-v)$ stands for the $n^{th}$ derivative of $uv$.

$$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + nc_r.$$

Problems:-

1. Find the $n^{th}$ derivative of $x^2 e^{5x}$.

$$y = x^2 e^{5x}.$$

$$u = e^{5x} \qquad v = x^2$$

Diff. w.r to $x$

$$u_1 = e^{5x}.(5) \qquad v_1 = 2x$$

$$u_2 = e^{5x}.5^2 \qquad v_2 = 2$$

$$\vdots$$

$$u_n = e^{5x}.5^n$$

$$u_{n-1} = e^{5x}.5^{n-1}$$

$$u_{n-2} = e^{5x}.5^{n-2}$$

$$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \dots$$

$$= e^{5x}5^n(x^2) + n e^{5x} 5^{n-1}(2x) + \frac{n(n-1)}{2!} e^{5x} 5^{n-2}(2) + \dots$$

$$= e^{5x}\left[x^2 5^n + n\, 2x\, 5^{n-1} + \frac{n(n-1)}{2} 5^{n-2} (2)\right]$$

$$= e^{5x} 5^{n-2}\left[x^2 5^2 + 2n x 5 + n(n-1)\right]$$

$$= e^{5x} 5^{n-2}\left[25x^2 + 10nx + n^2 - n\right]$$

$$= e^{5x} 5^{n-2}\left[25x^2 + (10x-1)n + n^2\right]$$

2. Find the $n^{th}$ derivative of $e^x \log x$.

$y = e^x \log x$

$u = e^x$ $\qquad v = \log x$

$u_1 = e^x$ $\qquad v_1 = 1/x$

$\vdots$

$u_n = e^x$ $\qquad v_2 = -1/x^2$

$u_{n-1} = e^x$ $\qquad v_3 = 2/x^3$

$u_{n-2} = e^x$ $\qquad \vdots$

$$v_n = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \cdots + u v_n$$

$$= e^x \log x + n e^x \frac{1}{x} + \frac{n(n-1)}{2!} e^x \left(\frac{-1}{x^2}\right) + \cdots + \frac{e^x (-1)^{n-1}(n-1)!}{x^n}$$

$$= e^x \left[\log x + \frac{n}{2} - \frac{n(n-1)}{2!}\frac{1}{x^2} + \cdots + \frac{(-1)^{n-1}(n-1)!}{x^n}\right]$$

3. Find the $n^{th}$ derivative of $x^2 \sin 5x$

$y = x^2 \sin 5x$

$u = \sin 5x$ $\qquad v = x^2$

$u_1 = \cos 5x \cdot 5$ $\qquad v_1 = 2x$

$u_1 = \sin\left(\frac{\pi}{2} + 5x\right) \cdot 5$ $\qquad v_2 = 2$

$u_2 = \cos\left(\frac{\pi}{2} + 5x\right) \cdot 5^2$

$u_2 = \sin\left[\frac{\pi}{2} + \left(\frac{\pi}{2} + 5x\right)\right] \cdot 5^2$

$u_2 = \sin\left(\frac{2\pi}{2} + 5x\right) 5^2$

$\vdots$

$u_n = \sin\left(5x + \frac{n\pi}{2}\right) \cdot 5^n$

$u_{n-1} = \sin\left(5x + (n-1)\frac{\pi}{2}\right) 5^{n-1}$

$u_{n-2} = \sin\left(5x + (n-2)\frac{\pi}{2}\right) 5^{n-2}$

$D^n(uv) = u_n v + nC_1 u_{n-1} v_1 + nC_2 u_{n-2} v_2 + \dots$

$= \sin\left(5x + \frac{n\pi}{2}\right) 5^n x^2 + n \sin\left(5x + (n-1)\frac{\pi}{2}\right)$

$5^{n-1} \cdot 2x + \frac{n(n-1)}{2!} \sin\left(5x + (n-2)\frac{\pi}{2}\right) 5^{n-2} (2)$.

$= 5^{n-2}\left[x^2 \sin\left(5x + \frac{n\pi}{2}\right) 5^2 + n \sin\left(5x + \right.\right.$

$\left.\left.(n-1)\frac{\pi}{2}\right)(5)(2x) + n(n-1)\sin\left(5x + (n-2)\frac{\pi}{2}\right)\right]$

$= 5^{n-2}\left[25x^2 \sin\left(5x + \frac{n\pi}{2}\right) + \cancel{\cos}\ 10nx \cdot \right.$

$\left. \sin\left(5x + (n-1)\frac{\pi}{2}\right) + n(n-1)\sin\left(5x + (n-2)\frac{\pi}{2}\right)\right]$

5. If $y = a\cos(\log x) + b\sin(\log x)$. S.T

$x^2 y_{n+2} + (2n+1) x y_{n+1} + (n^2+1) y_n = 0$.

$y = a\cos(\log x) + b\sin(\log x)$ —— ①

Diff eqn ① w.r.to $x$

$y_1 = a(-\sin)(\log x) \frac{1}{x} + b\cos(\log x) \frac{1}{x}$

$y_1 = \frac{-a\sin(\log x) + b\cos(\log x)}{x}$

$xy_1 = -a\sin(\log x) + b\cos(\log x)$ —— ②

Diff eqn ② w.r to $x$,

$xy_2 + y_1 = -a\cos(\log x)\frac{1}{x} + b(-\sin)\log x \frac{1}{x}$

$= -\left[\frac{a\cos\log x + b\sin\log x}{x}\right]$

$x(xy_2 + y_1) = -[a\cos(\log x) + b\sin(\log x)]$

$x^2 y_2 + xy_1 = -y$

$x^2 y_2 + xy_1 + y = 0$

By leibinity theorem of $n^{th}$ derivative.

$(x^2 y_2 + xy_1 + y_n)_n = 0$.

$(x^2 y_2)_n + (xy_1)_n + y_n = 0$

$I^{st}$ term $(x^2 y_2)_n$

$u_n = (y_2)_n$

$u_n = y_{n+2}$

$u_{n-1} = y_{n-1+2}$

$u_{n-1} = y_{n+1}$

$u_{n-2} = y_{n-2+2}$

$u_{n-2} = y_n$

$D^n(uv) = u_n v + nc_1 u_{n-1} v_1 + nc_2 u_{n-2} v_2 + \cdots$

$(x^2 y_2)_n = y_{n+2}\, x^2 + n y_{n+1}\, 2x + \frac{n(n-1)}{2} y_n \cdot 2$

$= y_{n+2}\, x^2 + 2nx\, y_{n+1} + (n^2 - n) y_n$

$\underline{\text{II}}^{nd}$ term.

$(xy_1)_n = y_{n+1}\, x + n y_n (1)$

$= y_{n+1}\, x + n y_n$ — ⑤

$u_n = (y_1)_n$

$u_n = y_{n+1}$

$u_{n-1} = y_{n-1+1}$

$= y_n$

Sub ④ & ⑤ in ③.

$y_{n+2}\, x^2 + 2nx y_{n+1} + n^2 y_n - n y_n + y_{n+1}\, x + n y_n + y_n = 0$

$y_{n+2}\, x^2 + (2nx + x) y_{n+1} + (n^2 + 1) y_n = 0$

$y_{n+2}\, x^2 + (2n+1)\, x\, y_{n+1} + (n^2 + 1) y_n = 0$

6. If $y = (\sin h^{-1} x)^2$ P.T $(1+x^2) y_{n+1} + (2n+1) x y_{n+1} + n^2 y_n = 0$.

$y = (\sinh^{-1}x)^2$

Diff w.r to $x$

$y_1 = 2\sinh^{-1}x \frac{1}{\sqrt{1+x^2}}$

$\sqrt{1+x^2}\, y_1 = 2\sinh^{-1}x$

squaring on b.s,

$(1+x^2)y_1^2 = 4(\sinh^{-1}x)^2$

$(1+x^2)y_1^2 = 4y$

Again diff. with r. to $x$

$(1+x^2)2y_1 \frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right) = 4\frac{dy}{dx}$

$\cancel{2y_1}\left[(1+x^2)y_2 + xy_1\right] = 2y_1 . 2$

$(1+x^2)y_2 + xy_1 - 2 = 0$

Applying leibinit~~z~~s theorem and differencing

$\left[(1+x^2)y_2\right]_n + \left[xy_1\right]_n - (2)_n = 0$

$(1+x^2)y_{n+2} + nc_1\, y_{n+1}(2x) + nc_2\, y_n(2) +$

$y_{n+1}\, x + nc_1\, y_n = 0$

$(1+x^2)y_{n+2} + 2nx\,y_{n+1} + n^2y_n - ny_n + xy_{n+1} + ny_n = 0$

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0.$

7.

7. $y = (x + \sqrt{1+x^2})^m$. P.T. $(1+x^2)\,y_{(n+2)} + (2n+1)x\,y_{n+1} + (n^2 - m^2)\,y_n = 0$.

$$y = (x + \sqrt{1+x^2})^m$$

$$y = (x + (1+x^2)^{1/2})^m$$

Diff. with r to $x$.

$$y_1 = m\,(x + (1+x^2)^{1/2})^{m-1} \cdot \left(1 + \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x\right)$$

$$= m\,(x + (1+x^2)^{1/2})^{m-1} \cdot (1 + x(1+x^2)^{-1/2})$$

$$= m\,(x + (1+x^2)^{1/2})^{m-1} \cdot \left(1 + \frac{x}{\sqrt{1+x^2}}\right)$$

$$y_1 = m\,(x + \sqrt{1+x^2})^{m-1} \left(\frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$$

$$\sqrt{1+x^2}\,y_1 = m\,(x + \sqrt{1+x^2})^m$$

$$\sqrt{1+x^2}\,y_1 = m\,y$$

Squaring on b. s,

$$(1+x^2)\,y_1^2 = m^2 y^2$$

Diff w. r to $x$.

$$(1+x^2)\,2y_1 \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 2x = m^2\, 2y \frac{dy}{dx}$$

$$\cancel{2}\cancel{\frac{dy}{dx}}\left[(1+x^2) + \frac{d^2y}{dx^2} \cdot x \frac{dy}{dx}\right] = m^2 \cancel{\frac{dy}{dx}}\,\cancel{2}y$$

$(1+x^2)\,y_2 + x\,y_1 - m^2 y = 0$

By Leibnitz theorem of $n^{th}$ derivative

$[(1+x^2)y_2]_n + [xy_1]_n - [m^2 y]_n = 0$

$(1+x^2)y_{n+2} + nc_1 y_{n+1} 2x + nc_2 y_n(2) +$

$y_{n+1}x + nc_1 y_n - (m^2 y_n) = 0$.

$(1+x^2)y_{n+2} + 2nx\, y_{n+1} + n^2 y_n - ny_n + xy_{n+1}$

$- m^2 y_n + ny_n = 0$.

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

8. If $y = e^{a\sin^{-1}x}$ P.T $(1-x^2)y_{n+2} - (2n+1)xy_{n+1}$
$-(n^2+a^2)y_n = 0$

$y = e^{a\sin^{-1}x}$.

Diff. w. r to x,

$y_1 = e^{a\sin^{-1}x} \cdot \frac{a}{\sqrt{1-x^2}}$

$y_1 = \frac{ay}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, y_1 = ay$

Sq. on b. s,

$(1-x^2)y_1^2 = a^2 y^2$

Diff. w. r to x,

$(1-x^2)2y_1y_2 + y_1^2(-2x) = a^2 y^2$

$\div 2y_1$

$[(1-x^2)y_2 - [xy_1]_n = a^2 y$

$(1-x^2)y_2 - xy_1 - a^2 y = 0$.

Applying Leibnitz theorem of $n^{th}$ derivatives.

$$[(1-x^2)y_2]_n - [xy_1]_n - [a^2y]_n = 0$$

$$(1-x^2)\,y_{n+2} + nC_1\,y_{n+1}(-2x) + nC_2\,y_n(-2) -$$

$$y_{n+1}\,x + nC_1\,y_n - a^2y_n = 0$$

$$(1-x^2)\,y_{n+2} + n\,y_{n+1}(-2x) + n(n-1)\,y_n - y_{n+1}\,x -$$

$$n\,y_n - a^2y_n = 0$$

$$(1-x^2)\,y_{n+2} + n\,y_{n+1}(-2x) - n^2y_n + ny_n - y_{n+1}\,x -$$

$$ny_n - a^2y_n = 0$$

$$(1-x^2)\,y_{n+2} - (2n+1)\,y_{n+1}\,x - (n^2+a^2)\,y_n = 0$$

9. If $y = \sin(m\sin^{-1}x)$. P.T $(1-x^2)\,y_{n+2} - (2n+1)\,x\,y_{n+1} + (m^2-n^2)\,y_n = 0$

$$y = \sin(m\sin^{-1}x)$$

$$\sin^{-1}y = m\sin^{-1}x$$

Diff w.r to $x$.

$$\frac{1}{\sqrt{1-y^2}}\,\frac{dy}{dx} = \frac{m}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\,\frac{dy}{dx} = m\sqrt{1-y^2}$$

sq. on b.s,

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1-y^2)$$

Diff. w.r to $x$

$$(1-x^2)\,2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) - 2x\left(\frac{dy}{dx}\right)^2 = -m^2\,2y\left(\frac{dy}{dx}\right)$$

$$2\frac{dy}{dx}\left[(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}\right]=-m^2\cancel{2}\left(\frac{dy}{dx}\right)y$$

$$(1-x^2)y_2-xy_1+m^2y=0$$

Applying leibnitz theorem,

$$[(1-x^2)y_2]_n-[xy_1]_n+[m^2y]_n=0$$

$$(1-x^2)y_{n+2}+nC_1\,y_{n+1}(-2x)+nC_2\,y_n(-2)-$$
$$[xy_{n+1}+nC_1\,y_n]+[m^2y_n]=0$$

$$(1-x^2)y_{n+2}+n(-2x)y_{n+1}+\frac{n(n-1)}{2}(-2)y_n-$$
$$xy_{n+1}-ny_n+m^2y_n=0$$

$$(1-x^2)y_{n+2}-2nxy_{n+1}-n^2y_n+\cancel{ny_n}-$$
$$xy_{n+1}-\cancel{ny_n}+m^2y_n=0$$

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$$

10. If $\cos^{-1}(y/b)=n\log(x/n)$. P.T

$$x^2y_{n+2}+(2n+1)y_{n+1}+2n^2y_n=0.$$

Let,

$$\cos^{-1}(y/b)=n\log(x/n)$$

$$\cos^{-1}(y/b)=n[\log x-\log n]$$

Diff w. r to $x$,

$$\frac{-1}{\sqrt{1-y^2/b^2}}\left(\frac{1}{b}\right)\frac{dy}{dx}=n\left(\frac{1}{x}\right)$$

$$\frac{-1}{\sqrt{\frac{b^2-y^2}{b^2}}}\left(\frac{1}{b}\right)\frac{dy}{dx}=\frac{n}{x}$$

$$\frac{-\cancel{b}}{\sqrt{b^2-y^2}}\left(\frac{1}{\cancel{b}}\right)\frac{dy}{dx}=\frac{n}{x}$$

$$\frac{-1}{\sqrt{b^2-y^2}}\frac{dy}{dx}=\frac{n}{x}$$

$$\therefore -x\frac{dy}{dx}=n\sqrt{b^2-y^2}$$

sq. on b.s. & diff w.r to $x$

$$x^2\left(\frac{dy}{dx}\right)^2+x^2\,2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=-n^2\,2y\,\frac{dy}{dx}$$

$$\cancel{2}\cancel{\frac{dy}{dx}}\left[x\frac{dy}{dx}+x^2\frac{d^2y}{dx^2}\right]=-n^2y\,\cancel{2}\cancel{\frac{dy}{dx}}$$

$$xy_1+x^2y_2+n^2y=0$$

Applying leibnitz's theorem of $n^{th}$ derivative

$$(x^2y_2)_n+(xy_1)_n+(n^2y)_n=0$$

$$x^2y_{n+2}+nC_1\,x^2y_{n+1}(2x)+nC_2\,2y_n+xy_{n+1}+y_n+n^2y_n=0$$

$$x^2y_{n+2}+2nxy_{n+1}+n^2y_n-ny_n+xy_{n+1}+ny_n+n^2y_n=0$$

$$x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0$$

Hence proved.

11. If $y^{1/m} + y^{-1/m} = 2x$. P.T. $(x^2-1)y_{n+2} + (2n+1)xy_{n+1}$ $+ (n^2-m^2)y_n = 0$

Let, $y^{1/m} + y^{-1/m} = 2x$

Diff w. r to $x$.

$$\frac{1}{m} y^{\frac{1}{m}-1} y_1 + \left(\frac{-1}{m}\right) y^{\frac{-1}{m}-1} y_1 = 2$$

$$\frac{1}{m} y^{1/m} \cdot y^{-1} y_1 - \frac{1}{m} y^{-1/m} \cdot y^{-1} y_1 = 2$$

$$\frac{y_1}{my}\left[y^{1/m} - y^{-1/m}\right] = 2$$

Sq. on b.s

$$\frac{y_1^2}{m^2y^2}\left[y^{1/m} - y^{-1/m}\right]^2 = 4$$

$$y_1^2\left[y^{1/m} - y^{-1/m}\right]^2 = 4m^2y^2$$

$$y_1^2\left[(y^{1/m} + y^{-1/m})^2 - 4\right] = 4m^2y^2$$

$$y_1^2\left[4x - 4\right] = 4m^2y^2$$

$$4y_1^2(x^2-1) = 4m^2y^2$$

$$(x^2-1)y_1^2 = m^2y^2$$

Diff w. r to $x$.

$$2x(y_1^2) + (x^2-1)2y_1y_2 = m^2 2yy_1$$

$$[xy_1 + (x^2-1)y_2] = m^2$$

$$(x^2-1)y_2 + xy_1 = m^2 y$$

$$(x^2-1)y_2 + xy_1 - m^2 y = 0$$

Applying leibnitz theorem of nth derivative

$$[(x^2-1)y_2]_n + [xy_1]_n - [m^2 y]_n = 0$$

$$(x^2-1)y_{n+2} + nC_1 y_{n+1}\, 2x + nC_2 y_n \cdot 2 + x y_{n+1} + n y_n - m^2 y_n = 0$$

$$(x^2-1)y_{n+2} + 2n y_{n+1} x + n^2 y_n - n y_n + x y_{n+1} + n y_n - m^2 y_n = 0$$

$$(x^2-1)y_{n+2} + (2n+1) x y_{n+1} + (n^2-m^2) y_n = 0$$

Hence proved.

12. If $y = [\log(x+\sqrt{1+x^2})]^2$. S.T $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$. Find $y_n(0)$

$1+\frac{1}{2}(1+x^2)^{-1/2}2x$

$1+\frac{x}{\sqrt{1+x^2}}$

$$y = [\log(x+\sqrt{1+x^2})]^2 \quad \text{--- (1)}$$

Diff w. r to x.

$$\frac{dy}{dx} = 2[\log(x+\sqrt{1+x^2})]\frac{1}{x+\sqrt{1+x^2}}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$\frac{dy}{dx} = 2[\log(x+\sqrt{1+x^2})]\frac{1}{x+\sqrt{1+x^2}}\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$$

$$\sqrt{1+x^2}\,\frac{dy}{dx} = 2\log(x+\sqrt{1+x^2}) \quad \text{--- (2)}$$

$y_1$

sq. on b.s.

$$(1+x^2)\left(\frac{dy}{dx}\right)^2 = 4\left[\log\left(x+\sqrt{1+x^2}\right)\right]$$

$$(1+x^2)\left(\frac{dy}{dx}\right)^2 = 4y$$

Again diff w.r to $x$.

$$2x\left(\frac{dy}{dx}\right)^2 + (1+x^2)2\,\frac{dy}{dx}\cdot\frac{d^2y}{dx^2} = \frac{dy}{dx}\cdot 4$$

$$2\frac{dy}{dx}\left[x\frac{dy}{dx} + (1+x^2)\frac{d^2y}{dx^2}\right] = \cancel{4}_2\,\frac{dy}{dx}$$

$$xy_1 + (1+x^2)y_2 = 2$$

$$(1+x^2)y_2 + xy_1 - 2 = 0 \quad \text{--- ③}$$

Applying leibnitz's theorem of nth derivati

$$[(1+x^2)y_2]_n + [xy_1]_n - [2]_n = 0$$

$$(1+x^2)y_{n+2} + nc_1\, y_{n+1}(2x) + nc_2\, y_n\cdot(2) + xy_{n+1} + ny_n - 0 = 0$$

$$(1+x^2)y_{n+2} + ny_{n+1}(2x) + n^2 y_n - ny_n + xy_{n+1} + ny_n = 0 \Rightarrow$$

$$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0 \quad \text{--- ④}$$

Put $x=0$ in eqn ①, ②, ③, ④

① $\Rightarrow y = y_0 \Rightarrow y = \log(1) = 0$

② $\Rightarrow y_1 = 0 \Rightarrow \frac{dy}{dx} = \frac{2\log(1)}{1} = 0$

③ $\Rightarrow y_2 = 2 \Rightarrow (1+0)y_2 + 0 - 2 = 0$

$$y_2 = 2$$

(4) $\Rightarrow y_{n+2} = -n^2 y_n \Rightarrow (1+0) y_{n+2} + (2n+1) y_{n+1} + n^2 y_n = 0$.

$y_{n+2} + 0 + n^2 y_n = 0$

$y_{n+2} = -n^2 y_n$

$n \Rightarrow 1,\ y_3 = -n^2 y_1$

$= (-1)^2\, 0$

$n = 2 \Rightarrow y_4 = (-2)^2 y_2$

$= -8$

$n = 3 \Rightarrow y_5 = (-3)^2 y_3 = 0$

$n = 4 \Rightarrow y_6 = -(4)^2 (-8)$

$y_6 = 128$ $\qquad y_n(0) = 0$

Proceeding in this way we get $n$ is odd

$y_n(0) = 0$

when $n$ is even

$y_n(0) = (-1)^{\frac{n-2}{2}}\, 2 \cdot 2^2 \cdot 4^2 \cdot 6^2 \cdots (n-2)^2$

4. Find the nth derivative of $e^x \cdot x^2$. P.T

$y_n = \frac{1}{2} n(n-1) y_2 - n(n-2) y_1 + \frac{1}{2}(n-1)(n-2) y$.

$y = e^x x^2$ $\qquad V = x^2$

$u_1 = e^x$ $\qquad V_1 = 2x$

$u_2 = e^x$ $\qquad V_2 = 2$

$\vdots$

$u_n = e^x$

By Leibnitz theorem.

$D^n(uv) = u_n V + nC_1 u_{n-1} V_1 + nC_2 u_{n-2} V_2$

$= e^x x^2 + n e^x 2x + \frac{n(n-1)}{\cancel{2}} e^x \cdot \cancel{2}$

$y_n = e^x(x^2+2nx+n^2-n)$ — ①

R.H.S,

$= \frac{1}{2}\, n(n-1)y_2 - n(n-1)y_1 + \frac{1}{2}(n-1)(n-2)y_n$

$= \frac{1}{2}\, n(n-1)\left[e^x(x^2+4x+4-2)\right] - n(n-2)$

$\left[e^x(x^2+2x)\right] + \frac{1}{2}\left[(n-1)(n-2)\left[e^x x^2\right]\right]$

$= \frac{1}{2}\, n(n-1)\, e^x\left[x^2+4x+2\right] - n(n-2)$

$e^x(x^2+2x) + \frac{1}{2}(n-1)(n-2)\, e^x . x^2$

$= \frac{e^x}{2}\left[(n^2-n)(x^2+4x+2)\right] - (2n^2-4n)(x^2+2x)$

$+ (n^2-2n-n+2)\, x^2\Big]$

$= \frac{e^x}{2}\left[x^2(n^2-n-2n^2+4n+n^2-3n+2)\right] +$

$x\left[4n^2-4n-4n^2+8n\right] + (2n^2-2n)$

$= \frac{e^x}{2}\left[2x^2+4nx+2n^2-2n\right]$

$= \frac{2e^x}{2}\left[x^2+2nx+n^2-n\right]$

$= e^x\left[x^2+2nx+n^2-n\right]$

$= y_n$ By ①.

$=$ LHS.

# Jacobians Total Differential

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Problems:-

1. If $x+y+z = u$, $y+z = uv$, $z = uvw$,

P.T, $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$x = u-y-z$

$y = uv-z$

$z = uvw$

$\frac{\partial x}{\partial u} = 1$, $\frac{\partial x}{\partial v} = 0$ $\frac{\partial x}{\partial w} = 0$

$\frac{\partial y}{\partial u} = v$, $\frac{\partial y}{\partial v} = u$ $\frac{\partial y}{\partial w} = 0$

$\frac{\partial z}{\partial u} = vw$, $\frac{\partial z}{\partial v} = uw$, $\frac{\partial z}{\partial w} = uv$

$$J = \begin{vmatrix} 1 & 0 & 0 \\ v & u & 0 \\ vw & uw & uv \end{vmatrix} = 1(u^2v - 0) - 0(\ ) + 0(\ )$$

$= 1(u^2v)$

$= u^2v.$

H.P

2. If $x = u^2 - v^2$, $y = 2uv$ to find $\frac{\partial(x,y)}{\partial(u,v)}$

$2u$ $2u$ $-2v$ $2v$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$\frac{\partial x}{\partial u} = 2u \qquad \frac{\partial y}{\partial u} = 2v$$

$$\frac{\partial x}{\partial v} = -2v \qquad \frac{\partial y}{\partial v} = 2u$$

$$J = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix}$$

$$= 4u^2 + 4v^2$$

$$= 4(u^2+v^2)$$

3. If $u = x+y$, $v = x-y$, to find $\frac{\partial(u,v)}{\partial(x,y)}$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = 1 \qquad \partial v = 1$$

$$\frac{\partial u}{\partial y} = 1 \qquad \frac{\partial v}{\partial y} = -1$$

$$J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1$$

$$J = -2$$

1. If $u = x+y$, $y = uv$. Find $\frac{\partial(x,y)}{\partial(u,v)}$

$u = x+y$, $x = u-y$ $\qquad y = uv$ $\qquad u = x+y$

$x = u - uv$ $\qquad u - y = x$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$= u(1-v) + uv$

$= u - uv + uv$

$J = u$

5. If $u = \frac{yz}{x}$, $v = \frac{2x}{y}$, $w = \frac{xy}{z}$ then S.T

$J(u,v,w) = 4$

$$J(u,v,w) = \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

$$= \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

$u = \frac{yz}{x}$, $u = zyx^{-1}$

$\frac{\partial u}{\partial x} = \frac{-yz}{x^2}$ $\quad \frac{\partial u}{\partial y} = \frac{z}{x}$ $\quad \frac{\partial u}{\partial z} = \frac{y}{x}$

$v = \frac{2x}{y} \Rightarrow 2xy^{-1}$

$\frac{\partial v}{\partial x} = \frac{2}{y}$, $\quad \frac{\partial v}{\partial y} = \frac{-2x}{y^2}$, $\quad \frac{\partial v}{\partial z} = \frac{x}{y}$

$w = \frac{xy}{z} \Rightarrow xyz^{-1}$

$\frac{\partial u}{\partial x} = \frac{y}{z}$, $\quad \frac{\partial w}{\partial y} = \frac{x}{z}$, $\quad \frac{\partial w}{\partial z} = \frac{-xy}{z^2}$

$$J(u,v,w) = \begin{vmatrix} -yz/x^2 & z/x & y/x \\ z/y & -2x/y^2 & x/y \\ y/z & x/z & -xy/z^2 \end{vmatrix}$$

$$= \frac{-y^2}{x^2}\left(\frac{x^2 y z}{y^2 z^2} - \frac{x^2}{zy}\right) - \frac{z}{x}\left(\frac{-xy^2}{yz^2} - \right)$$

$$+ \frac{y}{x}\left(\frac{xz}{yz} + \frac{xyz}{zy^2}\right) =$$

$$= \frac{-x^2y^2z^2}{x^2y^2z^2} + \frac{x^2yz}{x^2yz} + \frac{xyz^2}{xyz^2} + \frac{xyz}{xyz} +$$

$$\frac{xyz}{xyz} + \frac{\cancel{xy^2z}}{\cancel{xzyz}} \quad \frac{xy^2z}{xzy^2}$$

$$= -\cancel{1} + \cancel{1} + 1 + 1 + 1 + 1$$

$$= 4$$

$J(u, v, w) = 4$

(6.) If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$. S.T

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{y^2 - x^2}{2u^2 v} \qquad u^3 = x + y - v^3$$

$3u^2 = 1$

$$u^3 = x + y - v^3 \qquad \frac{\partial x}{\partial u} = 3u^2$$

$$v^2 = x^3 + y^3 - u^2$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \qquad \frac{\partial v}{\partial x} = \frac{1}{3u^2}$$

$$\frac{\partial y}{\partial x} = 3v^2 \qquad \frac{\partial u}{\partial z} = 1 \qquad 3u^2 \frac{\partial u}{\partial y} = 1$$

$$\frac{\partial u}{\partial x} = \frac{1}{3u^2} \qquad \frac{\partial u}{\partial y} = \frac{1}{3u^2}$$

$$2v \cdot \frac{\partial v}{\partial x} = 3x^2 \qquad 2v \frac{\partial v}{\partial y} = 3y^2$$

$$\frac{\partial v}{\partial x} = \frac{3x^2}{2v} \qquad \frac{\partial v}{\partial y} = \frac{3y^2}{2v}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1/3u^2 & 1/3u^2 \\ 3x^2/2v & 3y^2/2v \end{vmatrix}$$

$$= \frac{3y^2}{6u^2v} - \frac{3x^2}{6u^2v}$$

$$= \frac{3(y^2-x^2)}{6u^2v}$$

$$\partial(u,v) = \frac{y^2-x^2}{2u^2v}$$

7. If $u = x+y+z$, $v = xy+yz+zx$, $w = x^3+y^3+z^3-3xyz$

S.T $J(u,v,w) = 0$

$$J(u,v,w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

$u = x+y+z$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial u}{\partial z} = 1$$

$v = xy+yz+zx$

$$\frac{\partial v}{\partial x} = y+z, \quad \frac{\partial v}{\partial y} = x+z \quad \frac{\partial v}{\partial z} = y+x.$$

$w = x^3+y^3+z^3-3xyz$

$$\frac{\partial w}{\partial x} = 3x^2-3yz \quad \frac{\partial w}{\partial y} = 3y^2-3xz$$

$$\frac{\partial w}{\partial z} = 3z^2-3xy$$

$$J(u,v,w) = \begin{vmatrix} 1 & & \\ y+z & x+z & y+x \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$= 1[(x+z)(3z^2-3xy) - (y+x)(3y^2-3xz)]$

$-1[(y+z)(3z^2-3xy) - (y+x)(3x^2-3yz)]$

$+1[(y+z)(3y^2-3xz) - (x+z)(3x^2-3yz)]$

$= 3xz^2 - 3x^2y + 3z^3 - 3xyz + 3y^3 - 3xyz -$

$3xy^2 + 3x^2z - 3yz^2 - 3xy^2 - 3z^3 + 3xyz$

$3y^3 - 3xyz - 3y^3z - 3z^2x + 3x^3 + 3xyz$

$3x^2z + 3yz^2$

$= 0$

8. If $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix}$$

$y = r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial\theta} = r\cos\theta$

$x = r\cos\theta$

$\frac{\partial x}{\partial r} = \cancel{-\sin\theta}\ \cos\theta$. $\frac{\partial x}{\partial\theta} = -r\sin\theta$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

$\frac{\partial(x,y)}{\partial(r,\theta)} = r(\cos^2\theta + \sin^2\theta)$

$= r$

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

(9) Find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial r & \partial y/\partial\theta & \partial z/\partial\phi \\ \partial z/\partial r & \partial z/\partial\theta & \partial z/\partial\phi \end{vmatrix}$$

$x = r\sin\theta\cos\phi$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$ $\quad \frac{\partial x}{\partial\theta} = r\cos\theta\cos\phi$ $\quad \frac{\partial x}{\partial\phi} = -r\sin\phi$

$y = r\sin\theta\sin\phi$

$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$ $\quad \frac{\partial y}{\partial\theta} = r\cos\theta$ $\quad \frac{\partial y}{\partial\phi} = r\cos\phi$

$z = r\cos\theta$

$\frac{\partial z}{\partial r} = \cos\theta$ $\quad \frac{\partial z}{\partial\theta} = -r\sin\theta$ $\quad \frac{\partial z}{\partial\phi} = 0$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta & -r\sin\phi \\ \sin\theta\sin\phi & r\cos\theta & r\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= [\sin\theta\cos\phi\,(0 + r^2\sin\theta\cos\phi)] - [r\cos\theta\,(0 - r\cos\theta\cos\phi)] - [r\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta)]$

$= r^2\sin^2\theta\cos^2\phi + r^2\cos^2\theta\cos\phi + r^2\sin^2\phi\sin^3\theta + r^2\sin\phi\cos^2\theta$

$= r^2[\sin^2\theta\cos^2\phi + \cos^2\theta\cos\phi + \sin^2\phi\sin^3\theta + \sin\phi\cos^2\theta]$

1. If $x = u(1+v)$, $y = v(1+u)$. find $\frac{\partial(x,y)}{\partial(u,v)}$

$u + uv$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$$

$x = u(1+v)$

$$\frac{\partial x}{\partial u} = 1+v \qquad \frac{\partial x}{\partial v} = u$$

$y = v(1+u)$ $\quad v+uv$

$$\frac{\partial y}{\partial u} = v \qquad \frac{\partial y}{\partial v} = 1+u$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = \cancel{1}$$

$(1+v)(1+u) - vu$

$1+u+v+vu-$

$=$

$= 1$

2. If $u = \frac{y^2}{2x}$, $v = \frac{x^2+y^2}{2x}$, find $\frac{\partial(u,v)}{\partial(x,y)}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

$u = y^2/2x \qquad v = \frac{x^2+y^2}{2x}$

$$\frac{\partial u}{\partial x} = \frac{-y^2}{2x^2} \qquad \frac{\partial v}{\partial y} = \frac{x^2-y^2}{2x^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{2x} \qquad \frac{\partial v}{\partial y} = \frac{2y^2}{2x}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -y^2/2x^2 & 2y/2x \\ \frac{x^2-y^2}{2x^2} & 2y^2/2x \end{vmatrix}$$

$\frac{-x^2+y^2}{2x^2}$

$-2x$

$2x^2+2y^2$

$$= \frac{-y^3}{2x^3} - \frac{y}{x}\left(\frac{x^2-y^2}{2x^2}\right)$$

$$= \frac{-y^3}{2x^3} - \frac{y}{2x^3} + \frac{y^3}{2x^3}$$

$$= \frac{-yx^2}{2x^3}$$

$$= \frac{-y}{2x}$$

Maximum and Minimum of function of variables (6 sums)

Necessary condition:-

The Necessary condition for the existence of a maxima and minima of $f(x,y)$ at $x=a$ and $y=b$ are $f_x(a,b)=0$ and $f_y(a,b)=0$.

Where $f_x(a,b)$ and $f_y(a,b)$ respectively denoted the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $x=a$, $y=b$

Sufficient Condition:-

Let $f(x)(a,b)=0$ and $f_y(a,b)=0$

Let $r = f_{xx}(a,b)$

$s = f_{yx}(a,b)$

$t = f_{yy}(a,b)$

Then

i) If $(rt-s^2)>0$ and $r>0$ $f(x,y)$ is minimum at $(a,b)$

ii) If $(rt-s^2)>0$ and $r<0$ $f(x,y)$ is maximum at $(a,b)$

iii) If $(rt-s^2)<0$ and $r\leq 0$ $f(x,y)$ is neither maximum is maximum at $(a,b)$ [It is called sadetle point]

iv) If $(rt-s^2)=0$. The case is doubtfull.

Problem:-

1. Find the maximum and minimum value of $f(x,y)=2(x^2-y^2)-x^4+y^4$

$$f(x,y)=2(x^2-y^2)-x^4+y^4 \quad \text{--- (1)}$$

$$2x^2-2y^2-x^4+y^4$$

$$\frac{\partial f}{\partial x}=4x-4x^3 \quad \text{--- (2)} \qquad \frac{\partial f}{\partial y}=-4y+4y^3$$

$$\frac{\partial f}{\partial x}=0 \qquad \frac{\partial f}{\partial y}=0$$

$$4x-4x^3=0 \qquad -4y+4y^3=0$$

$$4x(1-x^2)=0 \qquad 4y(-1+y^2)=0$$

$$4x=0,\quad 1-x^2=0 \qquad 4y=0 \quad (-1)+y^2=0$$

$$x=0 \qquad x=\pm 1 \qquad y=0 \qquad y=\pm 1$$

The point for maximum or minima are $(0,0),(0,1)(0,-1),(1,0),(-1,0),(1,1),(-1,-1),(1,-1),(-1,1)$

from (2) $\Rightarrow \dfrac{\partial f}{\partial x}=4x-4x^3$

$r = f_{xx} = \frac{\partial^2 f}{\partial x^2} = 4 - 12x^2$

from ③ $\Rightarrow \frac{\partial f}{\partial y} = 4y + 4y^3$

$t = f_{yy} = \frac{\partial^2 f}{\partial y^2} = -4 + 12y^2$

from ② and ③.

$s = f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 0$

i) At $(0,0)$

$(rt - s^2) = (4 - 12x^2)(-4 + 12y^2) - (0)$

$= 4(-4)$

$= -16 < 0$

Sedelle point

ii) At $(0,1) = 4(-4+12)$

$= 4(8)$

$= 32 > 0$

$r = (4 - 12x^2)$

$r = 4 > 0$

Minimum point

iii) At $(0,-1)$

$(rt - s^2) = 4(-4+12) = 4(8) = 32 > 0$

$r = 4 > 0$

minimum point

iv) At (1,0)

$(rt - s^2) = (4-12)(-4)$

$= -8 \times -4$

$= 32 > 0$

$r = 4 - 12 = -8 < 0$

maximum point

v) (-1, 0)

$(rt - s^2) = (4-12)(-4)$

$= (-8)(-4)$

$= 32 > 0$

$r = (4-12) = -8 < 0$

maximum point

vi) (1,1)

$(rt - s^2) = (4-12)(-4-12)$

$= (-8)(8) = -64 < 0$

Seddele point

vii) (-1, -1)

$(rt - s^2) = (4-12)(-4+12)$

$= (-8)(8) = -64 < 0$

sedelle point

viii) (-1, 1)

$(rt - s^2) = (4-12)(-4+12)$

$= (-8)(8) = -64 < 0$

Sedelle point

ix) At $(1,-1)$

$(rt-s^2) = (4-12)(-4+12) = (-8)(8)$

$= -64 < 0$

Sedelle point.

The function is minimum at $(0,1), (0,-1)$

the minimum value is $f(x,y) = 2(x^2-y^2) - x^4+y^4$

i) At $(0,1)$

$f(0,1) = 2(0-1) - 0+1$

$= -1$

ii) At $(0,-1)$

$f(0,-1) = 2(0-1) - 0 - 1$

$= -1$

The minimum value is $-1$.

$f(x,y) = 2(x^2-y^2) - x^4+y^4$

At $(1,0)$

$f(1,0) = 2(1-0) - 1+0$

$= 2-1 = 1$

At $(-1,0)$.

$f(-1,0) = 2(1-0) - 1+0$

$= 2-1$

$= 1$

The maximum value is 1

Find the maximum and minimum value of $f(x,y) = x^4 + y^4 - 4xy + 1$

$f(x,y) = x^4 + y^4 - 4xy + 1$ —— ①

$\frac{\partial f}{\partial x} = 4x^3 - 4y$ — Ⓐ $\qquad \frac{\partial f}{\partial y} = 4y^3 + 4x$ — Ⓑ

$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$

$4x^3 - 4y = 0 \qquad 4y^3 - 4x = 0$

$4(x^3 - y) = 0 \qquad 4(y^3 - x) = 0$

$x^3 - y = 0 \qquad y^3 - x = 0$

cubic on b.s, $\qquad \boxed{y^3 = x}$

saddle

$(x^3)^3 - y^3 = 0$

$x^9 - y^3 = 0$

$x^9 - x = 0$

$x(x^8 - 1) = 0$

$x = 0. \qquad x^8 - 1 = 0$

$(x^4)^2 - (1)^2 = 0 \qquad a^2 - b^2 = (a+b)(a-b)$

$(x^4+1)[(x^2)^2 - (1^2)^2] = 0 \qquad (x^4+1)(x^4) - 1)$

$(x^2)^2 - (1)^2$

$(x^4+1)(x^2+1)(x^2-1) = 0$

$x^4 + 1 = 0 \qquad x^2 + 1 = 0 \qquad x^2 - 1 = 0$

$x = \pm 1 \qquad x^2 = \pm 1 \qquad x^2 = \pm 1$

The roots of $x$ are $0, -1, +1$

The roots of $y$ are $0 \quad -1 \quad +1$

The points are $(0,0)$ $(0,-1)$ $(0,1)$ $(-1,0)$

$(-1,-1)\ (-1,1)(1,0)\ (1,-1)\ (1,1)$

from (A) from (B) $\Rightarrow$

$$\frac{\partial f}{\partial x} = 4x^3 - 4y \qquad \frac{\partial f}{\partial y} = 4y^3 - 4x$$

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2 \qquad t = \frac{\partial^2 f}{\partial y^2} = 12y^2$$

from (A),

$$\frac{\partial f}{\partial x} = 4x^3 - 4y$$

$$s = \frac{\partial^2}{\partial x \partial y} = -4$$

i) At $(0,0)$

$$(rt - s^2) = (12x^2)(12y^2) - (4)^2$$
$$= (0)(0)(-16)$$
$$= -16 < 0$$

Sedelle point

ii) At $(0,-1)$

$$(rt - s^2) = (0)(12) - (4)^2 = -16 < 0$$

Sedelle point

iii) At $(0,1)$

$$(rt - s^2) = (0)(12) - (4)^2 = -16 < 0$$

Sedelle point.

iv) $(-1,0)$

$$(rt - s^2) = (12)(0) - 4^2 = -16 < 0$$

Sedelle point

v) $(-1,-1)$

$(rt-s^2) = (12)(12) - 4^2$

$= 144 - 16 = 128 > 0$

$r = 128 > 0$

minimum point.

vi) $(-1,1)$

$(rt-s^2) = (12)(12) - 4^2 = 144 - 16 = 128 > 0$

$r = 12 > 0$

minimum point

vii) $(1,0)$

$(rt-s^2) = (12)(12) - 4^2 = 144 - 16 = 128 > 0$

$r = 12 > 0$

minimum point.

viii) $(1,1)$

$(rt-s^2) = (12)(12) - 4^2 = 144 - 16 = 128 > 0$

$r = 12 > 0$

minimum point

The function is minimum of $(-1,-1)$ $(-1,1)$ $(1,-1)$ $(1,1)$

$f(x,y) = x^4 + y^4 - 4xy + 1$

At $(-1,-1)$

$f(-1,-1) = 1 + 1 - 4 + 1$

$= -1$

At $(-1,1)$

$f(-1,1) = 1+1+4+1 = 7$

At $(1,-1)$

$f(1,-1) = 1+1+4+1 = 7$

At $(1,1)$

$f(1,1) = 1+4+1+1 = 7$

The minimum value is $-1$

The maximum value is does not exist.

3. Find the minimum value of the function $f(x,y) = x^2+5y^2-6x+10y+12$.

$$f(x,y) = x^2+5y^2-6x+10y+12 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial x} = 2x-6 \quad \text{--- (2)} \qquad \frac{\partial f}{\partial x} = 10y+10 \quad \text{--- (3)}$$

$$\frac{\partial f}{\partial x} = 0 \qquad \frac{\partial f}{\partial y} = 0$$

$2x-6=0$ $\qquad$ $10y+10=0$

$x=3$ $\qquad$ $10y=-10$

$y=-1$

The points are $x=3$, $y=-1$

The minimum (or) maximum values $(3,-1)$

from (2) $\Rightarrow$ $\frac{\partial f}{\partial x} = 2x-6$

$$r = f_{xx} = \frac{\partial^2 f}{\partial x^2} = 2$$

$\frac{\partial f}{\partial y} = 10y+10$ from ③ ⇒

$t = f_{yy} = \frac{\partial^2 f}{\partial y^2} = 10$

from ② and ③ ⇒

$s = f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 0$

$(rt - s^2) = 2(10) - 0$

At $(3,-1)$

$(rt - s^2) = 2(10) - 0 = 20 > 0$

$r = 2 > 0$

minimum point.

The function is mini has $(3,-1)$

$f(3,-1) = 3^2 + 5(-1)^2 - 6(3) + 10(-1) + 12$

$= 9 + 5 - 18 - 10 + 12$

$= -2$

The maximum value is $-2$

4. Find the minimum value of $f(x,y)$ of $f(x,y) = 4x^2 + 6xy + 9y^2 - 8x - 24y + 4$

Soln:- $4x^2 + 6xy + 9y^2 - 8x - 24y + 4$ — ①

$\frac{\partial f}{\partial x} = 8x + 6y - 8$ — ② $\frac{\partial f}{\partial x} = 0$

$\frac{\partial f}{\partial x} = 0$

$8x + 6y - 8 = 0$ — Ⓐ

$\frac{\partial f}{\partial y} = 18y + 6x - 24$ — ③

$$\frac{\partial f}{\partial y} = 0$$

(A). $6x + 18y - 24 = 0$ — (B)

(A) $\times 3 \Rightarrow 24x + \cancel{18y} - 24 = 0$

(B) $\Rightarrow \underline{6x + \cancel{18y} - 24 = 0}$
$\quad - \quad\quad\quad + $

$18x = 0$

$x = 0 \quad (0, 4/3)$

(A) $\Rightarrow 8x + 6y - 8 = 0$

$8(0) + 6y - 8 = 0$

$6y = 8$

$y = 8/6$

$y = 4/3$

The points are $(0, 4/3)$ for the minimum or maximum values $(0, 4/3)$

from (2) $\Rightarrow$

$$\frac{\partial f}{\partial x} = 8x + 6y - 8$$

$$r = f_{xx} = \frac{\partial^2 f}{\partial x^2} = 8$$

from (3) $\Rightarrow$

$$\frac{\partial f}{\partial x} = 18y + 6x - 24$$

$$t = f_{yy} = \frac{\partial^2 f}{\partial y^2} = 18$$

from (2) and (3) $\Rightarrow$

$$s = f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(6x + 18y - 24)$$

$$= 6$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(18y + 6x - 24)$$

$S = 18$ $(rt - s^2) = (8)(18) - (18)$

$= 144 -$

The minimum value is

$f(x, y) = 4x^2 + 6xy + 9y^2 - 8x - 24y + 4$

At $(0, 4/3)$

$f(0, 4/3) = 4(0)^2 + 6(0)(4/3) + 9(4/3)^2 - 8(0) - 24() + 4$

$= 0 + 0 + 9(16/9) - 0 - 32 + 4$

$= 16 - 32 + 4$

$f(0, 4/3) = -12$

The minimum is $-12$.

5. Find the extreme values $xy(a - x - y)$

$f(x, y) = xy(a - x - y)$

$= xya - x^2y - xy^2$

$\frac{\partial f}{\partial x} = ya - 2xy - y^2$

$\frac{\partial f}{\partial y} = xa - x^2 - 2xy$

$r = \frac{\partial^2 f}{\partial x^2} = -2y$

$t = \frac{\partial^2 f}{\partial y^2} = -2x$

$$S = \frac{\partial f}{\partial x \partial y} = a - 2x - 2y$$

$$\frac{\partial f}{\partial x} = 0$$

$$ya - 2xy - y^2 = 0$$

$$y(a - 2x - y) = 0$$

$$y = 0, \quad a - 2x - y = 0$$

$$2x + y = a \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = 0$$

$$xa - x^2 - 2xy = 0$$

$$x(a - x - 2y) = 0 \quad \text{--- (2)}$$

$$2y + x = a \quad \text{--- (2)}$$

$$x + 2y = a$$

Put $x = 0$ in eqn (1),

$$y = a$$

Put $y = 0$ in eqn (2),

$$x = a$$

$x = 0, a$

$y = 0, a$

Solve the eqn (1) & (2),

$$(1) \Rightarrow \quad 2x + y = a$$

$$(2) \times 2 \Rightarrow \quad 2x + 4y = 2a$$

$$-3y = -a$$

$$y = a/3$$

| | |
|---|---|
| 0 | a |
| 0 | a |

$(0,0)$ $(0,a)$ $(a,0)$

$$(1) \Rightarrow 2x + \frac{a}{3} = a$$

$$2x = a - a/3$$

$$2x = 2a/3$$

$$x = \cancel{2}a/3 \times \cancel{2}$$

$$x = a/3$$

The critical points are $(0,0), (0,a)(a,0)$
$(a/3, a/3)$

At $(0,0)$

$$(rt-s^2) = (-2y)(-2x) - (a-2x-2y)^2$$

$$f(0,0) = (0) - a^2$$

$$= -a^2 < 0$$

Seddle point

At $(0,a)$

$$f(0,a) = 0 - (a-0-2a)^2$$

$$= -(-a)^2$$

$$= -a^2 < 0$$

saddle point $(-2y)($-

At $(a,0)$

$$f(a,0) = 0 - (a-2a-0)^2$$

$$= -a^2 < 0$$

Saddle point

At $(a/3, a/3)$

$$f(a/3, a/3) = [(-2(a/3)) (-2(a/3))] - [(a-2(\ldots (-2(a/3))]^2$$

$$= \frac{4a^2}{9} - \left(a - \frac{2a}{3} - \frac{2a}{3}\right)^2$$

$$= 4\frac{a^2}{9} - \left(\frac{3a-2a-2a}{3}\right)^2$$

$$= \frac{4a^2}{9} - \left(-\frac{a}{3}\right)^2$$

$$= 4a^2/9 - a^2/9 = 3a^2/9$$

$= \frac{a^2}{3} > 0$

$r = (-2y) = 2\left(a/3\right)$

$= \frac{-2a}{3} < 0$

maximum point

$\therefore$ f is neither maximum (or). minimum at $(0,0)$. $(0,a)$. $(a,0)$ and the function is maximum at $(a/3, a/3)$

The extreme value $= xy(a-x-y)$

$$= \left(\frac{a}{3}\right)\left(\frac{a}{3}\right)\left(a - \frac{a}{3} - \frac{a}{3}\right)$$

$$= \frac{a^2}{9} \cdot \frac{a}{3}$$

$$= a^3/27$$