

**MARUDHAR KESARI JAIN COLLEGE FOR WOMEN,VANIYAMBADI  
PG & RESEARCH DEPARTMENT OF MATHEMATICS**

**SUBJECT NAME:** MATHEMATICS FOR STATISTICS

**CLASS :** 1 B.Sc STATISTICS

**CODE:** 23UEST13

**SYLLABUS:**

**Unit-V** Successive differentiation: Leibnitz's theorem, nth derivatives of standard functions – simple problems. Partial differentiation: Successive partial differentiation.

## Successive Differentiation

\* The derivative of a function of  $x$  is also a function of  $x$

\* The New function may be differentiable, in which case, the derivative of the first derivative is called the second derivative of the original function.

\* Similarly, the derivative of the second derivative is called the third derivative and so on. upto the  $n^{\text{th}}$  derivative.

Lq:-  $y = 4x^5$

$$\frac{dy}{dx} = 20x^4$$

$$\frac{d}{dx} \left( \frac{dy}{dx} \right) = 80x^3$$

$$\frac{d}{dx} \left\{ \frac{d}{dx} \left( \frac{dy}{dx} \right) \right\} = 240x^2.$$

etc..

## $n^{\text{th}}$ Derivatives.

(4)

If  $y$  is a function of  $x$ , its derivatives  $dy/dx$  will be some other function of  $x$  and differentiation of this function with respect to  $x$  is called Second Derivatives. and it is denoted by  $\frac{d^2y}{dx^2}$ . [i.e.  $\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d^2y}{dx^2}$ ]

Similarly,  $\frac{d^3y}{dx^3}$  is called third derivative.

## Standard $n^{\text{th}}$ Derivative.

① Find the  $n^{\text{th}}$  derivative of  $e^{ax}$ .

Sol:

$$y = e^{ax}$$

$$y_1 = \frac{dy}{dx} = ae^{ax}$$

$$y_2 = \frac{d^2y}{dx^2} = a^2e^{ax}$$

$$y^n = \frac{d^n y}{dx^n} = e^{ax} a^n$$

② Find the  $n^{\text{th}}$  derivative of  $y = \frac{1}{ax+b}$ . (5)

Sol:

$$y_1 = (-1)(ax+b)^{-1-1} \cdot a$$

$$= (-1)(ax+b)^{-2} \cdot a$$

$$y_2 = (-1)(-2)(ax+b)^{-3} \cdot a^2 \cdot \frac{y_1 - (-1)^0 n! a^n}{(ax+b)^{n+1}}$$

$$y_n = (-1)(-2) \dots (-n)(ax+b)^{-(n+1)} \cdot a^n.$$

③ Find the  $n^{\text{th}}$  derivative of  $y = \frac{1}{(ax+b)^2}$ .

Sol:

$$y = (ax+b)^{-2}$$

$$y_1 = (-2)(ax+b)^{-3} \cdot a$$

$$y_2 = (-2)(-3)(ax+b)^{-4} \cdot a^2$$

$$y_n = (-2)(-3) \dots (-n)(ax+b)^{-(n+1)} \cdot a^n$$

$$y_n = \frac{(-1)^n (n+1)! a^n}{(ax+b)^{n+2}}$$

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Section - 1.3

(6)

Standard Results.

①  $y = (ax+b)^m$ . find  $y_n$ .

Sol:

$$y_1 = m(ax+b)^{m-1} \cdot a$$

$$y_2 = m(m-1)(ax+b)^{m-2} \cdot a^2$$

$$\vdots$$

$$y_n = m(m-1) \dots (m-n+1)(ax+b)^{m-n} \cdot a^n$$

②  $y = \log(ax+b)$ .

Sol:

$$y_1 = \frac{1}{ax+b} = (ax+b)^{-1} \cdot a$$

$$y_2 = (-1)(ax+b)^{-1-1} \cdot a^2$$

$$= (-1)(ax+b)^{-2} \cdot a^2$$

$$y_3 = (-1)(-2)(ax+b)^{-3} \cdot a^3$$

$$y_4 = \frac{(-1)(-2)}{(ax+b)^3} \cdot a^3$$

$$\vdots$$

$$y_n = (-1)(-2) \dots (n-1)(ax+b)^{-n} \cdot a^n$$

$$= (-1)^{n-1} (n-1)! a^n (ax+b)^{-n}$$

$$= \frac{(-1)^{n-1} (n-1)! a^n}{(ax+b)^n}$$

(3)

$$y = \sin(ax + b).$$

$$\underline{\text{Sol}} \quad y_1 = \cos(ax + b) \cdot a.$$

$$= a \cos(ax + b).$$

$$y_1 = a \sin\left(\frac{\pi}{2} + ax + b\right).$$

$$y_2 = a \cos\left(\frac{\pi}{2} + ax + b\right) \cdot a.$$

$$= a^2 \cos\left(\frac{\pi}{2} + ax + b\right)$$

$$y_2 = a^2 \sin\left(\frac{2\pi}{2} + ax + b\right)$$

$$y_2 = a^2 \sin\left(\frac{2\pi}{2} + ax + b\right)$$

$$y_3 = a^3 \sin\left(\frac{3\pi}{2} + ax + b\right).$$

$$y_n = a^n \sin\left(n\frac{\pi}{2} + ax + b\right)$$

$$\cos x = \sin\left(\frac{\pi}{2} + x\right)$$

$$\sin x = \sin\left(\frac{\pi}{2} + x\right)$$

Formula

$$\cos \theta = \sin(90^\circ + \theta)$$

$$= \sin\left(\frac{\pi}{2} + \theta\right)$$

$$\cos(\theta + b) = \sin\left(\frac{\pi}{2} + \theta + b\right)$$

(4)

$$e^{ax} \sin(bx + c)$$

Sol

$$y = e^{ax} \underline{\sin(bx + c)}.$$

$$\frac{d}{dx}(uv) = uv' + vu'$$

$$y_1 = e^{ax} \cos(bx + c) \cdot b + \sin(bx + c) a e^{ax}.$$

$$= ae^{ax} \sin(bx + c) + be^{ax} \cos(bx + c).$$

Put,

$$a = r \cos \theta, b = r \sin \theta.$$

$$r = \sqrt{a^2 + b^2} \text{ and } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$y_1 = r \cos \theta e^{ax} \sin(bx+c) + r \sin \theta e^{ax} \frac{(bx+c)}{\cos(bx+c)} \quad (8)$$

$$= r e^{ax} [\cos \theta \sin(bx+c) + \sin \theta \cos(bx+c)]$$

$$y_1 = \underline{r e^{ax}} \underline{[\sin(bx+c+\theta)]}.$$

$$y_2 = r [ae^{ax} \sin(bx+c+\theta) + e^{ax} [\cos(bx+c+\theta)]]$$

$$= r [r \cos \theta e^{ax} \sin(bx+c+\theta) + e^{ax} \cos(bx+c+\theta) r \sin \theta]$$

$$y_2 = r^2 e^{ax} [\cos \theta \sin(bx+c+\theta) + \sin \theta \cos(bx+c+\theta)].$$

$$y_2 = r^2 e^{ax} [\sin(bx+c+2\theta)].$$

$$y_3 = r^3 e^{ax} [\sin(bx+c+3\theta)].$$

$$\vdots \\ y_n = r^n e^{ax} [\sin(bx+c+n\theta)].$$

In General,

$$D^n \{e^{ax} \sin(bx+c)\} = (a^2 + b^2)^{n/2} e^{ax}$$

$$\sin \left[ (bx+c+n \tan^{-1} \frac{b}{a}) \right]$$

$$⑤ \cos(ax+b)$$

Sol:

$$y = \cos(ax+b)$$

$$y_1 = -a \sin(ax+b) \cdot a.$$

$$= a^2 \sin(ax+b).$$

$$y_1 = a^2 \cos(ax+b + \frac{\pi}{2})$$

$$y_2 = -a^2 \sin(ax+b + \frac{\pi}{2}) \cdot a.$$

$$= a^2 [-\sin(ax+b + \frac{\pi}{2})].$$

$$= a^2 (\cos(ax+b + \frac{2\pi}{2}))$$

$$y_2 = a^2 \cos(ax+b + \frac{2\pi}{2}).$$

$$y_3 = a^3 \cos(ax+b + \frac{3\pi}{2}).$$

$$y_n = a^n \cos(ax+b + n\frac{\pi}{2}).$$

$$e^{ax} \cos(bx+c).$$

Sol:

$$y = e^{ax} \cos(bx+c).$$

$$y_1 = ae^{ax} \cos(bx+c) - b e^{ax} \sin(bx+c).$$

put

$$a = r \cos \theta, b = r \sin \theta.$$

$$r = \sqrt{a^2 + b^2} \text{ & } \theta = \tan^{-1}\left(\frac{b}{a}\right).$$

$$y_1 = r e^{ax} [\cos \theta \cos(bx+c) - \sin \theta \sin(bx+c)]$$

$$y_1 = r e^{ax} \cos(bx+c+\theta)$$

$$y_2 = r [ae^{ax} \cos(bx+c+\theta) - \sin(bx+c+\theta)e^{ax} b]$$

$$= r [r \cos \theta e^{ax} \cos(bx+c+\theta) - \sin(bx+c+\theta) e^{ax} r \sin \theta]$$

$$= r^2 e^{ax} [\cos \theta \cos(bx+c+\theta) - \sin(bx+c+\theta) \sin \theta]$$

$$= r^2 e^{ax} [\cos(bx+c+2\theta)].$$

$$y_3 = r^3 e^{ax} [\cos(bx+c+3\theta)].$$

$$\vdots$$

$$y_n = r^n e^{ax} [\cos(bx+c+n\theta)].$$

For General.

$$D^n \{ e^{ax} \cos(bx+c) \} = (a^2 + b^2)^{n/2} e^{ax} \cos[(bx+c+n \tan^{-1} \frac{b}{a})].$$

#### 1.4 Fractional Expressions.

Fractional expressions of the form

$\frac{f(x)}{\phi(x)}$  both functions being algebraic

and rational, can be differentiated n times by splitting them into partial fractions.

Q9:

(11)

$$\frac{x^2}{(x+1)^2(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}.$$

① Find  $y_m$  when  $y = \frac{x^2}{(x+1)^2(x+2)}$ .

Sol:

$$y = \frac{x^2}{(x+1)^2(x+2)}$$

$$\frac{x^2}{(x+1)^2(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}.$$

$$x^2 = \frac{A(x+1)(x+2) + B(x+2) + C(x+1)^2}{(x+1)^2(x+2)}.$$

$$x^2 = A(x+1)(x+2) + B(x+2) + C(x+1)^2.$$

put  $x = -1$ .

$$x^2 = A(0)(-1+2) + B(-1+2) + C(-1+1)^2.$$

$$1^2 = A(0) + B(1) + C(0).$$

$$1 = B.$$

$$\boxed{B = 1}$$

$x = -2$ .

$$(-2)^2 = A(-2+1)(-2+2) + B(-2+2) + C(-2+1)^2$$

$$4 = A(-1)(0) + B(0) + C(+1).$$

$$\boxed{C = +4.}$$

put  $x=0$ : (12)

$$0 = A(0+1)(0+2) + B(0+2) + C(0+1)$$

$$0 = A(1)(2) + B(2) + C(1)$$

$$0 = A(2) + B(2) + C(1)$$

$$0 = A(2) + B(2) + C(1)$$

$$0 = A(2) + 2 + 4$$

$$0 = A(2) + 6$$

$$A(2) + 6 = 0$$

$$A(2) = -6 \quad | :2$$

$$A = -3$$

$$\boxed{A = -3}$$

$$\frac{1}{ax+b} = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$$

$$\frac{x^2}{(x+1)^2(x+2)} = \frac{-3}{(x+1)} + \frac{1}{(x+1)^2} + \frac{4}{(x+2)} \quad \frac{(ax+b)^2}{(ax+b)^{n+1}} = \frac{(-1)^n (n+1)! a^n}{(ax+b)^{n+1}}$$

$$y_n = \frac{-3(-1)^n \cdot n!}{(x+1)^{n+1}} + \frac{(-1)^n (n+1)!}{(x+1)^{n+2}} + \frac{4(-1)^n n!}{(x+2)^{n+1}}$$

(2) Find  $y_n$  when  $y = \frac{x^2}{(x-1)^2(x+2)}$ .

Solu:

$$y = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

$$\frac{x^2}{(x-1)^2(x+2)} = \frac{A(x-1)(x+2) + B(x+2) + C}{(x-1)^2(x+2)} \quad (13)$$

$$x^2 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

Put  $x = 1$ .

$$1^2 = A(1-1)(1+2) + B(1+2) + C(1-1)^2$$

$$1 = A(0) + B(3) + C(0)$$

$$B = \frac{1}{3}.$$

Put  $x = -2$ .

$$(-2)^2 = A(-2-1)(-2+2) + B(-2+2) + C(-2-1)^2$$

$$4 = A(0) + B(0) + C(9).$$

$$C = \frac{4}{9}.$$

Put  $x = 0$ .

$$0^2 = A(0-1)(0+2) + B(0+2) + C(0-1)^2$$

$$0 = A(-1)(2) + B(2) + C(1)$$

$$-2A + \frac{1}{3}(2) + \frac{4}{9} = 0.$$

$$-2A + \frac{2}{3} + \frac{4}{9} = 0.$$

$$\frac{10}{9} = 2A.$$

$$\frac{10}{18} = A.$$

$$A = \frac{5}{9}.$$

$$y = \frac{5}{9} \frac{1}{(x-1)} + \frac{1}{3} \frac{1}{(x-1)^2} + \frac{4}{9} \frac{1}{(x+2)}$$

$$y_n = \frac{5}{9} \frac{n!(-1)^n}{(x-1)^{n+1}} + \frac{(n+1)!(-1)^n}{3(x-1)^{n+2}} + \frac{4}{9} \frac{(-1)^n n!}{(x+2)^{n+1}}$$

$$y_n = (-1)^n n! \left\{ \frac{5}{9(x-1)^{n+1}} + \frac{n+1}{3(x-1)^{n+2}} + \frac{4}{9(x+2)^{n+1}} \right\}$$

(3)

$$y = \frac{x}{(x-1)^2(x+2)}$$

Sol:

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$$

$$= \frac{A(x-1)(x+2)}{(x-1)} + \frac{B(x+2)}{(x-1)^2} + \frac{C(x-1)^2}{(x+2)}$$

$$\frac{x}{(x-1)^2(x+2)} = \frac{A(x-1)(x+2) + B(x+2) + C(x-1)^2}{(x-1)(x-1)^2(x+2)}$$

$$x = A(x-1)(x+2) + B(x+2) + C(x-1)^2 \quad \text{--- (1)}$$

put  $x=1$  in (1)

$$1 = A(1-1)(1+2) + B(1+2) + C(1-1)^2$$

$$1 = A(0) + B(3) + C(0)$$

$$B = \frac{1}{3}$$

put  $x=-2$  in (1)

$$-2 = A(-2-1)(-2+2) + B(-2+2) + C(-2-1)^2$$

$$-2 = A(0) + B(0) + C(-3)^2$$

$$-R = C(1).$$

$$C = -\frac{2}{9}.$$

put  $x=0$  in ①

$$0 = A(0-1)(0+R) + B(0+R) + C(0-1)^2.$$

$$0 = A(-1)(R) + B(R) + C(-1)^2.$$

$$0 = A(-R) + \frac{R}{3} - \frac{R}{9}.$$

$$\frac{6}{9} - \frac{2}{9} = 2A.$$

$$\frac{4}{9} = 2A.$$

$$A = \frac{4}{18}.$$

$$\therefore A = \frac{4}{18}, B = \frac{1}{3}, C = -\frac{2}{9}.$$

$$y = \frac{4}{18} \frac{1}{(x-1)} + \frac{1}{3} \frac{1}{(x-1)^2} - \frac{2}{9} \frac{1}{(x+R)}.$$

$$y_n = \frac{4}{18} \frac{n!(-1)^n}{(x-1)^{n+1}} + \frac{(n+1)!(-1)^n}{3(x-1)^{n+2}} - \frac{2}{9} \frac{(-1)^n n!}{(x+R)^{n+1}}.$$

General form

$$y_n = (-1)^n n! \left\{ \frac{4}{18(x-1)^{n+1}} + \frac{(n+1)}{3(x-1)^{n+2}} - \frac{2}{9(x+R)^{n+1}} \right\}.$$

④ Find  $y_n$  when.  $y = \frac{1}{x^2 + a^2}$ .

Sol:

$$y = \frac{1}{x^2 + a^2}$$

$$y = \frac{1}{x^2 - (-a^2)}$$

$$= \frac{1}{x^2 - i^2 a^2}$$

$$y = \frac{1}{x^2 - (ia)^2} \Rightarrow (a^2 - b^2)$$

$$y = \frac{1}{(x+ia)(x-ia)}.$$

$$y = \frac{1}{2ia} \left\{ \frac{1}{x-ia} - \frac{1}{x+ia} \right\}$$

$$y_n = \frac{1}{2ia} \left\{ \frac{(-1)^n n! i^n}{(x-ia)^{n+1}} - \frac{(-1)^n n! i^n}{(x+ia)^{n+1}} \right\}.$$

$$y_n = \frac{1}{2ia} (-1)^n n! i^n \left\{ \frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}} \right\}.$$

⑤ Find the  $n^{\text{th}}$  derivative of  $\frac{x^3}{(x-a)(x-b)(x-c)}$

$$\frac{x^3}{(x-a)(x-b)(x-c)} = \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$$

$$= \frac{(x-b)(x-c)A}{(x-a)} + \frac{(x-a)(x-c)B}{(x-b)} + \frac{(x-a)(x-b)C}{(x-c)}$$

### (16) Partial Fraction.

$$\textcircled{1} \frac{1}{x^2 - a^2} = \frac{1}{(x+a)(x-a)}$$

$$= \frac{1}{2a} \left[ \frac{1}{(x-a)} - \frac{1}{(x+a)} \right],$$

$$\textcircled{2} \frac{1}{x^2 + a^2} = \frac{1}{(x-ia)(x+ia)}$$

$$= \frac{1}{2ia} \left[ \frac{1}{(x-ia)} - \frac{1}{(x+ia)} \right]$$

$$\frac{x^3}{(x-a)(x-b)(x-c)} = \frac{(x-b)(x-c)A + (x-a)(x-c)B + (x-a)(x-b)C}{(x-a)(x-b)(x-c)}. \quad (17)$$

$$x^3 = (x-b)(x-c)A + (x-a)(x-c)B + (x-a)(x-b)C.$$

Put  $x = a$ .

$$a^3 = (a-b)(a-c)A + (a-a)(a-c)B + (a-a)(a-b)C.$$

$$a^3 = (a-b)(a-c)A + B(0) + C(0).$$

$$a^3 = A(a-b)(a-c).$$

$$A = \frac{a^3}{(a-b)(a-c)}.$$

$$B = \frac{b^3}{(b-a)(b-c)}.$$

$$C = \frac{c^3}{(c-a)(c-b)}.$$

$$y = \sum \frac{a}{(a-b)(a-c)} \frac{1}{(x-a)}.$$

$$y_n = \sum \frac{a}{(a-b)(a-c)} \frac{(-1)^n n! 1^n}{(x-a)^{n+1}}.$$

(6.) Find the  $n^{th}$  derivative of  $y = \frac{3}{(x+1)(x-1)}$ .

Sol :

$$\frac{3}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}.$$

(18)

$$\frac{3}{(x+1)(x-1)} = \frac{A(x-1)}{(x+1)} + \frac{B(x+1)}{(x-1)}.$$

$$3 = A(x-1) + B(x+1).$$

Let,

$$x=1$$

$$3 = A(1-1) + B(1+1).$$

$$3 = B(2).$$

$$\boxed{B = \frac{3}{2}}.$$

$$B = \frac{3}{2}, x = -1.$$

$$3 = A(-1-1) + B(-1+1).$$

$$3 = A(-2) + B(0).$$

$$\boxed{A = -\frac{3}{2}}.$$

$$3 = -\frac{3}{2}/(x-1) + \frac{3}{2}/(x+1).$$

$$\boxed{3 = -\frac{3}{2} \frac{(-1)^n n! 1^n}{(x-1)^{n+1}} + \frac{3}{2} \frac{(-1)^n n! 1^n}{(x+1)^{n+1}}}.$$

(7)

If  $y = \tan^{-1}\left(\frac{x}{a}\right)$  show that  $y_n =$

$$(-1)^{n-1} (n-1)! a^{-n} \sin^n \theta \cos^n \theta.$$

Sol:

(19)

Prove:

$$\text{Given } y_n = \tan^{-1}\left(\frac{x}{a}\right).$$

$$\tan y = \frac{x}{a}.$$

Differentiate with respect to  $x$ .

$$\sec^2 y \cdot \frac{dy}{dx} = \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{1}{a \cdot \sec^2 y}$$

$$\sec^2 y = 1 + \tan^2 y$$

$$\frac{dy}{dx} = \frac{1}{a \left( \frac{a^2 + x^2}{a^2} \right)}$$

$$\begin{aligned} \sec^2 y &= 1 + \frac{x^2}{a^2} \\ &= \frac{a^2 + x^2}{a^2} \end{aligned}$$

$$= \frac{1}{\frac{a^2 + x^2}{a}}$$

$$\frac{dy}{dx} = \frac{a}{a^2 + x^2}$$

$$y_1 = \frac{d}{2i\alpha} \left\{ \frac{1}{x-i\alpha} - \frac{1}{x+i\alpha} \right\}$$

$$y_n = \frac{1}{2i} (-1)^{n-1} (n-1)! \left[ \frac{1}{(x-i\alpha)^n} - \frac{1}{(x+i\alpha)^n} \right]$$

(20)

Put,

$$x = r \cos \theta, \quad a = r \sin \theta.$$

$$\therefore y_n = \frac{1}{2^n} (-1)^{n-1} (n-1)!$$

$$\left[ r^{-n} (\cos \theta - i \sin \theta)^n = r^{-n} (\cos n\theta + i \sin n\theta) \right]$$

$$= \frac{1}{2^n} (-1)^{n-1} (n-1)! r^{-n} (\cos n\theta + i \sin n\theta)$$

$$y_n = (-1)^n (n-1)! a^{-n} \sin^n \theta + i \cos^n \theta.$$

### 1.5. TRIGONOMETRICAL TRANSFORMATION

It is possible to break up products of powers of sines and cosines into a sum by trigonometrical method.

Problems :

(1)  $\sin^3 x$

Sol:

$$\text{Let } y = \sin^3 x$$

$$y = \frac{3}{4} \sin 2x - \frac{1}{4} \sin 6x.$$

$$\therefore y_n = \frac{3}{4} 2^n \sin \left( 2x + \frac{n\pi}{2} \right) - \frac{1}{4} 6^n \sin \left( 6x + \frac{n\pi}{2} \right)$$

Formula:

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\sin^3 A = \frac{3 \sin A - \sin 3A}{4}$$

$$\sin(a \pm b)$$

$$a^n \sin(a \pm b \pm \frac{n\pi}{2})$$

(2) Find the derivative of  $\cos^4 x$ .

(21)

Sol:

$$y = \cos^4 x.$$

$$= (\cos^2 x)^2.$$

$$= \left( \frac{1 + \cos 2x}{2} \right)^2$$

$$= \frac{1}{4} [1 + 2\cos 2x + \cos^2 2x].$$

$$= \frac{1}{4} \left[ 1 + 2\cos 2x + \frac{1 + \cos 4x}{2} \right]$$

$$= \frac{1}{4 \times 2} [2 + 4\cos 2x + 1 + \cos 4x].$$

$$y = \frac{1}{8} [3 + 4\cos 2x + \cos 4x].$$

$$\therefore y_n = \frac{1}{8} \left[ 4 \cdot 2^n \cos \left( 2x + n\frac{\pi}{2} \right) + 4^n \cos \left( 4x + n\frac{\pi}{2} \right) \right].$$

(3) Find the derivative of  $\sin^4 x$ .

Sol:

$$y = \sin^4 x.$$

$$= (\sin^2 x)^2.$$

$$= \left( \frac{1 - \cos 2x}{2} \right)^2.$$

$$= \frac{1}{4} (1 - 2\cos 2x + \cos^2 2x).$$

$$= \frac{1}{4} (1 - 2\cos 2x + \frac{1 + \cos 4x}{2}).$$

Formula.

$$\sin^2 x = \frac{1 - \cos 2x}{2}.$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}.$$

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$$= \frac{1}{8} (2 - 4 \cos 2x + 1 + \cos 4x)$$

$$y = \frac{1}{8} (3 - 4 \cos 2x + \cos 4x).$$

$$y_m = \frac{1}{8} \left[ -4 \cdot 2^n \cos \left( 2x + \frac{n\pi}{2} \right) + 4^n \cos \left( 4x + \frac{n\pi}{2} \right) \right]$$

4.  $\sin^3 x \cos^2 x.$

$$y = \sin^3 x \cos^2 x.$$

$$= \left[ \frac{3 \sin x - \sin 3x}{4} \right] \left[ \frac{1 + \cos 2x}{2} \right].$$

$$y = \frac{1}{8} [3 \sin x - \sin 3x + 3 \cos 2x \sin x - \sin 3x \cos 2x],$$

$$y = \frac{1}{8} [3 \sin x - \sin 3x + \frac{3}{2} (\sin 3x - \sin x)],$$

$$\uparrow - \frac{1}{2} (\sin 5x + \sin x).$$

Formula

$$\therefore \cos A \sin B = \sin(A+B) + \sin(A-B)$$

sin A

$$y = \frac{1}{16} [6 \sin x - 2 \sin 3x + 3 \sin 3x - 3 \sin - \sin 5x - \sin x].$$

$$y = \frac{1}{16} [2 \sin x + \sin 3x - \sin 5x].$$

$$\therefore y_m = \frac{1}{16} \left[ 2 \sin \left( x + \frac{n\pi}{R} \right) + 3^{\circ} 280 \left( 2^{\circ} 74 \frac{n\pi}{R} \right) \right] \\ - 5^{\circ} 280 \left( 2x + \frac{n\pi}{R} \right)$$

$$R \cos A \sin B = \sin(A+B) - \sin(A-B)$$

$$\cos A \sin B = \frac{1}{2} [\sin(A+B) - \sin(A-B)]$$

$$R \sin A \cos B = \frac{1}{2} [\sin(A+B) + \sin(A-B)]$$

$$\cos A \sin B = \frac{1}{2} [\underline{\sin(A+B)} + \underline{\sin(A-B)}]$$

$$5. \quad \sin 2x \sin 4x \sin 6x.$$

$$\text{Sol:} \quad y = \sin 2x \sin 4x \sin 6x.$$

$\times^4$  &  $\div$  by 2.

$$= \frac{1}{2} [2 \sin 2x \sin 4x \sin 6x].$$

$$= \frac{1}{2} [\cos 2x - \cos 6x] (\sin 6x).$$

$$= \frac{1}{2} [\cos 2x \sin 6x - \cos 6x \sin 6x].$$

$\times^4$  &  $\div$  by 2.

$$= \frac{1}{4} [2 \cos 2x \sin 6x - 2 \cos 6x \sin 6x].$$

$$= \frac{1}{4} [\sin 8x + \sin 4x - (\sin 12x - \sin 12x)].$$

(2h)

$$\begin{aligned}
 &= \frac{1}{4} [\sin 8x + \sin 4x - \sin 12x] \\
 &= \frac{1}{4} \left( 8^n \sin \left( 8x + n\frac{\pi}{2} \right) - 4^n \sin \left( 4x + n\frac{\pi}{2} \right) - \right. \\
 &\quad \left. 12^n \sin \left( 12x + n\frac{\pi}{2} \right) \right).
 \end{aligned}$$

6.  $y = \frac{ax+b}{cx+d}$  find  $\frac{d^2y}{dx^2}$ .

Sol:

$$y = \frac{ax+b}{cx+d} \cdot \left( \frac{u}{v} \right) = \frac{vu' - uv'}{v^2}$$

Differential with respect to  $x$ .

$$\frac{dy}{dx} = \frac{(c(x+d))(a) - (ax+b)(c)}{(cx+d)^2}$$

$$= \frac{acx+ad - acx - bc}{(cx+d)^2}$$

$$\frac{dy}{dx} = \frac{ad - bc}{(cx+d)^2}$$

Again differential with respect to  $x$ .

(25)

$$\frac{d^2y}{dx^2} = \frac{(cx+d)^2(c)- (ad-bc)2(cx+d).c}{(cx+d)^4}$$

$$\frac{d^2y}{dx^2} = \frac{-2(ad-bc)}{(cx+d)^3}$$

10 mark

7.  $\sin^5 x \cos^4 x$ .

Sol:

$$y = \sin^5 x \cos^4 x.$$

Let,

$$z = \cos x + i \sin x.$$

$$\frac{1}{z} = \cos x - i \sin x.$$

$$z + \frac{1}{z} = 2 \cos x.$$

$$z - \frac{1}{z} = 2i \sin x. \quad z^2 + \frac{1}{z^2} = 2 \cos^2 x \\ z^2 - \frac{1}{z^2} = 2i \sin x. \quad z^4 + \frac{1}{z^4} = (a+b)^4$$

Consider

$$(z - \frac{1}{z})^5 (z + \frac{1}{z})^4 = (z^2 - \frac{1}{z^2})^4 (z - \frac{1}{z}).$$

Formula:

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

$$= [(z^2)^4 - 4(z^2)^3 \left(\frac{1}{z^2}\right) + 6(z^2)\left(\frac{1}{z^2}\right)^2 - 4(z^2) \left(\frac{1}{z^2}\right)^3 + \left(\frac{1}{z^2}\right)^4] \cdot z - \frac{1}{z}$$

$$\Rightarrow \left[ z^8 - 4z^4 + 6z^{-4} + \frac{1}{z^8} \right] \left[ z - \frac{1}{z} \right].$$

$$\Rightarrow z^9 - 4z^5 + 6z^{-1} + \frac{1}{z^7} - z^7 + 4z^3 - \frac{6}{z} + \frac{4}{z^5} - \frac{1}{z^9}.$$

$$\Rightarrow \left( z^9 - \frac{1}{z^9} \right) - \left( z^7 - \frac{1}{z^7} \right) - 4 \left( z^5 - \frac{1}{z^5} \right) + 4 \left( z^3 - \frac{1}{z^3} \right) + 6 \left( z - \frac{1}{z} \right)$$

$$(2i \sin x)^5 (2 \cos x)^4 = 2i \sin^9 x - 2i \sin^7 x - 4(2i \sin^5 x) + 4(2i \sin^3 x) + 6(2i \sin x),$$

$$(2i \sin x)^5 (2 \cos x)^4 = 2i (\sin^9 x - \sin^7 x - 4 \sin^5 x + 4 \sin^3 x + 6 \sin x).$$

$$(2i)^5 \sin^5 x \cdot 2^4 \cos^4 x = 2i (\sin^9 x - \sin^7 x - 4 \sin^5 x + 4 \sin^3 x + 6 \sin x).$$

$$(2i)^4 \sin^5 x \cdot 2^4 \cos^4 x = \sin^9 x - \sin^7 x - 4 \sin^5 x + 4 \sin^3 x + 6 \sin x,$$

$$\sin^5 x \cos^4 x = \frac{1}{2^8} (\sin^9 x - \sin^7 x - 4 \sin^5 x + 4 \sin^3 x + 6 \sin x).$$

$$\sin^5 \alpha \cos^3 \alpha = \frac{1}{2^8} \left[ 4 \cdot 5 \cdot 1^n \sin \left( 4x + \frac{n\pi}{2} \right) + 4 \cdot 3 \cdot 1^n \sin \left( 3x + \frac{n\pi}{2} \right) + 6 \cdot 1^n \sin \left( x + \frac{n\pi}{2} \right) \right].$$

$$8. \cos x \cos 2x \cos 3x.$$

Sol:

$$y = \cos x \cos 2x \cos 3x.$$

$$\therefore 2 \cos A \cos B = \cos(A+B) + \cos(A-B)$$

$\times^y$  and  $\div$  by 2.

$$= \frac{1}{2} [2(\cos x \cos 2x \cos 3x)].$$

$$= \frac{1}{2} [2 \cos x \cos 2x (\cos 3x)].$$

$$= \frac{1}{2} [\cos 3x + (-\cos x)] (\cos 3x)$$

$$= \frac{1}{2} [\cos 3x - \cos x] \cos 3x.$$

$$= \frac{1}{2} [\cos 3x \cos 3x + \cos x \cos 3x].$$

$x$  and  $\div$  by 2.

$$= \frac{1}{2} [\cos 6x + (0) - \cos 4x + \cos 2x].$$

$$= \frac{1}{4} [\cos 6x - \cos 4x + \cos 2x].$$

$$\frac{1}{4} \left[ 6^n \cos\left(6x + \frac{n\pi}{2}\right) - 1^n \cos\left(1x + \frac{n\pi}{2}\right) - 2^n \cos\left(2x + \frac{n\pi}{2}\right) \right] \quad (20)$$

### 1.6 Formations of Equations involving Derivatives.

A relation between  $x$  and  $y$  is given, we can deduce from it a relation between the variables  $x, y$  and the derivatives of  $y$  with respect to  $x$ .

$$\text{Eq: } xy = ae^{ax} + be^{-x}$$

#### Problems.

- 1) If  $xy = ae^x + be^{-x}$  Prove that  
 $x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - xy = 0$ .

Sol: Given:

$$xy = ae^x + be^{-x}$$

Differential with respect to  $x$ :

$$x \frac{dy}{dx} + y = ae^x + be^{-x}$$

Differential with respect to  $x$ . (31)

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = ae^x + be^{-x}$$

$$x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} = 2y$$

$$x \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 2y = 0$$

Hence Proved.

- 2.) Prove that if  $y = \sin(m \sin^{-1}x)$ ,  
 $(1-x^2)y_2 - xy_1 + m^2 y = 0$ .

Sol:  $y = \sin(m \sin^{-1}x)$ .  $\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$ .

$$\sin^{-1}y = m \sin^{-1}x$$

Differential with respect to  $x$ .

$$\frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} = \frac{m}{\sqrt{1-x^2}}$$

Squaring & Transposing, we have

$$\left( \frac{1}{(1-y^2)^{1/2}} \right)^2 \left( \frac{dy}{dx} \right)^2 = \left( \frac{m}{(1-x^2)^{1/2}} \right)^2$$

$$\left(\frac{1}{1-y^2}\right) \left(\frac{dy}{dx}\right)^2 = \frac{m^2}{(1-x^2)}. \quad (1-y^2)^{2m+1} - 2y$$

$$(1-x^2) \left(\frac{dy}{dx}\right)^2 = (1-y^2)m^2.$$

Differential with respect to  $x$ .

$$\text{Div. } (1-x^2)^2 \frac{dy}{dx} \frac{d^2y}{dx^2} - 2x \left(\frac{dy}{dx}\right)^2 = -2m^2y \frac{dy}{dx}.$$

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \left(\frac{dy}{dx}\right) = -m^2y.$$

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \left(\frac{dy}{dx}\right) + m^2y = 0.$$

$$(1-x^2)y_2 - 2xy_1 + m^2y = 0.$$

Hence Proved.

3) If  $x = m \sin \theta$ ,  $y = \cos p\theta$  Prove that

$$(1-x^2)y_2 - 2xy_1 + p^2y = 0.$$

Sol: Let,  
 $x = m \sin \theta$ ,  $y = \cos p\theta$ .

$$\frac{dx}{d\theta} = m \cos \theta, \quad \frac{dy}{d\theta} = -p \sin p\theta.$$

$$\frac{dy}{dx} = -\frac{P \sin p\theta}{\cos \theta} = \frac{-Py}{x}$$

(33)

$$= -\frac{P\sqrt{1-y^2}}{\sqrt{1-x^2}} \cdot \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\cos \theta = \sqrt{1-\sin^2 \theta}.$$

Squaring on both sides.

$$\left(\frac{dy}{dx}\right)^2 = P^2 \left(\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\right)^2$$

$$\left(\frac{dy}{dx}\right)^2 = P^2 \left(\frac{1-y^2}{1-x^2}\right).$$

$$\therefore (1-x^2) \left(\frac{dy}{dx}\right)^2 = P^2 (1-y^2).$$

Differential with respect to  $x$ .

$$(1-x^2)^2 \frac{dy}{dx} \left(\frac{dy}{dx}\right)^2 - 2x \left(\frac{dy}{dx}\right) = -2P^2 y \frac{dy}{dx}.$$

$$2 \frac{dy}{dx} \left[ (1-x^2) \frac{d^2y}{dx^2} - \frac{x dy}{dx} \right] = 2 \frac{dy}{dx} [-P^2 y]$$

$$(1-x^2) y_2 - xy_1 + P^2 y = 0.$$

Hence proved.

Another method

$$x = \sin \theta, y = \cos p\theta.$$

$$\frac{dx}{d\theta} = \cos \theta, \quad \frac{dy}{d\theta} = -P \sin p\theta.$$

$$\frac{dy}{dx} = p \sin p\theta$$

$$\frac{d^2y}{dx^2} = \cos p\theta$$

$$(uv)' = uv' + vu'$$

$$\cos p\theta \frac{dy}{dx} = -p \sin p\theta.$$

$$\cos p\theta \frac{d^2y}{dx^2} + \frac{dy}{dx} (-\sin p\theta) \frac{d\theta}{dx} = -p^2 \cos p\theta \frac{d\theta}{dx}$$

$$\frac{d\theta}{dx} \left[ \cos p\theta \frac{d^2y}{dx^2} - \sin p\theta \frac{dy}{dx} \right] = -p^2 \cos p\theta.$$

$$\cos p\theta \frac{d^2y}{dx^2} \frac{dx}{d\theta} - \sin p\theta \frac{dy}{dx} \frac{d\theta}{dx} \left( \frac{dx}{d\theta} \right) = -p^2 \cos p\theta.$$

$$\cos p\theta \frac{d^2y}{dx^2} \frac{dx}{d\theta} - \sin p\theta \frac{dy}{dx} = -p^2 \cos p\theta.$$

$$\cos p\theta \cdot \cos p\theta \frac{d^2y}{dx^2} - \sin p\theta \frac{dy}{dx} = -p^2 \cos p\theta.$$

$$\cos^2 p\theta \frac{d^2y}{dx^2} - \sin p\theta \frac{dy}{dx} = -p^2 \cos p\theta.$$

$$(1 - \sin^2 p\theta) \frac{d^2y}{dx^2} - \sin p\theta \frac{dy}{dx} = -p^2 \cos p\theta.$$

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = -p^2 y.$$

$$(1 - x^2)y_2 - xy_1 + p^2 y = 0.$$

Hence Proved.

4) If  $y = -x^3 \log x$  prove that (35)

$$x \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 3x^2 = 0.$$

Sol:

Diff with respect to  $x$ :

$$\frac{dy}{dx} = -x^3 \left(\frac{1}{x}\right) + \log x (-3x^2).$$

$$\Rightarrow -\frac{x^3}{x} - 3x^2 \log x.$$

$$\frac{dy}{dx} = -x^2 - 3x^2 \log x.$$

Again diff. with respect to  $x$ .

$$\frac{d^2y}{dx^2} = -2x - \left[ 3x^2 \left(\frac{1}{x}\right) + \log x (6x) \right].$$

$$\frac{d^2y}{dx^2} = -2x - \left[ \frac{3x^2}{x} + 6x \log x \right]$$

$$= -2x - [3x + 6x \log x].$$

$$\therefore -2x - 3x - 6x \log x.$$

$\times^4$  by  $x$ .

$$x \frac{d^2y}{dx^2} = -2x^2 - 3x^2 - 6x^2 \log x.$$

$$x \frac{d^2y}{dx^2} + 3x^2 = -2x^2 - 6x^2 \log x.$$

$$x \frac{d^2y}{dx^2} + 3x^2 = 2(-x^2 - 3x^2 \log x).$$

$$\frac{x \cdot d^2y}{dx^2} + 3x^2 - 2 \frac{dy}{dx} = 0.$$

Hence Proved.

(10) 5.) If  $x = a(\cos t + t \sin t)$ ,  $y = a(\sin t - t \cos t)$

Find  $\frac{d^2y}{dx^2}$ .

Sol:

Given.

$$\Rightarrow x = a(\cos t + t \sin t).$$

$$\frac{dx}{dt} = a(-\sin t + \sin t + t \cos t) = a(t \cos t).$$

$$\frac{dx}{dt} = a(t \cos t).$$

$$\Rightarrow y = a(\sin t - t \cos t).$$

$$\frac{dy}{dt} = a[\cos t - (\sin t + t \cos t + \cos t)] = a[\cos t + t \sin t - \cos t].$$

$$\frac{dy}{dt} = a t \sin t.$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{at \sin t}{at \cos t} = \frac{\sin t}{\cos t} = \tan t.$$

$$\frac{dy}{dx} = \frac{at \sin t}{at \cos t} = \frac{\sin t}{\cos t} = \tan t.$$

$$\frac{dy}{dx} = \tan t.$$

Differential with respect to  $x$ . (Q)

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right)$$

$$= \frac{d}{dt} \left( \frac{dy}{dx} \right) \frac{dt}{dx}$$

$$= \frac{d}{dt} (\tan t) \frac{dt}{dx} \Rightarrow \frac{dt}{dx} = \frac{1}{\text{at cost}}$$

$$= \sec^2 t \frac{1}{\text{at cost}} \cdot \text{sextant lost}$$

$$= \frac{\sec^3 t}{\text{at}}$$

6)  $y = a \cos 5x + b \sin 5x$  Show that

$$\frac{d^2y}{dx^2} + 25y = 0.$$

Sol:

$$y = a \cos 5x + b \sin 5x.$$

Differential with respect to  $x$ .

$$\frac{dy}{dx} = -5a \sin 5x + 5b \cos 5x.$$

$$\frac{d^2y}{dx^2} = -5^2 a \cos 5x - 5^2 b \sin 5x.$$

$$= -25 a \cos 5x - 25 b \sin 5x.$$

$$\frac{d^2y}{dx^2} = -25 a \cos 5x + 25 b \sin 5x.$$

$$\frac{d^2y}{dx^2} = -25y.$$

$$\frac{d^2y}{dx^2} + 25y = 0.$$

Hence Proved.

7.  $x = \sin t, y = \sin pt$ . Prove that

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + p^2 y = 0.$$

Sol:

Given.

$$x = \sin t.$$

$$\frac{dx}{dt} = \cos t.$$

$$y = \sin pt.$$

$$\frac{dy}{dt} = p \cos pt.$$

$$\frac{dy}{dx} = \frac{p \cos pt}{\cos t}$$

$$\frac{d^2y}{dx^2} = \left[ \frac{\cos t (-p^2 \sin pt) + p \cos pt \sin t}{\cos^2 t} \right]$$

$$= \left[ \frac{p \cos pt \sin t - p^2 \sin pt \cos t}{\cos^2 t} \right]$$

$$\frac{d^2y}{dx^2} = \left[ \frac{p [\cos pt \sin t - p \cos t \sin pt]}{\cos^2 t} \right]$$

$$= \left[ \frac{p (\cos pt \sin t - p \cos t \sin pt)}{(1 - \sin^2 t)} \right] \frac{dt}{dx}$$

$$\frac{d^2y}{dx^2}(1-x^2) = p \cos pt \sin t \frac{dt}{dx} - p^2 \sin pt \quad (3)$$

$\cos t \frac{dt}{dx}$

$$\frac{d^2y}{dx^2}(1-x^2) = p \frac{\cos pt}{\cos t} x - p^2 y \frac{\cos t}{\cos t}.$$

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + p^2 y = 0.$$

∴ Hence proved.

8.)  $y = e^{-x} \cos x$  prove that  $\frac{d^4y}{dx^4} + 4y = 0$ .

Given:

$$y = e^{-x} \cos x.$$

$$\frac{dy}{dx} = -e^{-x} \sin x - \cos x e^{-x}.$$

$$\frac{d^2y}{dx^2} = -e^{-x} [\sin x + \cos x].$$

$$= -e^{-x} [\cos x - \sin x] + e^{-x} [\sin x + \cos x]$$

$$= e^{-x} [\sin x + \cos x - \cos x + \sin x].$$

$$= e^{-x} [2 \sin x].$$

$$\frac{d^3y}{dx^3} = e^{-x} [2 \cos x] - e^{-x} [2 \sin x].$$

$$= e^{-x} [2 \cos x - 2 \sin x].$$

$$= 2e^{-x} [\cos x - \sin x]$$

$$\begin{aligned} \frac{d^4y}{dx^4} &= 2 \left\{ e^{-x} [-\sin x - \cos x] + (-e^{-x}) \right. \\ &\quad \left. [\cos x - \sin x] y \right\}. \\ &= 2 e^{-x} [\sin x - \cos x - (\cos x - \sin x)]. \\ &= 2 e^{-x} [-2 \cos x]. \\ &= -4y. \end{aligned}$$

$$\frac{d^4y}{dx^4} + 4y = 0.$$

Idence Proved.

Leibnitz formula for the  $n^{\text{th}}$  Derivative of a product.

If small  $u$  and  $v$  are function of  $x$ , we have.

$$\frac{d}{dx}(uv) = v \frac{du}{dx} + u \frac{dv}{dx}.$$

$$\text{i.e } D(uv) = vD(u) + uD(v).$$

Differential with respect to  $x$ , we get,

$$\begin{aligned} D^2(uv) &= D(v \cdot D(u)) + D(u \cdot D(v)) \\ &= vD^2(u) + D(u)D(v) + uD^2(v) + \\ &\quad D(v)D(u). \end{aligned}$$

$$D^2(uv) = vD^2(u) + 2uD(u)v + uD^2(v) \quad (4)$$

$$D^n(uv) = U_n v + nC_1 U_{n-1} v_1 + nC_2 U_{n-2} v_2 + \dots + UV_n.$$

Problem:

1) Find the  $n^{\text{th}}$  derivative of  $x^2 e^{5x}$ .

Let,  $y = x^2 e^{5x}$

$$u = e^{5x}, \quad v = x^2.$$

$$U_n = 5^n e^{5x}, \quad v_1 = 2x.$$

$$U_{n-1} = 5^{n-1} e^{5x}, \quad v_2 = 2.$$

$$U_{n-2} = 5^{n-2} e^{5x}.$$

$$\frac{d^n y}{dx^n} = U_n v + nC_1 U_{n-1} v_1 + nC_2 U_{n-2} v_2 + \dots$$

$$= 5^n x^2 e^{5x} x^2 + nC_1 5^{n-1} e^{5x} 2x + nC_2$$

$$5^{n-2} e^{5x} \cdot 2.$$

$$nC_1 = n.$$

$$nC_2 = \frac{n(n-1)}{2} = 5^n e^{5x} x^2 + n 5^{n-1} e^{5x} \cdot 2x + \frac{n(n-1) 5^{n-2}}{2}$$

$$e^{5x} \cdot 2.$$

$$= e^{5x} \left[ 5^n x^2 + 2n x 5^{n-1} + n(n-1) 5^{n-2} \right]$$

$$= e^{5x} 5^{n-2} \left[ 5^2 x^2 + 2n x 5 + n(n-1) \right]$$

$$= e^{5x} 5^{n-2} \left[ 25 x^2 + 10 n x + n(n-1) \right].$$

2. Find the  $n^{\text{th}}$  Derivative of

$$x^2 \sin 5x.$$

Let  $y = x^2 \sin 5x$ .

$$u = \sin 5x \quad v = x^2$$

$$U_n = 5^n \sin\left(5x + n\frac{\pi}{2}\right) \quad V_1 = 2x$$

$$U_{n-1} = 5^{n-1} \sin\left(5x + (n-1)\frac{\pi}{2}\right) \quad V_2 = 2.$$

$$U_{n-2} = 5^{n-2} \sin\left(5x + (n-2)\frac{\pi}{2}\right)$$

$$\frac{d^n y}{dx^n} = U_n v + n C_1 U_{n-1} V_1 + n C_2 U_{n-2} V_2 + \dots$$

$$= 5^n \sin\left(5x + \frac{n\pi}{2}\right) x^2 + n C_1 5^{n-1} \sin\left(5x + (n-1)\frac{\pi}{2}\right)$$

$$2x + n C_2 5^{n-2} \sin\left(5x + (n-2)\frac{\pi}{2}\right) - 2.$$

$$= 5^{n-2} \left[ 5^2 \sin\left(5x + \frac{n\pi}{2}\right) x^2 + 5n \sin\left(5x + (n-1)\frac{\pi}{2}\right) \right]$$

$$+ \frac{n(n-1)}{2} \sin\left(5x + (n-2)\frac{\pi}{2}\right) - 2$$

$$= 5^{n-2} \left[ 25x^2 \sin\left(5x + \frac{n\pi}{2}\right) + 10x n \sin\left(5x + (n-1)\frac{\pi}{2}\right) \right]$$

$$+ n(n-1) \sin\left(5x + (n-2)\frac{\pi}{2}\right) - 2$$

$$= 5^{n-2} \left[ 25x^2 \sin\left(5x + \frac{n\pi}{2}\right) + 10x n \sin\left(5x + (n-1)\frac{\pi}{2}\right) \right] + n(n-1) \sin\left(5x + (n-2)\frac{\pi}{2}\right) - 2$$

$$= 5^{n-2} \left[ 25x^2 \sin\left(5x + \frac{n\pi}{2}\right) + 10x n \sin\left(5x + (n-1)\frac{\pi}{2}\right) \right] + n(n-1) \sin\left(5x + (n-2)\frac{\pi}{2}\right) - 2$$

$$= 5^{n-2} \left[ 25x^2 \sin\left(5x + \frac{n\pi}{2}\right) + 10x n \sin\left(5x + (n-1)\frac{\pi}{2}\right) \right] + n(n-1) \sin\left(5x + (n-2)\frac{\pi}{2}\right) - 2$$

3) If  $y = a \cos(\log x) + b \sin(\log x)$  (43)  
 Prove That  $x^2 y_{n+2} + (2n+1)x y_{n+1} + (n^2 + 1)y_n = 0$ .

$$y = a \cos(\log x) + b \sin(\log x).$$

Dif. with respect to  $x$ .

$$y_1 = -\frac{a \sin(\log x)}{x} + \frac{b \cos(\log x)}{x}$$

$$x y_1 = -a \sin(\log x) + b \cos(\log x).$$

Again,

$$x(y_2 + y_1) = -\frac{a \cos(\log x)}{x} - \frac{b \sin(\log x)}{x}$$

Multiply by  $x$ .

$$\begin{aligned} x^2 y_2 + x y_1 &= -a \cos(\log x) - b \sin(\log x). \\ &= -y. \end{aligned}$$

$$\therefore x^2 y_2 + x y_1 + y = 0.$$

Applying Leibnitz theorem to differentiate this  $n$  times, we get.

$$(x^2 y_0)_n + (x y_1)_n + y_n = 0.$$

$$\Rightarrow [y_{n+2} x^2 + n C_1 y_{n+1} (2x) + n (2 y_n \cdot 2)]$$

$$+ [y_{n+1} x + n C_1 y_n \cdot 1] + y_n = 0.$$

$$\Rightarrow x^2 y_{n+2} + 2n x y_{n+1} + n(n-1) y_n + x y_{n+1} +$$

$$n y_n + y_n = 0$$

$$\Rightarrow \alpha^2 y_{n+2} + 2\alpha y_{n+1} + \alpha^2 y_n - \alpha y_n +$$

$$2y_{n+1} + \alpha y_n + y_n = 0.$$

$$\Rightarrow \alpha^2 y_{n+2} + (2n+1)y_{n+1} + (n^2+1)y_n = 0.$$