MARUDHAR KESARI JAIN COLLEGE FOR WOMEN,VANIYAMBADI
PG & RESEARCH DEPARTMENT OF MATHEMATICS

SUBJECT NAME: MATHEMATICS FOR STATISTICS

CLASS : 1 B.Sc STATISTICS

CODE: 23UEST13

SYLLABUS:

Unit-IV Differential calculus: Functions – Different types – simple valued and many valued – Implicit and Explicit functions, Odd and even functions, periodic functions, algebraic and transcendental functions

DIFFERENTIAL CALCULAS

single valued and many valued functions:

Single valued:

Each value of x there corresponds one and only one value of y. then y is called one valued or single valued function of x

E.g:- ★ $y = \frac{2x+3}{7x-4}$

★ $y = \cos x$

★ $y = [(2-4x)^3]^2$ are single valued function of x.

Many valued function:

Each value of x there correspond more than one value of y then y is called a many or multiple value function of x.

E.g:- ★ $y^2 = a^2 - x^2$

★ $y = \pm\sqrt{a^2 - x^2}$

Implicit and Explicit function.

Explicit function:-

* If the functional relation b/w two variables x and y is expressed in the form $y = f(x)$ than y is called an explicit function of x.

e.g:- * $y = x^3 + 2x - 3$

* $y = \sin^2 x - \cos x$

* In the case of explicit function the value of dependent variables for any given value of the independent variables can be obtained by direct substition in the functional relation.

Implicit function:

* If the relation b/w two variables x and y is expressed in the form $f(x,y) = 0$

* We may consider y as a function of x or x as a function of y in such cases dependent variable is an implicit function of the independent variable.

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Implicit function:

* If the relation b/w two variables x and y is expressed in the form $f(x,y)=0$

* We may consider y as a function of x or x as a function of y in such cases dependent variable is an implicit function of the independent variable.

* In the case of implicit function the value or value of the dependent variable for the given value on value of the independent variables cannot be obtained by direct substitution.

e.g.:- $\Rightarrow x^3 + y^3 = 3axy$

$\Rightarrow \cos x + ay = b \tan y$

$\Rightarrow Y = e^{xy}$

Odd and Even function:-

Even functions:-

Their is no change in the sign of fun $f(x)$ when x is changed to $-x$ that function is called an even function

e.g.:- $\Rightarrow Y = \cos x$

$\Rightarrow Y = 4x^4 - 3x^2 + 6$

$\Rightarrow Y = e^x + e^{-x}$

Odd function:-

If $f(x)$ is an even function then $f(x) = f(-x)$ is the sign of $f(x)$ is changed when x is changed to $-x$ is called an odd function so if $f(x)$ is odd we get
$f(x) = -f(-x)$

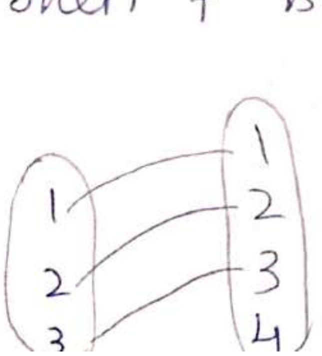
Functions :

A function $f: A \to B$ is a rule which associates every element of A with unique element of B.

Into or Injective function:

If $f: A \to B$ there exists atleast one element in B is not the image of any element in A then 'f' is into function

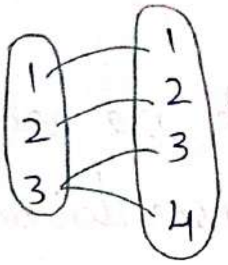
e.g.:-



Onto / surjective:

If $f: A \to B$ there exists each element in B is image of altleast one element in A than 'f' is onto

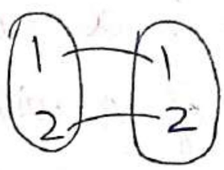
e.g:



one - to - one function.

The function $f: A \to B$ is one - to - one if $x_1, x_2 \in A$, $f(x_1 = x_2)$

e.g:-

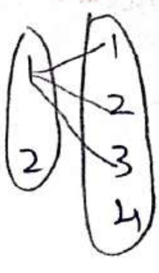


Many - to - one function.

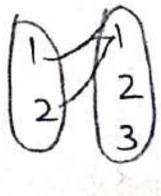
If f is function $f: A \to B$ this many-to - one ~~a~~ if distinct element of A as the same element in.

e.g:

1.



2.



One - to - one correspondents and Bijective :-

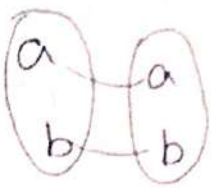
If $f: A \to B$ is one - to - one and onto then f is called one - to - one correspondent b/w A & B

e.g:

Identity mapping:-

If $f: A \to B$ defined by $f(x) = x, \ \forall x \in A$ is called an identity mapping

e.g:



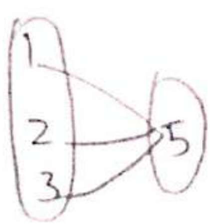
$f(a) = 0$

$f(b) = 0$

Constant function:

If $f: A \to B$ is defined $f(x) = c \ \forall x \in A$ is called constant function

e.g:



projection:-

If $f: S \times T \to S$ is defined by $f(A, B), A \ \forall x \in S, b \in$ is called projection $S \times T$ onto S.

Equal function:

Two function $f: A \to B$ & $g: A \to B$ are said to be equal if $f(x) = g(x)\ \forall x \in A$

Inverse function:

let $f: A \to B$ be one-to-one and onto then the inverse function $f^{-1}: B \to A$ is $y = f(x)\ \forall x \in A, y \in B$.

Composition:

Let $f: A \to B$ and $g: B \to C$ be two function the composition of f and g is a function.

$g \circ f: A \to x$ is defined by

$$(g \circ f)x = g[f(x)]\ \forall x \in A.$$

① Example:

If $f(x) = 3x^3 - 5x^2 + 6x - 4$ find the value of $f(1)$, $f(2)$, $f(0)$, $f(-1)$, $f(-2)$.

Solution:

Given: $f(x) = 3x^3 - 5x^2 + 6x - 4$

Put $\cancel{f(1)}$ $x = 1$

$$f(1) = 3(1)^3 - 5(1)^2 + 6(1) - 4$$

$$= 3 - 5 + 6 - 4$$

$$= 3 + 6 - 9$$

$$= 9 - 9$$

$$\boxed{f(1) = 0}$$

Put $x = 2$

$$f(2) = 3(2)^3 - 5(2)^2 + 6(2) - 4$$

$$= 3(8) - 5(4) + 12 - 4$$

$$= 24 - 20 + 12 - 4$$

$$= 24 + 12 - 24$$

$$\boxed{f(2) = 12}$$

Put $x = 0$

$$f(0) = 3(0)^3 - 5(0)^2 + 6(0) - 4$$

$$= 0 - 0 + 0 - 4$$

$$\boxed{f(0) = -4}$$

Put $x = -1$

$$f(+1) = 3(-1)^3 - 5(-1)^2 + 6(-1) - 4$$

$$= 3(-1) - 5(1) + (-6) - 4$$

$$= -3 - 5 - 6 - 4$$

$$= -8 - 10$$

$$\boxed{f(-1) = -18}$$

Put $x = -2$

$$f(-2) = 3(-2)^3 - 5(-2)^2 + 6(-2) - 4$$

$$= 3(-8) - 5(4) + (-12) - 4$$

$$= -24 - 20 - 12 - 4$$

$$= -24 - 24 - 12$$

$$= -48 - 12$$

$$\boxed{f(-2) = -60}$$

② If $f(x) = A\cos x + B\sin x$ S.T $f(x+2\pi) = f(x)$

solution:-

GT. To show that $f(x+2\pi) = f(x)$

Put: $x = x + 2\pi$

$$f(x+2\pi) = A\cos(x+2\pi) + B\sin(x+2\pi)$$

$$= A\cos x + \cancel{A}\cos 2\pi + B\sin x + B\sin 2\pi$$

$$= A\cos x + B\sin x$$

$$\boxed{f(x+2\pi) = f(x)}$$

(3) If $f(x) = \log x$ S.T $f(a, b, c) = f(a) + f(b) + f(c)$

solution:

To show $f(a, b, c) = f(a) + f(b) + f(c)$

$f(x) = \log x$

Put $x = a$

$f(a) = \log a$

$f(b) = \log b$

$f(c) = \log c$

$f(a) + f(b) + f(c) = \log a + \log b + \log c$

$= \log(abc)$

$= f\cancel{(x)}(abc)$

$$\boxed{f(abc) = f(a) + f(b) + f(c)}$$

(4) If $3x^2 - 7xy + 2y^2 + 2x - y + 3 = 0$ find y when $x = 3$

solution:-

~~3(~~

Put $x = 3$

$3(3)^2 - 7(3)y + 2y^2 + 2(3) - y + 3 = 0$

$3(9) - 21y + 2y^2 + 6 - y + 3 = 0$

$27 - 21y + 2y^2 + 6 - y + 3 = 0$

$27 - 22y + 2y^2 + 9 = 0$

$36 - 22y + 2y^2 = 0$

$2y^2 - 22y + \frac{36}{2} = 0$

$2(y^2 - 11y + 18) = 0$

18
-2 -9
-11

$2(y-2)(y-9) = 0$

$2(y-2) = 0 \quad y-9 = 0$

$2y - 4 = 0 \quad \boxed{y = 9}$

$2y = 4$

$y = \frac{4}{2}$

$\boxed{y = 2}$

$\boxed{y = 2 \text{ (or) } 9}$

If $f(x) = (2x-1)(x-3)$ find the value of $f(0), f(1), f(2), f(1/2), f(3)$.

solution:-

given:- $f(x) = (2x-1)(x-3)$

Put $x = 0$

$f(0) = (2(0)-1)(0-3)$

$= (0-1)(0-3)$

$= -1 - 3$

$\boxed{f(0) = -4}$

Put $x = 1$

$$f(1) = (2(1)-1)(1-3)$$
$$= (2-1)(-2)$$
$$= (1)(-2)$$
$$\boxed{f(1) = -2}$$

Put $x = 2$

$$f(2) = (2(2)-1)(2-3)$$
$$= (4-1)(-1)$$
$$= (3)(-1)$$
$$\boxed{f(2) = -3}$$

Put $x = 1/2$

$$f(1/2) = (2(1/2)-1)(1/2-3)$$
$$= \left(\frac{2}{2}-1\right)\left(-\frac{5}{2}\right)$$
$$= (1-1)\left(-\frac{5}{2}\right)$$
$$= 0\left(-\frac{5}{2}\right)$$
$$= 0$$

Put $x = 3$

$$f(3) = (2(3)-1)(3-3)$$
$$= (6-1)(0)$$
$$\boxed{f(3) = 0}$$

Result:

$f(1) = -2$, $f(2) = -3$, $f(1/2) = 0$, $f(0) = -4$

$f(3) = 0$

⑤ If $f(x) = x^3 + 2x + 1$ find $f(\alpha^2)$

Solution:-

Given: $f(x) = x^3 + 2x + 1$

$f \propto x = \alpha^2$

$f(\alpha^2) = (\alpha^2)^3 + 2\alpha^2 + 1$ //-

Implicit Function:

If the relation betw x & y be given in the form $f(x,y)=c$, where c is constant, then the total differential co-efficient with respects to x is 0

Since, differential co-efficient of a constant is (0) zero.

Therefore $\left[\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\,\frac{dy}{dx}=0\right]$

$[f(x,y)=0]$, $\therefore \cancel{\frac{dy}{dx}} = \cancel{+\frac{df}{dx}}$

$$\therefore \frac{dy}{dx}=\frac{-\partial f/\partial x}{\partial f/\partial y}$$

find $\frac{du}{dt}$ where $u=x^2+y^2+z^2$, $x=e^t$, $y=e^t \sin t$, $z=e^t \cos t$.

solution:

Problem: implicit

① find dy/dx if x and y are related as follows

$y^2 = 4ax$

Solution:-

W.K.T

$$dy/dx = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

$$y^2 = 4ax \; ; \; y^2 - 4ax = 0$$

$$f(x,y) = y^2 - 4ax$$

$$\frac{\partial f}{\partial x} = -4a, \quad \frac{\partial f}{\partial y} = 2y$$

$$\frac{dy}{dx} = \frac{-(-4a)}{2y} \Rightarrow \frac{4a}{2y}$$

$$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

② find $\frac{dy}{dx}$, when $x^3 + y^3 + 3axy = f$

Solution:-

W.K.T

$$\frac{dy}{dx} = \frac{-\partial f/\partial x}{\partial f/\partial y}$$

$$f = x^3 + y^3 + 3axy$$

$$\frac{\partial f}{\partial x} = 3x^2 + 3ay \;, \quad \frac{\partial f}{\partial y} = 3y^2 + 3ax$$

$$\frac{dy}{dx} = \frac{-(3x^2 + 3ay)}{3y^2 + 3ax}$$

$$= \frac{-3(x^2 + ay)}{3(y^2 + ax)}$$

$$\frac{dy}{dx} = \frac{-x^2 + ay}{y^2 + ax}$$

③ Find $\frac{dy}{dx}$, $xy = c^2$

Solution :-

WKT

$$\frac{dy}{dx} = \frac{-\partial f / \partial x}{\partial f / \partial y}$$

$$xy - c^2 = \cancel{0}\,0$$

$$\frac{\partial f}{\partial x} = y \;, \quad \frac{\partial f}{\partial y} = x$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

④ $(x-a)^2 + (y-b)^2 = r^2$ find $\frac{dy}{dx}$.

Solution :-

W.K.T.

$$\frac{dy}{dx} = \frac{-\partial f / \partial x}{\partial f / \partial y}$$

$$(x-a)^2 + (y-b)^2 - r^2 = 0$$

$$\frac{df}{dx} = 2(x-a), \quad \frac{df}{dy} = 2(y-b)$$

$$\frac{dy}{dx} = \frac{-2(x-a)}{2(y-b)} = \frac{-(x-a)}{(y-b)}$$

Odd & even: problem:

① The following function which are odd function which are even function.

i) $ax^7 + b^5 + cx^3 + dx$, (ii) $ax^6 + bx^4 + cx^2 + d$

(iii) a) $\sin x$ b) $\cos x$ c) $\tan x$ d) $\operatorname{cosec} x$

e) $\sec x$ f) $\cot x$

(iv) a) $\frac{e^x - e^{-x}}{x}$ b) $\frac{e^x - e^{-x}}{2}$.

Solution:

i) $f(-x) = a(-x)^7 + b^5 + c(-x)^3 + d(-x)$

$= -ax^7 + b^5 * - cx^3 - dx$

It is odd function.

ii) $f(-x) = a(-x)^6 + b(-x)^4 + c(-x)^2 + d$

$f(-x) = a(-x^6) + b(-x)^4 + c(-x)^2 + d$

$= ax^6 + bx^4 + cx^2 + d$

$= +ax^6 + bx^4 + cx^2 + d$

It is even function.

it is ~~odd~~ function

(iii) a) $\sin x$

$f(-x) = \sin(-x)$

$= -\sin x$

It is odd function

b) $\cos x$

$f(-x) = \cos(-x)$

$= \cos x.$

It is even fuction

c) $\tan x$

$f(-x) = \tan(-x)$

$= -\tan x$

It is odd function

d) $\operatorname{cosec} x$

$f(-x) = \operatorname{cosec}(-x)$

$= -\operatorname{cosec} x$

It is odd function

e) $\sec x$

$f(-x) = \sec(-x)$

$= -\sec x$

It is even function

f) $\cot x$

$f(-x) = \cot(-x)$

$= -\cot x$

It is odd function

(iv) a) $\frac{e^x - e^{-x}}{x}$

$$f(-x) = \frac{e^{-x} - e^{-(-x)}}{-x}$$

$$= \frac{e^{-x} - e^{x}}{-x}$$

$$= \frac{-(e^{-x} - e^{x})}{x}$$

$$= \frac{-e^{-x} + e^{x}}{x}$$

$$= \frac{e^{x} - e^{-x}}{x}$$

$$= f(x)$$

It is even function.

v) b) $\frac{e^x - e^{-x}}{2}$

$$f(-x) = \frac{e^{-x} - e^{-(-x)}}{2}$$

$$= \frac{e^{-x} - e^{x}}{2}$$

It is odd function.

odd & even: problem.

① Idenify whether the function $f(x) = \sin x \cdot \cos x$ is an even or odd function. verify using the even and odd function definitions.

Solution:-

Given function $f(x) = \sin x \cdot \cos x$

we need to check if $f(x)$ is even or odd

we know that $\sin x$ is an odd function and $\cos x$ is an even function.

Also, the product of an even and an odd function is odd

Hence, $f(x) = \sin x \cdot \cos x$ is an odd function

Now, let us verify this using the definition of an odd function

consider

$$f(-x) = \sin(-x) \cdot \cos(-x)$$

$$-f(x) = -\sin x \cdot \cos x$$

Therefore, $f(x)$ is an odd function.

Hence, verified.

(2) Determine if the function $f(x) = \cosh x$ is even or not using even and odd function definition

Solution:-

Given function : $f(x) = \cosh x = (e^x + e^{-x})/2$

To determine if $f(x)$ is even or not,

Sub $x = -x$

$f(-x) = \cosh(-x)$

$= e^{-x} + e^{-(-x)}/2$

$= e^{-x} + e^{x}/2$

$f(x) = \cosh x$

$\therefore f(x) = \cosh x$ is a even function.

(3) Is $f(x) = x/(x^2-1)$ even or odd or neither?

Solution:-

Given: $f(x) = \frac{x}{(x^2-1)}$

$f(-x) = \frac{-x}{((-x)^2-1)}$

$= \frac{-x}{x^2-1}$

$= -f(x)$

$\therefore f(-x) = -f(x)$ is a odd function.

(4) $f(x) = x^3+1$

Solution:

$f(-x) = (-x)^3+1 = -x^3+1$

$f(-x) \neq f(x);\ -x^3+1 \neq x^3+1$

(not even).

$f(-x) \neq -f(x);\ -x^3+1 \neq -x^3-1$

(not odd)

(5) $f(x) = 10x^3 - 4x^2+3x-8$.

solution:-

$f(-x) = 10(-x)^3 - 4(-x)^2 + 3(-x) - 8$

$= 10(-1)^3(x)^3 - 4(-1)^2(x^2) + 3(-1)(x) - 8$

$= 10(-1)x^3 - 4(1)x^2 - 3x - 8$

$= -10x^3 - 4x^2 - 3x - 8$

$= -1(10x^3+4x^2+3x+8)$

$f(-x) = -(10x^3+4x^2+3x+8)$

(6) $f(x) = 2x^2-3$

Solution:-

$f(-x) = 2(-x)^2 - 3$

$= 2(-1)^2(x)^2 - 3$

$= 2(1)x^2 - 3$

$f(-x) = 2x^2 - 3$ is even function

Formula.

even functions $f(-x) = f(x)$	odd functions $f(-x) = -f(x$
$\cos(-x) = \cos x$ $\sec(-x) = \sec x$	$\sin(-x) = -\sin x$ $\operatorname{cosec}(-x) = -\operatorname{cosec} x$ $\tan(-x) = -\tan x$ $\cot(-x) = -\cot x$

functions	even, odd or neither?
$f(x) = 3x^2+8$	$f(-x) = 3(-x)^2+8$ $= 3x^2+8 = f(x)$ even
$f(x) = x^5-4x$	$f(-x) = (-x)^5-4(-x)$ $= -x^5+4x = -(x^5-4x)$ $= -f(x)$ odd
$f(x) = 2x^2-x-1$	$f(-x) = 2(-x)^2-(-x)-1$ $= 2x^2+x-1$ $= -(2x^2-x+1)$ $= -2x^2-x+1$ $f(-x) \neq f(x) \neq -f(x)$ neither

even+odd

$h(x) = f(x) + g(x)$

$h(-x) = f(-x) + g(-x)$

$h(-x) = f(x) - g(x)$

neither

even + odd = neither

odd + odd

$h(x) = g(x) + q(x)$

$h(-x) = g(-x) + q(-x)$

$h(-x) = -g(x) + -q(x)$

$h(-x) = -[g(x) + q(x)]$

$= -1 \cdot h(x)$

odd

odd + odd = odd.

even − even

$h(x) = f(x) - p(x)$

$h(-x) = f(-x) - p(-x)$

$h(-x) = f(x) - p(x)$

$= h(x)$

even

even − even = even

even + even.

$h(x) = f(x) + p(x)$

$h(-x) = f(-x) + p(-x)$

$h(-x) = f(x) + p(x)$

$= h(x)$

even

~~even~~ even + even = even

even − odd.

$h(x) = f(x) - g(x)$

$h(-x) = f(-x) - g(-x)$

$h(-x) = f(x) + f(x)$

neither

even − odd =

neither

odd − odd

$h(x) = g(x) - q(x)$

$h(-x) = g(-x) - q(-x)$

$= -[g(x) + q(x)]$

$=$

① $f(x) = x^{-4}$

Solution:-

$f(-x) = (-x)^{-4}$

$= 1/(-x)^4$

$= 1/x^4$

$= (-x)^4$

$= x^4$

$= f(x)$ is a even

② $g(x) = x^{-5}$

solution:

$g(-x) = (-x)^{-5}$

$= 1/(-x)^5$

$= -1/x^5$

$= -x^{-5}$

$= -g(x)$ is a odd.

③ $h(x) = x^3 + 2x^2$

Solution:-

$h(-x) = (-x)^3 + 2(-x)^2$

$= -x^3 + 2x^2$

$\neq h(x)$ or $-h(x)$

neither

The hyperbolic sine and cosine function are defined

$$\boxed{\sinh x = \frac{e^x - e^{-x}}{2} \quad \& \quad \cosh x = \frac{e^x + e^{-x}}{2}}$$

Formula:

$\cosh^2 x - \sinh^2 x = 1$

$\frac{d}{dx}(\sinh x) = \cosh x$

$\frac{d}{dx}(\cosh x) = \sinh x$

① use this definition to S.T $\sinh x$ is odd.

solution :-

$$\sinh(-x) = \frac{e^{(-x)} - e^{-(-x)}}{2}$$

$$= \frac{e^{-x} - e^{x}}{2}$$

$$= \frac{-(e^{x} - e^{-x})}{2}$$

$\sinh(-x) = -\sinh x$ is a odd function

② use this definition S.T $\cosh x$ is even.

solution:-

$$\cosh(-x) = \frac{e^{(-x)} + e^{-(-x)}}{2}$$

$$= \frac{e^{-x} + e^{x}}{2}$$

$$= \frac{e^{x} + e^{-x}}{2}$$

$\cosh(-x) = \cosh x$ is a even function

$f(x) = g(x) \quad \forall\, x \in A$.

54. Prove that the mapping $f: Z^+ \to Z^+$ defined by $f(x) = x^2$ where Z^+ is the set of positive integers is 1-1 & onto.

$$Z^+ = \{1, 2, 3, \dots\}$$

$\because f(x) = x^2 \Rightarrow f(x) = \{1, 4, 9, \dots\}$

$\therefore x_1, x_2 \in Z^+$

$$f(x_1) = f(x_2)$$
$$x_1^2 = x_2^2$$
$$x_1 = x_2$$

$y = f(x)$

$y = x^2$

$\Rightarrow x = \sqrt{y}$

$\therefore f$ is 1-1 vs onto.

Prove that $f: X \to X$ where $X = \{x \in R \;\&\; x \neq 0\}$ defined by $f(x) = \frac{1}{x}$ is 1-1 & onto.

<u>one to one</u>:

$X = R - \{0\}$, $f: X \to X$ defined by

$$f(x) = \frac{1}{x}$$

$\therefore$ let $f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$

<u>Onto</u>: let $x \in X$ then

$$f\left(\frac{1}{x}\right) = \frac{1}{(1/x)} = x$$

$\therefore$ hence f is 1-1 & onto.

Inverse function:

Let $f: A \to B$ be 1-1 & onto. Then the I.f. $f^{-1}: B \to A$ is $y = f(x)$ $\forall\, x \in A, y \in B$.

1. Prove that $f: R \to R$ defined by $f(x) = ax+b$ where $a, b, x \in R$, $a \neq 0$ is invertible

One to one

$f: R \to R$ by $f(x) = ax+b$

$x_1, x_2 \in R$

$$f(x_1) = f(x_2)$$
$$ax_1 + b = ax_2 + b$$
$$ax_1 = ax_2$$
$$x_1 = x_2$$

Onto:

If $y \in R$ then $y = f(x)$

$y = ax+b \Rightarrow x = \frac{y-b}{a}$

Now $x \in R$ $\exists$ $\frac{y-b}{a} = x \in R$ $\ni$

$$f(x) = f\left(\frac{y-b}{a}\right)$$
$$= \cancel{a}\left(\frac{y-\cancel{b}}{\cancel{a}}\right) + \cancel{b}$$
$$f(x) = y$$

$\therefore$ Hence f^{-1} exists. defined by

$$f^{-1}(y) = \frac{y-b}{a}$$

Composition:

Let $f: A \to B$ & $g: B \to C$ be two funs. The composition of f & g is the fun $g \circ f: A \to C$ defined by

$$(g \circ f)\, x = g[f(x)] \quad \forall x \in A.$$

If $x \in$ {($f : R \to R$ & $g : R \to R$ are define
by $f(x) = x^2$ & $g(x) = \sin x$ find fog & g

$$(f \circ g)\,x = f[g(x)]$$
$$= f[\sin x]$$
$$= \sin x^2$$

& $$(g \circ f)\,x = g[f(x)]$$
$$= g[x^2]$$
$$= \sin x^2$$

Q. If $f(x) = x+2$, $g(x) = x-2$ & $h(x) = 3x^2$
for $x \in R$, find $f \circ g \circ h$ & $f \circ h \circ g$

$$f \circ g \circ h = f[g(h(x))]$$
$$= f[g(3x^2)]$$
$$= f[3x^2 - 2]$$
$$= 3x^2 - 2 + 2$$
$$= 3x^2$$

$$f \circ h \circ g = f[h(g(x))]$$
$$= f[h(x-2)]$$
$$= f[3(x-2)^2]$$
$$= f[3(x-2)^2 + 2]$$

) $f(x) = 3x+4$, $g(x) = x^2+2$

$$f \circ g = f[g(x)] = f[x^2+2]$$
$$= 3(x^2+2)+4$$
$$= 3x^2+6+4$$
$$= 3x^2+10$$

$$g \circ f = g[f(x)] = g[3x+4]$$
$$= (3x+4)^2 + 2$$
$$= (3x)^2 + 4^2 + 2(3x)(4) + 2$$
$$= 9x^2 + 16 + 24x + 2$$
$$= 9x^2 + 24x + 18$$

An algebraic ~~and~~ expression is one containing algebraic

① Show that $f(x) = 3\sqrt{x} - 2$ is algebraic function.

Method : 1

We write this function f into an algebraic form

$$P_n(x)y^n + P_{n-1}(x)y^{n-1} + \dots + P_1(x)y + P_0(x) = 0$$

Now let $y = f(x) = 3\sqrt{x} - 2$

$$y = 3\sqrt{x} - 2$$

$$y + 2 = 3\sqrt{x}$$

$$(y+2)^3 = 3\sqrt{x}$$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$y^3 + 3y^2(2) + 3y(2)^2 + 2^3 = x$$

$$y^3 + 6y^2 + 12y + 8 = x$$

$$y^3 + 6y^2 + 12y + 8 - x = 0$$

where $P_3(x) = 1$, $P_2(x) = 6$, $P_1(x) = 12$, $P_0(x) = 8 - x$

So $f(x) = 3\sqrt{x} - 1$ is algebraic function

Method 2 :

$$f(x) = 3\sqrt{x} - 2$$

Let $f_1(x) = 3\sqrt{x}$ and $f_2(x) = 2$

(i) Show that $f_1(x) = 3\sqrt{x}$ is algebraic

we have $f_1(x) = 3\sqrt{x}$

$$y = 3\sqrt{x}.$$

$y^3 = x$

$y^3 - x = 0$ is algebraic form

where $P_3(x) = 1$, $P_0(x) = -1$

So $f_1(x) = 3\sqrt{x}$ is algebraic function.

(ii) show $f_2(x) = 2$ is a algebraic function

we have $f_2(x) = 2$

$y = 2$

$y - 2 = 0$ is algebraic function

form where $P_1(x) = 1$ $P_0(x) = -2$

So $f_2(x) = 2$ is algebraic

Since $f_1(x) = 3\sqrt{x}$ and $f_2(x) = 2$ are

algebraic function

So, $f(x) = f_1(x) - f_2(x) = 3\sqrt{x} - 2$ is

algebraic function

(2) Show that function $g(x) = \frac{x}{1+\pi\sqrt{x}}$ is

algebraic function

solution:

$$g(x) = \frac{x}{1+\pi\sqrt{x}}$$

Let $g_1(x) = x$ and $g_2(x) = 1 + \pi\sqrt{x}$.

(i) show that $g_1(x) = x$ is algebraic function

we have $g_1(x) = x$

$y = x$

$y - x = 0$ is an algebraic

form

$g_1(x) = x$ is an algebraic function

(+) Show that $g_2(x) = 1+\pi\sqrt{x}$ is algebraic function

we have $g_2(x) = 1+\pi\sqrt{x}$

$$y = 1+\pi\sqrt{x}$$
$$y-1 = \pi\sqrt{x}$$
$$(y-1)^2 = \pi^2 x$$
$$y^2-2y+1 = \pi^2 x$$
$$y^2-2y+(1-\pi^2 x) = 0$$ is an algebraic form

$g_2(x) = 1+\pi\sqrt{x}$ is algebraic function.

Since

$g_1(x) = x$ and $g_2(x) = 1+\pi\sqrt{x}$ are algebraic functions.

So $g(x) = \dfrac{g_1(x)}{g_2(x)} = \dfrac{x}{1+\pi\sqrt{x}}$ is an algebraic function.

③ Show that $h(x) = 1-|x|$ is algebraic function

Sol

$h(x) = 1-|x|$

we have $|x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$

Hence $h(x) = \begin{cases} 1-x, & x>0 \\ 1+x, & x<0 \end{cases}$

let $h_1(x) = 1-x$ and $h_2(x) = 1+x$

(+) Show that

$h_1(x) = 1-x$ is algebraic function

we have $h_1(x) = 1-x$

$y = 1-x$

$y + (x-1) = 0$ is an algebraic form

$\Rightarrow h_1(x) = 1-x$ is algebraic function

(+) show that

$h_2(x) = 1+x$ is algebraic function

we have $h_2(x) = 1+x$

$\Leftrightarrow y = 1+x$

$\Rightarrow y + (-x-1) = 0$ is an algebraic form

$h_2(x) = 1+x$ is algebraic function

since $h_1(x) = 1-x$ and $h_2(x) = 1+x$ are algebraic function.

So, $h(x) = \begin{cases} h_1(x) = 1-x, & x > 0 \\ h_2(x) = 1+x & x < 0 \end{cases}$ is an algebraic function

(4). Show that $K(x) = \operatorname{sgn} x$ is algebraic function

soln

$K(x) = \operatorname{sgn} x$

$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } \boxed{x < 0} \end{cases}$

Since obviously $-1, 0, 1$ are algebraic functions.

So $K(x) = \operatorname{Sgn} x$ is an algebraic function

⑤ Show that $l(x) = \begin{cases} \sqrt{x} - 1, & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}$ is algebraic function

soln:

$$l_1(x) = \begin{cases} \sqrt{x} - 1, & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0. \end{cases}$$

Let $l_1(x) = \sqrt{x} - 1$ and $l_2(x) = x^2$

$l_1(x) = \sqrt{x} - 1$ is algebraic function

because $f(x) = \sqrt{x} - 1$

$$y = \sqrt{x} - 1$$

$$y + 1 = \sqrt{x}$$

$$(y+1)^2 = x$$

$$y^2 + 1^2 + 2y = x$$

$$y^2 + 2y + 1 = x$$

$y^2 + 2y + (1 - x) = 0$ is an algebraic form.

$l_2(x) = x^2$ is algebraic function because

$$l_2(x) = x^2$$

$$y = x^2$$

$\boxed{y - x^2 = 0}$ is an algebraic form.

Since $l_1(x) = \sqrt{x} - 1$ and $l_2(x) = x^2$ are algebraic functions

(4). So, $l(x) = \begin{cases} \sqrt{x} - 1, & x \geq 0 \\ x^2, & x < 0 \end{cases}$ is an algebraic function

(6) Solve the equation $4^x - 5\times 2^x + 6 = 0$

Soln:

$4^x - 5\times 2^x + 6 = 0$

Recall $\begin{cases} a^{p(x)} = a^{q(x)} \\ a>0, a\neq 1 \end{cases}$ $p(x) = g(x)$

$4^x - 5\times 2^x + 6 = 0$.

$(2^x)^2 - 5\times 2^x + 6 = 0 \longrightarrow$ (A)

Let $t = 2^x$, where $t > 0$.

Hence (A) $\Leftrightarrow$ $t^2 - 5t + 6 = 0$.

$\Rightarrow t^2 - 2t - 3t + 6 = 0$

$\Rightarrow (t^2 - 2t) - (3t - 6) = 0$

$\Rightarrow t(t-2) - 3(t-2) = 0$

$\Rightarrow (t-2)(t-3) = 0$

$\Rightarrow \begin{cases} t - 2 = 0 \\ t - 3 = 0 \end{cases}$

$= \begin{cases} t = 2 \\ t = 3. \end{cases}$

We have $t = 2^x$, where $t > 0$.

Now if $t = 2$, then $2 = 2^x$

$\Rightarrow x = 1$

if $t = 3$

then $3 = 2^x$

$\Leftrightarrow x = \log_2 3$

So $x = \{1, \log_2 3\}$ are roots of the equation

(4) Solve the system of eqn $\begin{cases} 8^x = y \\ 2^x = 5y \end{cases}$

<u>Soln</u>:-

Solve of Given $\begin{cases} 8^x = y \rightarrow (1) \\ 2^x = 5y \rightarrow (2) \end{cases}$

Take $y = 8^x$ substitute in (2)

Then (2) $\Leftrightarrow 2^x = 5 \times 8^x$

$$\frac{2^x}{8^x} = 5$$

$$\left(\frac{2}{8}\right)^x = 5$$

$$\left(\frac{1}{4}\right)^x = 5$$

$$x = \log_{1/4} 5$$

$$x = \log_{2^{-2}} 5^1$$

$\log_{a^m} b^n = \frac{n}{m}$

$= \log_{2^{-2}} 5$

$= -\frac{1}{2} \log_2 5$

By ② $\Leftrightarrow$

$2^{-1/2 \log_2 5} = 5y$

$\Leftrightarrow 2^{\log_2 5^{-1/2}} = 5y$. $\boxed{a^{\log_a n} = n}$

$\Leftrightarrow 5^{-1/2} = 5y$.

$\Leftrightarrow \frac{1}{\sqrt{5}} = 5y$.

$y = \frac{1}{5\sqrt{5}}$

$= \sqrt{5}/25$

So, $x = -\frac{1}{2} \log_2 5$ and $y = \frac{\sqrt{5}}{25}$ are roots of the system equations,

⑧. Solve the equation $x^{\sqrt{x}} = (\sqrt{x})^x$.

Soln :-

$x^{\sqrt{x}} = (\sqrt{x})^x$

$x^{\sqrt{x}} = x^{1/2 x}$

$\ln x^{\sqrt{x}} = \ln x^{1/2 x}$

$\Leftrightarrow \sqrt{x} \ln x = \frac{1}{2} x \ln x$ where $x > 0$

$\Leftrightarrow \sqrt{x} \ln x - \frac{1}{2} x \ln x = 0$.

$\Leftrightarrow \ln x (\sqrt{x} - \frac{1}{2} x) = 0$

$\Leftrightarrow \begin{cases} \ln x = 0 \longrightarrow ① \\ \sqrt{x} - 1/2 x = 0 \longrightarrow ② \end{cases}$.

By (1) $\Leftrightarrow x=1$

And (2) $\Leftrightarrow \sqrt{x} - \frac{1}{2}x = 0$

$$\sqrt{x} = \frac{1}{2}x$$

$$x = \frac{1}{4}x^2$$

$$\frac{1}{4}x^2 - x = 0.$$

$$x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$\Rightarrow \begin{cases} x=0 \\ x-4=0 \end{cases} \Leftrightarrow \begin{cases} x=0 & \text{not required because } x>0 \\ x=4. \end{cases}$$

So $x = \{1, 4\}$ are root of the equation.

(9) Solve the equation $2^{4\cos^2 x+1} + 16 \times 2^{4\sin^2 x-3} = 20$

Soln

eqn $2^{4\cos^2 x+1} + 16\times 2^{4\sin^2 x-3} = 20$

$$2^{4\cos^2 x+1} + 16\times 2^{4(1-\cos^2 x)-3} = 20$$

$$2^{4\cos^2 x+1} + 16\times 2^{1-4\cos^2 x} = 20$$

$$\Leftrightarrow 2^{4\cos^2 x+1} + 16\times 2^{1-4\cos^2 x} = 20.$$

$$2 \times 2^{4\cos^2 x} + 32 \times 2^{-4\cos^2 x} = 20$$

$$2^{4\cos^2 x} + 16 \times 2^{-4\cos^2 x} = 10 \rightarrow (A)$$

let $t = 2^{4\cos^2 x}$ where $t > 0$

Hence (A) $\Leftrightarrow$ $t + 16 \times t^{-1} = 10$

$$t + \frac{16}{t} = 10$$

$$\frac{t^2 + 16}{t} = 10$$

$$t^2 + 16 = 10t$$

$$t^2 - 10t + 16 = 0$$

using $\Delta = b^2 - 4ac$

$$= (-10)^2 - 4(1)(16)$$

$$= 100 - 64$$

$$\Delta = 36$$

the $\sqrt{\Delta} = \sqrt{36}$

$\boxed{\sqrt{\Delta} = \sqrt{6}}$ $\sqrt{\Delta} = 6$

Now $t_1 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{10+6}{2} = 8$

$$= \frac{-b - \sqrt{\Delta}}{2a} = \frac{10-6}{2} = 2$$

Since $t = 2^{4\cos^2 x}$

where $t > 0$

If $t = 8$, then $2^{4\cos^2 x} = 8 = 2^3$

$\Leftrightarrow 4\cos^2 x = 3$

$\Leftrightarrow \cos^2 x = 3/4$

$$\cos x = \pm \frac{\sqrt{3}}{2}$$

Now

$$\cos x = \frac{\sqrt{3}}{2}$$

$$\boxed{\begin{array}{l}\cos x = \cos \alpha \\ x = \pm \alpha + 2k\pi, k \in \mathbb{Z}\end{array}}$$

$$\cos x = \frac{\sqrt{3}}{2}$$

$$\cos x = \cos \frac{\pi}{6}$$

$$x = \pm \frac{\pi}{6} + 2k\pi, \; k \in \mathbb{Z}$$

$$\cos x = -\frac{\sqrt{3}}{2}$$

$$\cos x = \cos \frac{5\pi}{6}$$

$$x = \pm \frac{5\pi}{6} + 2k\pi, \; k \in \mathbb{Z}$$

(4) If $t = 2$, then

$$2^{4\cos^2 x} = 2$$

$$4\cos^2 x = 1$$

$$\boxed{\cos^2 x = 1/4}$$

$$\cos x = \pm \frac{1}{2}$$

Now

$$\cos x = \frac{1}{2} = \cos \frac{\pi}{3}$$

$$x = \pm \frac{\pi}{3} + 2k\pi, \; k \in \mathbb{Z}$$

So $x = \left\{ \frac{\pi}{6} + 2k\pi, -\frac{\pi}{6} + 2k\pi, \pm\frac{5\pi}{6} \right.$

$$\cos x = -\frac{1}{2} = \cos \frac{2\pi}{3}$$

$$x = \pm \frac{2\pi}{3} + 2k\pi, \; k \in \mathbb{Z}$$

So $x = \left\{ \frac{\pi}{6} + 2k\pi, \frac{-\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi, \right.$

$-\frac{5\pi}{6} + 2k\pi, \frac{\pi}{3} + 2k\pi, \frac{-\pi}{3} + 2k\pi$

$\left. \frac{2\pi}{3} + 2k\pi, -\frac{2\pi}{3} + 2k\pi, \text{ where } k \in \mathbb{Z} \right\}$ are

roots of the equation.