MARUDHAR KESARI JAIN COLLEGE FOR WOMEN,VANIYAMBADI
PG & RESEARCH DEPARTMENT OF MATHEMATICS

SUBJECT NAME: MATHEMATICS FOR STATISTICS

CLASS : 1 B.Sc STATISTICS

CODE: 23UEST13

SYLLABUS:

Unit-II Series: Summation and approximations related to Binomial, Exponential and Logarithmic series -Taylor‘s series.

Unit-II
Series

Binomial series:

when n is a rational number

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \dots \quad \forall x$$

Such that $-1 < x < 1$

Result:

1. $(1-x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \dots$

2. $(1-x)^{-n} = 1 - \frac{(-n)}{1}x + \frac{(-n(1-n)-1)}{1\cdot 2}x^2 \dots$

When $-1 < x < 1$ and n is a positive integer

3. $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

4. $\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

5. $\frac{1}{(1-x)^3} = (1-x)^{-3} = \frac{1}{1\cdot 2}\left[1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + 4\cdot 5x^3 + \dots\right]$

6. $\frac{1}{(1-x)^4} = (1-x)^{-4} = \frac{1}{1\cdot 2\cdot 3}\left[1\cdot 2\cdot 3 + 2\cdot 3\cdot 4x + 3\cdot 4\cdot 5x^2 \dots\right]$

7. $\frac{1}{(1-x)^n} = (1-x)^{-n} = \frac{1}{1\cdot 2\cdot 3 (n-1)}\left[1\cdot 2 \dots (n-1) + 2\cdot 3\cdot 4 \dots n \dots\right]$

8. $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

9. $\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

10. $\frac{1}{(1+x)^3} = (1+x)^{-3} = \frac{1}{1\cdot 2}\left[1\cdot 2 - 2\cdot 3x + 3\cdot 4x^2 \dots\right]$

11. $\frac{1}{(1+x)^4} = (1+x)^{-4} = \frac{1}{1\cdot2\cdot3}[1\cdot2\cdot3 - 2\cdot3\cdot4x + 3\cdot4\cdot5x^2 \cdots]$

12. $\frac{1}{(1+x)^n} = (1+x)^{-n} = \frac{1}{1\cdot2\cdot3\cdots n}[1\cdot2\cdot3\cdots(n-1) - 2\cdot3\cdots n\cdot x + \cdots]$

When n is a positive number.

1 Find the co-efficient of x^n the expansion of $\frac{1}{(1-x^2)}$

Solution:-

$$\frac{1}{(1-x^2)} = (1-x^2)^{-1}$$

$$(1-x)^{-1} = 1+x+x^2+x^3+\cdots+x^n$$

$$= 1+x^2+x^4+x^6+\cdots x^{2n}$$

co-efficient of $x^n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

2 find the co-efficient of x^{2n} in the expansion of $(1-x^2)^{-1}$

Solution:- $=(1-x^2)^{-1}$

$$(1-x)^{-1} = 1+x+x^2+x^3+\cdots+x^n$$

$\boxed{x = x^2}$

$$(1-x^2)^{-1} = 1+x^2+(x^2)^2+(x^2)^3+\cdots+x^{2n}$$

$$= 1+x^2+x^4+x^6+\cdots+x^{2n}$$

co-efficient of $x^{2n} = 1$

x^{3n} $\quad x^{3n} = 0$

③ find the co-efficient of x^2 in expansion of $(1+x)^{-3}$

Solution:-

$$(1+x)^{-3} = \frac{1}{1\cdot2}\left[1\cdot2 - 2\cdot3x + 3\cdot4x^2 + \overset{4\cdot5x^3}{\cdots}\right]$$

$$\text{co-efficient of } x^2 = \frac{1}{1\cdot2}[3\cdot4]$$

$$\text{co-efficient of } x^2 = 6.$$

④ find the co-efficient of x^n in $\dfrac{1}{1-2x} + \dfrac{1}{1-3x}$

Solution:-

$$(1-x)^{-1} = 1 + x + x^2 + \cdots + x^n$$

$$(1-2x)^{-1} + (1-3x)^{-1}$$

$$x = 2x, \quad x = 3x$$

$$(1-2x)^{-1} + (1-3x)^{-1} = (1 + 2x + (2x)^2 + \cdots + (2x)^n$$
$$+ (1 + 3x + (3x^2) + \cdots + (3x)^n)$$

$$\text{co-efficient of } x^n = 2^n + 3^n$$

⑤ find the sum of the following series:

i) $1 + 2(\overset{x}{1/2}) + 3(1/2)^2 + \cdots + \infty$

ii) $1 + 2(\overset{x}{1/3}) + 3(1/3)^2 + \cdots + \infty$

Solution:-

$$(1+x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$x = 1/2$$

$$1+2(1/2)+3(1/2)^2+4(1/2)^3=(1-1/2)^{-2}$$

$$=\left(\frac{2-1}{2}\right)^{-2}=\left(\frac{1}{2}\right)^{-2}=(2)^2=4$$

$$(1-x)^{-2}=1+2x+3x^2+4x^3\ldots\ldots$$

$x=1/3$

$$1+2(1/3)+3(1/3)^2+4(1/3)^3=(1-1/3)^{-2}$$

$$=\left(\frac{3-1}{3}\right)^{-2}=\left(\frac{2}{3}\right)^{-2}=\left(\frac{3}{2}\right)^2=\frac{9}{4}$$

(6) find the co-efficient of x^n in expansion of $(2-3x)^{-1}$ in ascending power of x.

Solution:-

$$(2+3x)^{-1}=2^{-1}\left(1+\frac{3x}{2}\right)^{-1}$$

$$=\frac{1}{2}\left(1+\frac{3x}{2}\right)^{-1}$$

WKT

$$(1+x)^{-1}=1-x+x^2-x^3+\ldots\ldots$$

$$\frac{1}{2}\left[1+\frac{3}{2}x\right]=\frac{1}{2}\left[1-\frac{3}{2}x+\left(\frac{3}{2}x\right)^2-\left(\frac{3}{2}x\right)^3+\ldots\ldots+(-1)^n\left(\frac{3}{2}x\right)^n+\ldots\right.$$

$$=\frac{1}{2}\left[1-\frac{3}{2}x+\frac{3^2}{2^2}x^2+\frac{3^3}{2^3}x^3\ldots(-1)^n\frac{3^n}{2^n}x^n+\ldots\right]$$

Co-efficient $x^n=(-1)^n\frac{3^n}{2^n}$

$$=\frac{1}{2}\left[(-1)^n\frac{3^n}{2^n}\right]$$

(7) find the co-efficient of x^n is expansion of

$[1+2x+3x^2+4x^3+\dots]^2$

Solution:-

$$1+2x+3x^2+4x^3+\dots = (1-x)^{-2}$$

$$[1+2x+3x^2+4x^3+\dots]^2 = ((1-x))^{-2(2)}$$

$$= (1-x)^{-4}$$

(9)

$$(1-x)^{-4} = \frac{1}{1\cdot2\cdot3}[1\cdot2\cdot3+2\cdot3\cdot4x+3\cdot4\cdot5x^2+4\cdot5\cdot6x^3+$$
$$5\cdot6\cdot7x^4+\dots x^n]$$

$$\text{co-eff of } x^n = \frac{1}{1\cdot2\cdot3}[(n+1)(n+2)(n+3)x^n]$$

(8) write the n+1th term in the expansion $(3-2x)^{-2}$ when x small

Solution:-

$$(3-2x)^{-2} = 3^{-2}\left(1-\frac{2x}{3}\right)^{-2}$$

$$= \frac{1}{3^2}\left(1-\frac{2x}{3}\right)^{-2}$$

$$= \frac{1}{9}\left(1-\frac{2x}{3}\right)^{-2}$$

$$(1-x)^{-2} = 1+2x+3x^2+4x^3+\dots$$

$$\frac{1}{9}\left(1-\frac{2x}{3}\right)^{-2} = \frac{1}{9}\left[1+2\left(\frac{2x}{3}\right)+3\left(\frac{2}{3}x\right)^2+4\left(\frac{2}{3}x\right)^3+\dots(n+1)\left(\frac{2}{3}x\right)^n\right]$$

$$= \frac{1}{9}\left[1+2\left(\frac{2}{3}x\right)+3\left(\frac{2^2}{3^2}x^2\right)+4\left(\frac{2^3}{3^3}x^3\right)+\dots(n+1)\left(\frac{2^n}{3^n}x^n\right)\right]$$

$(n+1)^{th}$ term $= \frac{1}{9} \cancel{x} \left[\frac{2^n}{3^n} x^n\right]$

$= \frac{1}{3^2} \left[\frac{2^n}{3^n} x^n\right]$

$= \left(\frac{2^n}{3^{n+2}}\right) x^n \ (n+1) x^n$

(9) find co-efficient of x^2 in expansion of $(1+2/3x)^{3/2}$

Solution:-

$$(1+x)^n = 1 + \frac{n}{1} x + \frac{n(n-1)}{1.2} x^2 + \frac{n(n-1)(n-2)}{1.2.3} x^3 + \dots$$

$$\left(1+\frac{2}{3}x\right)^{3/2} = 1 + \frac{3/2}{1}(2/3x) + \frac{3/2(3/2-1)}{1.2}(2/3x)^2 + \frac{3/2(3/2-1)(3/2-2)}{1.2.3}(2/3 \dots$$

$$\text{co-eff of } x^2 = \cancel{\tfrac{x}{3}}\cancel{x}\ \frac{3/2(3/2-1)}{1.2}(2/3\cancel{x})^2$$

$$= \frac{3/2(1/2)}{2}\left(\frac{4}{9}\right)$$

$$= \frac{\cancel{3}^{1}}{\cancel{4}_{2}} \times \frac{\cancel{2}^{1}}{\cancel{9}_{3}}$$

Co.eff of x^2 $= \frac{1}{6}$ //-

(10) find the co-efficient of x^6 in the expansion of $\frac{1}{(1-x^2)^3}$

Solution:-

$$(1-x)^{-3} = \frac{1}{1\cdot 2}[1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + 4\cdot 5x^3 + 5\cdot 6x^4 + 6\cdot 7x^5 + 7\cdot 8x^6 + \dots]$$

Put $x = x^2$

$$(1-x^2)^{-3} = \frac{1}{1\cdot 2}[1\cdot 2 + 2\cdot 3(x^2) + 3\cdot 4(x^2)^2 + 4\cdot 5(x^2)^3 + 5\cdot 6(x^2)^4 + 6\cdot 7(x^2)^5 + 7\cdot 8(x^2)^6 + \dots]$$

$$= \frac{1}{1\cdot 2}[1\cdot 2 + 2\cdot 3x^2 + 3\cdot 4x^4 + 4\cdot 5x^6 + 5\cdot 6x^8 + 6\cdot 7x^{10} + 7\cdot 8x^{12} + \dots]$$

co-efficient of $x^6 = \frac{1}{1\cdot 2}[4\cdot 5]$

co-efficient of $x^6 = 10$

(11) If x is small, what is the value of a if $\sqrt{x^2+4} - \sqrt{x^2+1} = 1 - ax^2$ nearly

Solution:-

$$\sqrt{x^2+4} - \sqrt{x^2+1} = (x^2+4)^{1/2} - (x^2+1)^{1/2}$$

$$= 4^{1/2}\left(1+\frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

$$= 2\left(1+\frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

WKT

$$(1+x)^n = \left[1 + \frac{n}{1}x + \frac{n(n-1)}{1.2}x^2 + \cdots\cdots\right]$$

Put $x = \frac{x^2}{4}$, $n = 1/2$

$$= 2\left[1 + \frac{1/2}{1}\left(\frac{x^2}{4}\right) + \frac{1/2(1/2-1)}{1.2}\left(\frac{x^2}{4}\right)^2 + \cdots\cdots\right]$$

$$-\left[1 + \frac{1/2}{1}x^2 + \frac{1/2(1/2-1)}{1.2}(x^2)^2 + \cdots\cdots\right]$$

$$= 2\left[1 + 1/2\left(\frac{x^2}{4}\right)\right] - \left[1 + 1/2\, x^2\right] \text{ nearly}$$

$$= 2\left[1 + \frac{x^2}{8}\right] - 1 - 1/2\, x^2$$

$$= 2 + \frac{2x^2}{8} - 1 - 1/2\, x^2$$

$$= 1 + \frac{x^2}{4} - \frac{x^2}{2}$$

$$= 1 + \frac{x^2 - 2x^2}{4}$$

$$= 1 - \frac{x^2}{4}$$

$$\therefore \sqrt{x^2+4} - \sqrt{x^2+1} = 1 - \frac{x^2}{4}$$

$$a = \frac{1}{4}$$

(12) When x is small p.T $(1-x)^{1/2}+(1+x)^{1/2}$

$= 2+x+\frac{x^2}{4}$ (nearly)

Solution:-

WKT

$$(1-x)^{-n} = 1-\frac{(-n)x}{1}+\frac{(-n)((-n)-1)}{1\cdot 2}x^2+\cdots$$

$$(1-x)^{-n} = 1+\frac{n}{1}x+\frac{n(n+1)}{1\cdot 2}x^2+\cdots$$

$$(1+x)^{n} = 1+\frac{n}{1}x+\frac{n(n-1)}{1\cdot 2}x^2+\cdots$$

$$(1-x)^{-1/2}+(1+x)^{1/2} = \left[1+\cancel{\tfrac{1}{1}}\cdot\frac{1/2}{1}x+\frac{1/2(1/2+1)}{1\cdot 2}x^2+\cdots\right]$$

$$+\left[1+\frac{1/2}{1}x+\frac{1/2(1/2-1)}{1\cdot 2}x^2+\cdots\right]$$

$$=\left[1+\frac{1}{2}x+\frac{1/2(3/2)}{1\cdot 2}x^2+\cdots\right]+\left[1+\frac{1}{2}x\ \frac{1/2(-1/2)}{1\cdot 2}x^2\right]$$

$$=\left[1+\frac{1}{2}x+\frac{3}{4\times 2}x^2\right]+\left[1+\frac{1}{2}x-\cancel{\tfrac{1}{4}}\frac{1/4}{2}x^2\right]$$

$$\doteq 1+\frac{1}{2}x+\frac{3}{8}x^2+1+\frac{1}{2}x-\frac{1}{8}x^2 \text{ (nearly)}$$

$$= 2+x+\frac{2}{8}x^2 \quad \text{(nearly)}$$

$$(1-x)^{-1/2}+(1+x)^{1/2} = 2+x+\frac{x^2}{4} \quad \text{(nearly)}$$

(13) When x is small P.T $\sqrt{x^2+4} - \sqrt{x^2+1} = 1 - \frac{1}{4}x^2 + \frac{7}{64}x^4$ nearly.

Solution:-

$$\sqrt{x^2+4} - \sqrt{x^2+1} = (x^2+4)^{1/2} - (x^2+1)^{1/2}$$

$$= 4^{1/2}\left(1+\frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

$$= 2\left(1+\frac{x^2}{4}\right)^{1/2} - (1+x^2)^{1/2}$$

W.K.T

$$(1+x)^n = 1 + \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \dots$$

Put $x = \frac{x^2}{4}$ and $x = x^2$, $n = 1/2$

$$= 2\left[1 + \frac{1/2}{1}\left(\frac{x^2}{4}\right) + \frac{1/2(1/2-1)}{1\cdot 2}\left(\frac{x^2}{4}\right)^2 + \cancel{\frac{1/2(1/2-1)(1/2-2)}{1\cdot 2\cdot 3}x^3} + \dots\right]$$

$$- \left[1 + \frac{1/2}{1}x^2 + \frac{1/2(1/2-1)}{1\cdot 2}(x^2)^2 + \dots\right]$$

$$= 2\left[1 + \frac{x^2}{8} + \frac{1/2(-1/2)}{2}\left(\frac{x^4}{16}\right) + \dots\right]$$

$$- \left[1 + 1/2\,x^2 + \frac{1/2(-1/2)}{2}x^4 + \dots\right]$$

$$= 2\left[1 + \frac{x^2}{8} - \frac{1}{8}\frac{x^4}{16}\right] - \left[1 + 1/2\,x^2 - \frac{1}{8}x^4\right]$$

$$= 2 + \frac{2x^2}{8} - \frac{2}{8}\frac{x^4}{16} - 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4$$

$$= 1 + \frac{x^2 - 2x^2}{4} + \left(\frac{8-1}{64}\right)x^4 \text{ nearly}$$

$$= 1 - \frac{1}{4}x^2 + \frac{7}{64}x^4 \text{ nearly}$$

4. find the co-effi of x^n in the expansion of $(1+x+x^2+x^3+\dots)^{-n}$.

Solution:-

$$(1+x+x^2+x^3+\dots)^{-n} = \left((1-x)^{-1}\right)^{-n}$$

$$= (1-x)^n$$

$$(1-x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 - \dots +$$

$$\frac{(-1)\,n(n-1)(n-2)\cdots 2\cdot 1}{1\cdot 2\cdot 3\cdots (n-2)(n-1)n}x^n$$

$$= 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \dots (-1)\,x^n$$

co-eff of $x^n = (-1)^n$ //-

$= 1 + \frac{x^2 - 2x^2}{4} + \left(\frac{8-1}{64}\right)x^4$ nearly

$= 1 - \frac{1}{4}x^2 + \frac{7}{64}x^4$ nearly.

4. find the co-effi of x^n in the expansion of $(1+x+x^2+x^3+\ldots)^{-n}$.

Solution:-

$$(1+x+x^2+x^3+\ldots)^{-n} = ((1-x)^{-1})^{-n}$$

$$= (1-x)^n$$

$$(1-x)^n = 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 - \cancel{n(n} \ldots\ldots + \frac{(-1)\,n(n-1)(n-2)\ldots 2\cdot 1}{1\cdot 2\cdot 3\ldots(n-2)(n-1)n}x^n$$

$$= 1 - \frac{n}{1}x + \frac{n(n-1)}{1\cdot 2}x^2 + \ldots (-1)x^n$$

co-eff of $x^n = (-1)^n$ //-

Summation : Binomial series

The formula for find the sum of binomial series

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2 + \cdots$$

1 −

(1) $1 + \frac{1}{3} + \frac{1\cdot 3}{3\cdot 6} + \frac{1\cdot 3\cdot 5}{3\cdot 6\cdot 9}.$

Solution:-

$$(1-x)^{-p/q} = 1 + \frac{p}{1}\left(\frac{x}{q}\right) + \frac{p(p+q)}{1\cdot 2}\left(\frac{x}{q}\right)^2 + \cdots$$

$p = 1, \quad p + q = 3$

$\cancel{p}\,1 + q = 3$

$q = 3 - 1$

$\boxed{q = 2}$

$$= 1 + \frac{1}{1}\left(\frac{1}{3}\right) + \frac{1\cdot 3}{1\cdot 2}\left(\frac{1}{3}\right)^2 + \frac{1\cdot 3\cdot 5}{1\cdot 2\cdot 3}\left(\frac{1}{3}\right)^3 + \cdots$$

$\frac{x}{q} = \frac{1}{3}$

$\frac{x}{2} = \frac{1}{3}$

$\boxed{x = \frac{2}{3}}$

$$(1-x)^{-p/q} = \left(1 - 2/3\right)^{-1/2}$$

$$= \left(\frac{3-2}{3}\right)^{-1/2}$$

$$= \left(\frac{1}{3}\right)^{-1/2}$$

add & subtract $= (3^{-1})^{-1/2}$

$$= (3)^{1/2}$$

$$\boxed{(1-x)^{-p/q} = \sqrt{3}}$$

H.W

(2) $\frac{1}{10} + \frac{1.4}{10.20} + \frac{1.4.7}{10.20.30}$.

Solution:-

$$(1-x)^{-p/q} = 1 + \frac{p}{1}(x/q) + \frac{p(p+q)}{1.2}(x/q)^2 + \ldots$$

$\boxed{p=1}$, $p+q=4$

$1+q=4$

$q=4-1$

$\boxed{q=3}$

add & sub 1

$$\left[1 + \frac{1}{10} + \frac{1.4}{10.20} + \frac{1.4.7}{10.20.30}\right] - 1$$

$$= \left[1 + \frac{1}{1}(1/10) + \frac{1.4}{1.2}(1/10)^2 + \frac{1.4.7}{1.2.3}(1/10)^3\right] - 1$$

$x/q = \frac{1}{10}$

$\frac{x}{3} = \frac{1}{10} \Rightarrow x = \frac{3}{10}$

$(1-x)^{-p/q} = \left(1-\frac{3}{10}\right)^{-1/3} - 1$

$= \left(\frac{10-3}{10}\right)^{-1/3} - 1$

$= \left(\frac{7}{10}\right)^{-1/3} - 1$

$= \left(\frac{1}{7/10^{1/3}}\right)^{?} - 1$

$= \left(\frac{10}{7}\right)^{1/3} - 1.$ //

③ $\frac{5}{1\cdot2}(1/3) + \frac{5\cdot7}{1\cdot2\cdot3}(1/3)^2 + \frac{5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}(1/3)^3 + \cdots$

Solution:

$S = \frac{5}{1\cdot2}(1/3) + \frac{5\cdot7}{1\cdot2\cdot3}(1/3)^2 + \frac{5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}(1/3)^3 + \cdots$

multiple 3 on b.s

$3S = \frac{3\cdot5}{1\cdot2}(1/3) + \frac{3\cdot5\cdot7}{1\cdot2\cdot3}(1/3)^2 + \frac{3\cdot5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}(1/3)^3 + \cdots$

multiple 1/3 on b.s

$\left(\frac{1}{3}\right)3S = \frac{3\cdot5}{1\cdot2}(1/3)(1/3) + \frac{3\cdot5\cdot7}{1\cdot2\cdot3}(1/3)^2(1/3) + \frac{3\cdot5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}(1/3)^3(1/3) + \cdots$

$S = \frac{3\cdot5}{1\cdot2}(1/3)^2 + \frac{3\cdot5\cdot7}{1\cdot2\cdot3}(1/3)^3 + \frac{3\cdot5\cdot7\cdot9}{1\cdot2\cdot34}(1/3)^4 + \cdots$

Add & Sub $1+\frac{3}{1}(1/3)$ on b.s

$= \left[1+\frac{3}{1}(1/3) + \frac{3\cdot5}{1\cdot2}(1/3)^2 + \frac{3\cdot5\cdot7}{1\cdot2\cdot3}(1/3)^2 + \frac{3\cdot5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}(1/3)^4 + \cdots\right]$

$- \left[1+\frac{3}{1}(1/3)\right]$

$P=3$

$P+q=5$

$3+q=5$

$q=5-3$

$q=2$

$\frac{x}{q}=\frac{1}{3} \Rightarrow \frac{x}{2}=\frac{1}{3} \Rightarrow \boxed{x=\frac{2}{3}}$

$(1-x)^{-P/q}$ $\cancel{=(1-x)^{-3/2}}$ $-2-[1+1]$

$\left(1-\frac{2}{3}\right)^{-3/2}$ $-1-1$

$\left(\frac{3-2}{3}\right)^{-3/2}$ -2

$\left(\frac{1}{3}\right)^{-3/2}-2$

$(3)^{3/2}-2$

$\sqrt[3]{3}-2$

(4) $\frac{7}{9}+\frac{7\cdot 9}{9\cdot 12}+\frac{7\cdot 9\cdot 12}{9\cdot 12\cdot 15}+\cdots$

Solution:-

$S=\frac{7}{9}+\frac{7\cdot 9}{9\cdot 12}+\frac{7\cdot 9\cdot 12}{9\cdot 12\cdot 15}+\cdots$

$=\frac{7}{3}(1/3)+\frac{7\cdot 9}{3\cdot 4}(1/3)^2+\frac{7\cdot 9\cdot 12}{3\cdot 4\cdot 5}(1/3)^3+\cdots$

Multiplying by b.s $3\cdot 5/1\cdot 2$

$\frac{3\cdot 5}{1\cdot 2}S=\frac{3\cdot 5\cdot 7}{1\cdot 2\cdot 3}(1/3)+\frac{3\cdot 5\cdot 7\cdot 9}{1\cdot 2\cdot 3\cdot 4}(1/3)^2+\cdots$

Multiplying by $(1/3)^2$ on b.s

$$\frac{15}{2}s\left(1/3\right)^2 = \frac{3\cdot5\cdot7}{1\cdot2\cdot3}(1/3)(1/3)^2 + \frac{3\cdot5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}(1/3)^2(1/3)^2 + \cdots$$

$$\frac{15s}{2}\left(\frac{1}{9}\right) = \frac{3\cdot5\cdot7}{1\cdot2\cdot3}(1/3)^3 + \frac{3\cdot5\cdot7\cdot9}{1\cdot2\cdot3\cdot4}(1/3)^4 + \cdots$$

Add & sub by $1+\frac{3}{1}(1/3)+\frac{3\cdot5}{1\cdot2}(1/3)^2$ in R.H.S

$$\frac{\cancel{15}^{5}s}{2}\times\frac{1}{\cancel{9}_3} = \left[1+\frac{3}{1}(1/3)+\frac{3\cdot5}{1\cdot2}(1/3)^2+\frac{3\cdot5\cdot7}{1\cdot2\cdot3}(1/3)^3+\cdots\right]-\left[1+\frac{3}{1}(1/3)+\frac{3\cdot5}{1\cdot2}(1/3)^2\right]$$

$$\frac{5}{6}s = \left[1+\frac{3}{1}(1/3)+\frac{3\cdot5}{1\cdot2}(1/3)^2+\frac{3\cdot5\cdot7}{1\cdot2\cdot3}(1/3)^3+\cdots\right]-\left[1+\frac{3}{1}(1/3)+\frac{15}{2}(1/9)\right]$$

$\boxed{p=3}$, $p+q=5$, $\frac{x}{q}=1/3$

$q=5-3$

$\boxed{q=2}$

$\frac{x}{2}=1/3$

$\boxed{x=2/3}$

$$\frac{5}{6}s = \left(1-\frac{2}{3}\right)^{-3/2}-\left[2+\frac{5}{6}\right]$$

$$=\left(\frac{1}{3}\right)^{-3/2}-\left[\frac{12+5}{6}\right]$$

$$=(1/3)^{-3/2}-\left(\frac{17}{6}\right)$$

$$=(3)^{3/2}-\frac{17}{6}$$

$$=3\sqrt{3}-\frac{17}{6}$$

$$\frac{5}{6}s = 3\sqrt{3}-\frac{17}{6}$$

$$s=\left(3\sqrt{3}-\frac{17}{6}\right)\frac{6}{5}$$

$$=\frac{3\sqrt{3}\times6}{5}-\frac{17}{6}\times\frac{6}{5}$$

$$=\frac{3\sqrt{3}}{5}\times6-\frac{17}{5}$$

$$s=\frac{18\sqrt{3}}{5}-\frac{17}{5}$$

5) $\frac{1}{3\cdot6}+\frac{1\cdot3}{3\cdot6\cdot9}+\frac{1\cdot3\cdot5}{3\cdot6\cdot9\cdot12}+\cdots$

Solution:-

Multiple by -1 on b.s

$$-1s = \frac{-1\cdot1}{3\cdot6}+\frac{-1\cdot1\cdot3}{3\cdot6\cdot9}+\frac{-1\cdot1\cdot3\cdot5}{3\cdot6\cdot9\cdot12}$$

$$=\frac{-1\cdot1}{1\cdot2}(1/3)^2+\frac{-1\cdot1\cdot3}{1\cdot2\cdot3}(1/3)^3+\frac{-1\cdot1\cdot3\cdot5}{1\cdot2\cdot3\cdot4}(1/3)^4+\cdots$$

Add & sub $1+\frac{-1}{1}(1/3)$ on . b.s

$$=\left[1+\frac{-1}{1}(1/3)+\frac{-1.1}{1.2}(1/3)^2+\frac{-1.1.3}{1.2.3}(1/3)^3+\frac{-1.1.3.5}{1.2.3.4}(1/3)\right.$$

$$-\left[1+\frac{-1}{1}(1/3)\right]$$

$p=-1$, $p+q=1$, $\frac{x}{q}=1/3$

$-1+q=1$

$q=1+1$ $\qquad \frac{x}{2}=1/3$

$\boxed{q=2}$ $\qquad \boxed{x=2/3}$

$(1-x)^{-p/q}$

$(1-2/3)^{1/2}$

$\left(\frac{3-2}{3}\right)^{1/2}$

$\left(\frac{1}{3}\right)^{1/2}$

$(3)^{-1/2}$

$\frac{1}{3^{1/2}}$

$-\frac{2}{3}+\frac{1}{\sqrt{3}}$

$\frac{2}{3}-\frac{1}{\sqrt{3}}$ //

⑥ S.T $\sqrt{8} = 1 + \frac{3}{4} + \frac{3\cdot 5}{2\cdot 4^2} + \frac{3\cdot 5\cdot 7}{2\cdot 3\cdot 4^3} + \cdots$

Solution:-

$$= 1 + \frac{3}{1}\left(\frac{1}{4}\right) + \frac{3\cdot 5}{2}\left(\frac{1}{4}\right)^2 + \frac{3\cdot 5\cdot 7}{2\cdot 3}\left(1/4\right)^3$$

$P = 3$, $p + q = 5$, $\frac{x}{q} = \frac{1}{4}$

$3 + q = 5$

$q = 5 - 3$, $\frac{x}{2} = \frac{1}{4}$

$\boxed{q = 2}$, $x = \frac{2}{4} \Rightarrow \boxed{x = \frac{1}{2}}$

$$(1-x)^{-p/q} = \left(1 - 1/2\right)^{-3/2}$$

$$= \left(\frac{2-1}{2}\right)^{-3/2}$$

$$= \left(1/2\right)^{-3/2}$$

$$= (2)^{3/2} \Rightarrow (2^3)^{1/2}$$

$$= \sqrt{8}$$

⑦ $1 + \frac{2}{1}\left(\frac{1}{72}\right) + \frac{2\cdot 5}{1\cdot 2}\left(1/74\right) + \cdots$

Solution:-

$$1 + \frac{2}{1}\left(\frac{1}{72}\right) + \frac{2\cdot 5}{1\cdot 2}\left(1/74\right) + \cdots$$

$$1 + \frac{2}{1}\left(1/49\right) + \frac{2\cdot 5}{1\cdot 2}\left(1/49\right)^2 + \frac{2\cdot 5\cdot 8}{1\cdot 2\cdot 3}\left(1/49\right)^3 + \cdot$$

$P = 2$, $p + q = 5$

$2 + q = 5$

$q = 5 - 2$

$\boxed{q = 3}$

$$\frac{x}{q} = \frac{1}{49}$$

$$\frac{x}{3} = \frac{1}{49}$$

$$x = \frac{3}{49}$$

$$(1-x)^{-p/q} = \left(1 - \frac{3}{49}\right)^{-2/3}$$

$$= \left(\frac{49-3}{49}\right)^{-2/3}$$

$$= \left(\frac{46}{49}\right)^{-2/3}$$

$$= \left(\frac{49}{46}\right)^{2/3}$$

$(1-x)$

⑧ Sum the series $1-\frac{1}{4}+\frac{1\cdot3}{4\cdot8}-\frac{1\cdot3\cdot5}{4\cdot8\cdot12}$

Solution:-

$$S=1-\frac{1}{4}+\frac{1\cdot3}{4\cdot8}+\frac{1\cdot3\cdot5}{4\cdot8\cdot12}\cdots$$

WKT

$$(1-x)^{-p/q}=1+\frac{p}{1}(x/q)+\frac{p(p+q)}{1\cdot2}(x/q)^2+\cdots$$

$$S=1+1\left(-\frac{1}{4}\right)+\frac{1\cdot3}{1\cdot2}\left(-\frac{1}{4}\right)^2+\frac{1\cdot3\cdot5}{1\cdot2\cdot3}\left(-\frac{1}{4}\right)^3+$$

$p=1,\ p+q=3$

$1+q=3$

$q=3-1$

$\boxed{q=2}$

$\frac{x}{q}=-\frac{1}{4}$

$\frac{x}{2}=-\frac{1}{4}$

$x=-\frac{2}{4}\Rightarrow\boxed{x=-1/2}$

$$S=(1-x)^{-p/q}$$

$$=(1+1/2)^{-1/2}$$

$$=\left(\frac{2+1}{2}\right)^{-1/2}$$

$$=\left(\frac{3}{2}\right)^{-1/2}$$

$$S=(2/3)^{1/2}$$

⑨ Sum the series $1-\frac{1}{2}(1/2)+\frac{1\cdot3}{2\cdot4}(1/2)^2-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}(1/2)^3+\cdots$

Solution:-

$$1-\frac{1}{2}(1/2)+\frac{1\cdot3}{2\cdot4}(1/2)^2-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}(1/2)^3+\cdots$$

~~WKT~~ $1+\frac{1}{1}(-1/4)+\frac{1\cdot3}{1\cdot2}(-1/4)^2-\frac{1\cdot3\cdot5}{1\cdot2\cdot3}(1/4)^3$

$\cancel{(1-x)}^{-p/q}=1+$

W.k.T

$$(1-x)^{-p/q} = 1 + \frac{p}{1}(x/q) + \frac{p(p+q)}{1\cdot 2}(x/q)^2 + \cdots$$

$p=1$, $\quad p+q=3$, $\quad \frac{x}{q} = -1/4$

$1+q=3$ $\quad \frac{x}{2} = -1/4$

$q = 3-1$

$\boxed{q=2}$ $\quad x = -2/4 \Rightarrow \boxed{x = -1/2}$

$$S = (1-x)^{-p/q}$$

$$= (1+1/2)^{-1/2}$$

$$= \left(\frac{2+1}{2}\right)^{-1/2}$$

$$= \left(\frac{3}{2}\right)^{-1/2}$$

$$= \left(\frac{2}{3}\right)^{1/2}$$

$$= \sqrt{\frac{2}{3}}$$

10. $\frac{1\cdot 4}{5\cdot 10} - \frac{1\cdot 4\cdot 7}{5\cdot 10\cdot 15} + \frac{1\cdot 4\cdot 7\cdot 10}{5\cdot 10\cdot 15\cdot 20} + \cdots$

Solution:-

$$S = \frac{1\cdot 4}{1\cdot 2}(-1/5)^2 + \frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}(-1/5)^3 + \frac{1\cdot 4\cdot 7\cdot 10}{1\cdot 2\cdot 3\cdot 4}(-1/5)^4 + \cdots$$

Add & sub $1 + \frac{1}{1}(-1/5)$ on b.s

$$S = \left[1 + \frac{1}{1}(-1/5) + \frac{1\cdot 4}{1\cdot 2}(-1/5)^2 + \frac{1\cdot 4\cdot 7}{1\cdot 2\cdot 3}(-1/5)^3 + \cdots\right]$$
$$-\left[1 + \frac{1}{1}(-1/5)\right]$$

$p = 1$, $p + q = 4$, $\frac{x}{q} = -\frac{1}{5}$

$1 + q = 4$

$q = 4 - 1$ $\qquad$ $\frac{x}{3} = -\frac{1}{5}$

$\boxed{q = 3}$

$\boxed{x = -\frac{3}{5}}$

$$S = (1 - x)^{-p/q}$$

$$= \left(1 + \frac{3}{5}\right)^{-1/3}$$

$$= \left(\frac{5+3}{5}\right)^{-1/3}$$

$$= \left(\frac{8}{5}\right)^{-1/3}$$

$$= \frac{1}{\left(\frac{8}{5}\right)^{1/3}}$$

$$= \left(\frac{5}{8}\right)^{1/3} - \frac{4}{5}$$

$$= \sqrt[3]{\frac{5}{8}} - \frac{4}{5}$$

Exponential Series :-

1. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

2. $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\dots$

3. $\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

4. $\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

5. $e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$

6. $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$

7. $\frac{e + e^{-1}}{2} = 1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots$

8. $\frac{e - e^{-1}}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \dots$

① Sum the series $\frac{1+3x}{1!} + \frac{(1+3x)^2}{2!} + \frac{(1+3x)^3}{3!} + \dots$

Solution:-

$$x = 1 + 3x$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$S = \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Add & Sub ①

$$e^x = \left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right] - 1$$

$$e^x - 1 = \left[1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right]$$

$$e^{1+3x} - 1 \Rightarrow \boxed{S = e^{1+3x} - 1}$$

(2) Sum the series $1 - \log e^{2} + \frac{(\log e^{2})^{2}}{2!} - \frac{(\log e^{2})^{3}}{3!} + \cdots$

Solution:-

$$x = \log e^{2}$$

$$S = 1 - \frac{x}{1!} + \frac{x^{2}}{2!} - \frac{x^{3}}{3!} + \cdots$$

$$= e^{-x}$$

$$= e^{-\log e^{2}}$$

$$= e^{\log e^{2^{-1}}}$$

$$= 2^{-1}$$

$$\boxed{S = 1/2}$$

(3) S.T $\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$

Solution:-

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\cdots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3} + \cdots\cdots$$

$$e - e^{-1} = \left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3} + \cdots\right) - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots\right)$$

$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots - 1 + \frac{1}{1!} - \frac{1}{2} + \frac{1}{3} + \cdots$$

$$= 2\frac{1}{1!} + 2\frac{1}{3!} + 2\frac{1}{5!} + \cdots$$

$$= 2\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right)$$

$$\frac{e - e^{-1}}{2} = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

$$\frac{1}{2}\, e - e^{-1} = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

(4) S.T $\dfrac{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}} = \dfrac{e^2+1}{e^2-1}$

Solution :-

$$\frac{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}} = \frac{\frac{e+e^{-1}}{2}}{\frac{e-e^{-1}}{2}}$$

$$= \frac{e+e^{-1}}{2} \times \frac{2}{e-e^{-1}}$$

$$= \frac{e+e^{-1}}{e-e^{-1}}$$

$$= \frac{e+\frac{1}{e}}{e-\frac{1}{e}}$$

$$= \frac{\frac{e^2+1}{e}}{\frac{e^2-1}{e}}$$

$$= \frac{e^2+1}{e} \times \frac{e}{e^2-1}$$

$$= \frac{e^2+1}{e^2-1}$$

LHS = RHS.

(5) S.T $\dfrac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots} = \dfrac{e-1}{e+1}$

Solution :-

Add & Sub 1 in numerator

$$\frac{\left[1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\right]-1}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots} = \frac{\frac{e+e^{-1}}{2}-1}{\frac{e-e^{-1}}{2}} = \frac{\frac{e-e^{-1}-2}{2}}{\frac{e-e^{-1}}{2}}$$

(4) S.T $\dfrac{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}} = \dfrac{e^2+1}{e^2-1}$

Solution :-

$$\frac{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}} = \frac{\frac{e+e^{-1}}{2}}{\frac{e-e^{-1}}{2}}$$

$$= \frac{e+e^{-1}}{2} \times \frac{2}{e-e^{-1}}$$

$$= \frac{e+e^{-1}}{e-e^{-1}}$$

$$= \frac{e+\frac{1}{e}}{e-\frac{1}{e}}$$

$$= \frac{\frac{e^2+1}{e}}{\frac{e^2-1}{e}}$$

$$= \frac{e^2+1}{e} \times \frac{e}{e^2-1}$$

$$= \frac{e^2+1}{e^2-1}$$

LHS = RHS

(5) S.T $\dfrac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots} = \dfrac{e-1}{e+1}$

Solution :-

Add & sub 1 in numerator

$$\frac{\left[1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\right]-1}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots} = \frac{\frac{e+e^{-1}}{2}-1}{\frac{e-e^{-1}}{2}} = \frac{\frac{e+e^{-1}-2}{2}}{\frac{e-e^{-1}}{2}}$$

$$= \frac{e+e^{-1}-2}{2} \times \frac{2}{e-e^{-1}}$$

$$= \frac{e+e^{-1}-2}{e-e^{-1}}$$

$$= \frac{e+\frac{1}{e}-2}{e-\frac{1}{e}}$$

$$= \frac{\frac{e^2+1-2e}{e}}{\frac{e^2-1}{e}}$$

$$= \frac{e^2+1-2e}{e} \times \frac{e}{e^2-1}$$

$$= \frac{e^2+1-2e}{e^2-1}$$

$\therefore (a-b)^2 = a^2+b^2-2ab$

$(a^2-b^2) = (a+b)(a-b)$

$$= \frac{(e+1)^2}{(e+1)(e-1)} = \frac{e+1}{e-1)} \;//-$$

⑥ S.T
$$\frac{1+\frac{1}{2!}+\frac{2}{3!}+\frac{2^2}{4!}+\cdots}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots} = e/2$$

Solution :-

$$\frac{1+\frac{1}{2!}+\frac{2}{3!}+\frac{2^2}{4!}+\cdots}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}$$

Multiple and divided by 2^2 on the numerator only

$$\frac{\frac{1}{2^2}\left[\frac{2^2}{1}+\frac{2^2}{2!}+\frac{2^3}{3!}+\frac{2^4}{4!}+\cdots\right]}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}$$

Put in $\frac{2^2}{1!} = \left(1+\frac{2}{1!}\right)+1$

$$= \frac{\frac{1}{2^2}\left[1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\cdots\right]1}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}$$

$$= \frac{\frac{1}{2^2}\left(e^2+1\right)}{\frac{1}{2}\left(e+e^{-1}\right)}$$

$$= \frac{2}{2^2}\left(\frac{e^2+1}{e+\frac{1}{e}}\right)$$

$$= \frac{1}{2}\left(\frac{e^2+1}{\frac{e^2+1}{e}}\right)$$

$$= \frac{1}{2} \times e$$

$$= \frac{e}{2} //-$$

LHS = RHS.

7) S.T $1+\frac{x\log_e a}{1!}+\frac{(x\log_e a)^2}{2!}+\frac{(x\log_e a)^3}{3!}+\cdots = a^x$

Solution:-

$$x = x\log_e a$$

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$= e^x$$

$$= e^{x\log_e a}$$

$$= e^{\log_e a^x}$$

$$= a^x$$

Put in $\frac{2^2}{1!} = \left(1+\frac{2}{1!}\right)+1$

$$= \frac{\frac{1}{2^2}\left[1+\frac{2}{1!}+\frac{2^2}{2!}+\frac{2^3}{3!}+\cdots\right]1}{1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots}$$

$$= \frac{\frac{1}{2^2}(e^2+1)}{\frac{1}{2}(e+e^{-1})}$$

$$= \frac{2}{2^2}\left(\frac{e^2+1}{e+\frac{1}{e}}\right)$$

$$= \frac{1}{2}\left(\frac{e^2+1}{\frac{e^2+1}{e}}\right)$$

$$= \frac{1}{2} \times e$$

$$= \frac{e}{2} //-$$

LHS = RHS.

(7) S.T $1+\frac{x\log_e a}{1!}+\frac{(x\log_e a)^2}{2!}+\frac{(x\log_e a)^3}{3!}+\cdots = a^x$

Solution:-

$$x = x\log_e a$$

$$e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$= e^x$$

$$= e^{x\log_e a}$$

$$= e^{\log_e a^x}$$

$$= a^x$$

⑧ S.T $\log_e 2 + \frac{(\log_e 2)^2}{2!} + \frac{(\log_e 2)^3}{3!} + \cdots = 1$

Solution :-

Add & sub by 1

$$\left[1 + \log_e 2 + \frac{(\log_e 2)^2}{2!} + \frac{(\log_e 2)^3}{3!} + \cdots\right] - 1$$

$x = \log_e 2$

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} = e^x$$

$$= e^{\log_e 2} - 1$$

$$= 2 - 1$$

$$= 1$$

Summation: Exponential series:

(1) Sum the series $\frac{2^2}{1!}+\frac{3^2}{2!}+\frac{4^2}{3!}+\frac{5^2}{4!}+\dots$

Solution:-

let t_n be the n^{th} term

$$t_n=\frac{(n+1)^2}{n!}$$

Degree of numerator is 2

Let $(n+1)^2 = A+Bn+Cn(n-1)$

$n^2+2n+1 = A+Bn+Cn(n-1)$ — (1)

equ the co-eff of n^2

$$\boxed{C=1}$$

equ the co-eff of n

$2=B-C$

$2=B-1$

$$\boxed{B=3}$$

equ the constant form

$$\boxed{A=1}$$

Sub A, B, C value in (1)

$(n+1)^2 = 1+3n+n(n-1)$

$$t_n=\frac{(n+1)^2}{n!}$$

$$=\frac{1+3n+n(n-1)}{n!}$$

$$=\frac{1}{n!}+\frac{3n}{n!}+\frac{n(n-1)}{n!}$$

$$= \frac{1}{n!} + \frac{3n}{n(n-1)!} + \frac{n(n-1)}{n(n-1)(n-2)!}$$

$$t_n = \frac{1}{n!} + \frac{3}{(n-1)!} + \frac{1}{(n-2)!}$$

Put $n = 1, 2, 3 \dots$

$$t_1 = \frac{1}{1!} + \frac{3}{0!}$$

$$t_2 = \frac{1}{2!} + \frac{3}{1!} + \frac{1}{0!}$$

$$t_3 = \frac{1}{3!} + \frac{3}{2!} + \frac{1}{1!}.$$

$$t_3 = \frac{1}{4!} + \frac{3}{3!} + \frac{1}{2!}$$

Adding vertically

$$S = \left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\right] + 3\left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots\right] + \left[\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots\right]$$

$$\cancel{S = \left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\right] + 3[}$$

$$= (e^1 - 1) + 3e^1 + e^1$$

$$= e - 1 + 3e + e$$

$$S = 5e - 1$$

② Sum the series $\sum_{n=0}^{\infty} \frac{5n+1}{(2n+1)!}$

Solution:-

let $t_n = \frac{5n+1}{(2n+1)!}$ $n=0,1,2,3$

Degree of the numerator is 1

$5n+1 = A+B(2n+1)$ — ①

coeff of n

$5 = 2B$

$\boxed{B = 5/2}$

co-eff of constant

$1 = A+B$

$1 = A + 5/2$

$A = 1 - 5/2$

$A = \frac{2-5}{2}$

$\boxed{A = -3/2}$

Sub A, B value in ①

$5n+1 = -3/2 + 5/2\,(2n+1)$

$$\frac{5n+1}{(2n+1)!} = \frac{-3/2 + 5/2\,(2n+1)}{(2n+1)!}$$

$$= \frac{-3/2}{(2n+1)!} + \frac{5/2\,(2n+1)}{(2n+1)!}$$

$$= \frac{-3/2}{2n+1} + \frac{5/2\,(2n+1)}{(2n+1)\,2n!}$$

$$t_n = \frac{-3/2}{(2n+1)!} + \frac{5/2}{2n!}$$

Put $n = 0, 1, 2, 3 \ldots\ldots$

$$t_0 = \frac{-3/2}{1!} + \frac{5/2}{0!}$$

$$t_1 = \frac{-3/2}{3!} + \frac{5/2}{2!}$$

$$t_2 = \frac{-3/2}{5!} + \frac{5/2}{4!}$$

$$t_3 = \frac{-3/2}{7!} + \frac{5/2}{6!}$$

Adding vertically.

$$S = \left[\frac{-3/2}{1!} + \frac{-3/2}{3!} + \frac{-3/2}{5!} + \cdots\right] + \left[\frac{5/2}{0!} + \frac{5/2}{2!} + \frac{5/2}{4!} + \cdots\right]$$

$$= -3/2\left[\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right] + 5/2\left[\frac{1}{0!} + \frac{1}{2!} + \frac{1}{4!} + \cdots\right]$$

$$= -3/2\left[\frac{e - e^{-1}}{2}\right] + 5/2\left[\frac{e + e^{-1}}{2}\right]$$

$$= \frac{-3}{2}\left[\frac{e - 1/e}{2}\right] + \frac{5}{2}\left[\frac{e + 1/e}{2}\right] = \frac{-3}{2}\left[\frac{e^2 - 1}{2e}\right] + \frac{5}{2}\left[\frac{e^2 + 1}{2e}\right]$$

$$= \frac{1}{4e}\left[-3e^2 + 3 + 5e^2 + 5\right]$$

$$= \frac{1}{4e}\left[2e^2 + 8\right] = \frac{2(e^2 + 4)}{4e}$$

$$= \frac{e^2 + 4}{e}$$

$$S = \frac{e^2}{2e} + \frac{4}{2e}$$

$$S = \frac{e}{2} + \frac{2}{e}$$

④ Sum the series $\frac{1^2}{3!} + \frac{2^2}{5!} + \frac{3^2}{7!} + \cdots$

Solution:-

let t_n be the n^{th} term

$$t_n = \frac{n^2}{(2n+1)!}$$

let $n^2 = A + B(2n+1) + C\{2n(2n+1)\}$

equate the co-eff of n^2

$1 = 4C$

$\boxed{C = 1/4}$

co-eff of n

$0 = 2B + 2C$

$0 = 2B + 2(1/4)$

$2B + \frac{1}{2} = 0$

$2B = -1/2 \Rightarrow \boxed{B = -1/4}$

Co-eff of constant

$$0 = A + B$$

$$0 = A - 1/4$$

$$\boxed{A = 1/4}$$

$$t_n = \frac{n^2}{(2n+1)!}$$

$$= \frac{\frac{1}{4} - 1/4(2n+1) + 1/4\, 2n(2n+1)}{(2n+1)!}$$

$$= \frac{1/4}{(2n+1)!} - \frac{1/4(2n+1)}{(2n+1)2n!} + \frac{1/4\, 2n(2n+1)}{(2n+1)2n(2n-1)!}$$

$$= \frac{1/4}{(2n+1)!} - \frac{1/4}{2n!} + \frac{1/4}{(2n-1)!}$$

Put n = 1, 2, 3

$$t_1 = \frac{\frac{1}{4}}{3!} - \frac{1/4}{2!} + \frac{1/4}{1!}$$ 5

$$t_2 = \frac{1/4}{5!} - \frac{1/4}{4!} + \frac{1/4}{3!}$$

$$t_3 = \frac{1/4}{7!} - \frac{1/4}{6!} + \frac{1/4}{5!}$$

Adding vertically

$$\cancel{S = \frac{1}{4}\left[\frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} - \frac{1/4}{2!}\ \frac{1}{4}\right.}$$

$$S = \frac{1}{4}\left[\frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots\right] - \frac{1}{4}\left[\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots\right]$$

$$+ \frac{1}{4}\left[\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \cdots\right]$$

$$= \frac{1}{4}\left[\frac{e-e^{-1}}{2} - \frac{1}{1!}\right] - \frac{1}{4}\left[\frac{e+e^{-1}}{2} - 1\right] + \frac{1}{4}\left[\frac{e-e^{-1}}{2}\right]$$

$$= \frac{1}{4}\left\{\frac{e-1/e}{2} - 1 - \left[\left(\frac{e+1/e}{2}\right) - 1\right] + \frac{e-1/e}{2}\right\}$$

$$= \frac{1}{4}\left\{\frac{e^2-1}{2e} - 1 - \left[\left(\frac{e^2+1}{2e}\right) - 1\right] + \frac{e^2-1}{2e}\right\}$$

$$= \frac{1}{8}\left\{\frac{e^2-1}{e} - 1 - \frac{e^2-1}{2e} + 1\ \frac{e^2-1}{e}\right\}$$

$$= \frac{1}{8e}\left[e^2 - 1 - e^2 - 1 + e^2 - 1\right]$$

$$= \frac{1}{8e}(e^2 - 3)$$

$$= \frac{e^2}{8e} - \frac{3}{8e}$$

$$\boxed{S = \frac{e - 3e^{-1}}{8}}$$

5 Sum the series $\frac{1}{1!} + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!} + \cdots \infty$

Solution:-

let

$$t_n = \frac{2n-1}{n!}$$

numerator degree is 1

$$2n-1 = A + Bn$$

co-eff of n

$$\boxed{2 = B}$$

co-eff of constant

$$\boxed{-1 = A}$$

$$t_n = \frac{2n-1}{n!} = \frac{A+Bn}{n!}$$

$$= \frac{-1+2n}{n!}$$

$$= \frac{-1}{n!} + \frac{2n}{n(n-1)!}$$

$$t_n = \frac{-1}{n!} + \frac{2}{(n-1)!}$$

Put $n = 1, 2, 3, \ldots\ldots$

$$t_1 = \frac{-1}{1!} + \frac{2}{0!}$$

$$t_2 = \frac{-1}{2!} + \frac{2}{1!}$$

$$t_3 = \frac{-1}{3!} + \frac{2}{2!}$$

$$\ldots\ldots$$

Adding vertically

$$S = \left[\frac{-1}{1!} - \frac{1}{2!} - \frac{1}{3!} \cdots\cdots\right] + \left[\frac{2}{0!} + \frac{2}{1!} + \frac{2}{2!} + \cdots\cdots\right]$$

$$= -1\left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right] + 2\left[1 + \frac{1}{1!} + \frac{1}{2!} + \cdots\cdots\right]$$

$$= -1(e-1) + 2e$$

$$= -e + 1 + 2e$$

$$\boxed{S = e+1}$$

Logarithmic Series

Definition

The infinite series

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$ is called

the logarithmetic series. If the value of x is such that $-1 < x < 1$.

The sum is

$$\log(1+x) = x - \frac{x^2}{x} + \frac{x^3}{x} - \frac{x^4}{x} + \cdots$$

1) $\log(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right]$

2) $-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

3) $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

4) $\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$

Problems:

1. If $|x|>1$ then find $\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\cdots$

Solution.

W.K.T

$$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)=x+\frac{x^3}{3}+\frac{x^5}{3}+\cdots$$

$$\frac{1}{x}+\frac{1}{3x^3}+\frac{1}{5x^5}+\cdots=\frac{1/x}{1}+\frac{1/x^3}{3}+\frac{1/x^5}{5}$$

$$=\frac{1}{2}\log\left[\frac{1+1/x}{1-1/x}\right]$$

$$=\frac{1}{2}\log\left[\frac{\frac{x+1}{x}}{\frac{x-1}{x}}\right]$$

$$=\frac{1}{2}\log\left[\frac{x+1}{x}\times\frac{x}{x-1}\right]$$

$$=\frac{1}{2}\log\left[\frac{x+1}{x-1}\right]$$

2. S.T $\log 10=3\log 2+\frac{1}{4}-\frac{1}{2}\cdot\frac{1}{4^2}+\frac{1}{3}\cdot\frac{1}{4^3}+\cdots\infty$

Solution:-

$\boxed{x=1/4}$

$$RHS=3\log 2+x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$$

$$=3\log 2+\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots\right]$$

$$=3\log 2+\log(1+x)$$

$$= 3\log 2 + \log(1 + 1/4)$$

$$= 3\log 2 + \log\left(\frac{4+1}{4}\right)$$

$$= 3\log 2 + \log 5/4$$

$$= \log 2^3 + \log 5/4$$

$$= \log(2^3 + 5/4)$$

$$= \log\left(8 * \frac{5}{4}\right)$$

$$= \log 10.$$